Change of Basis in Polynomial Interpolation

A Seminar.sty Demonstration

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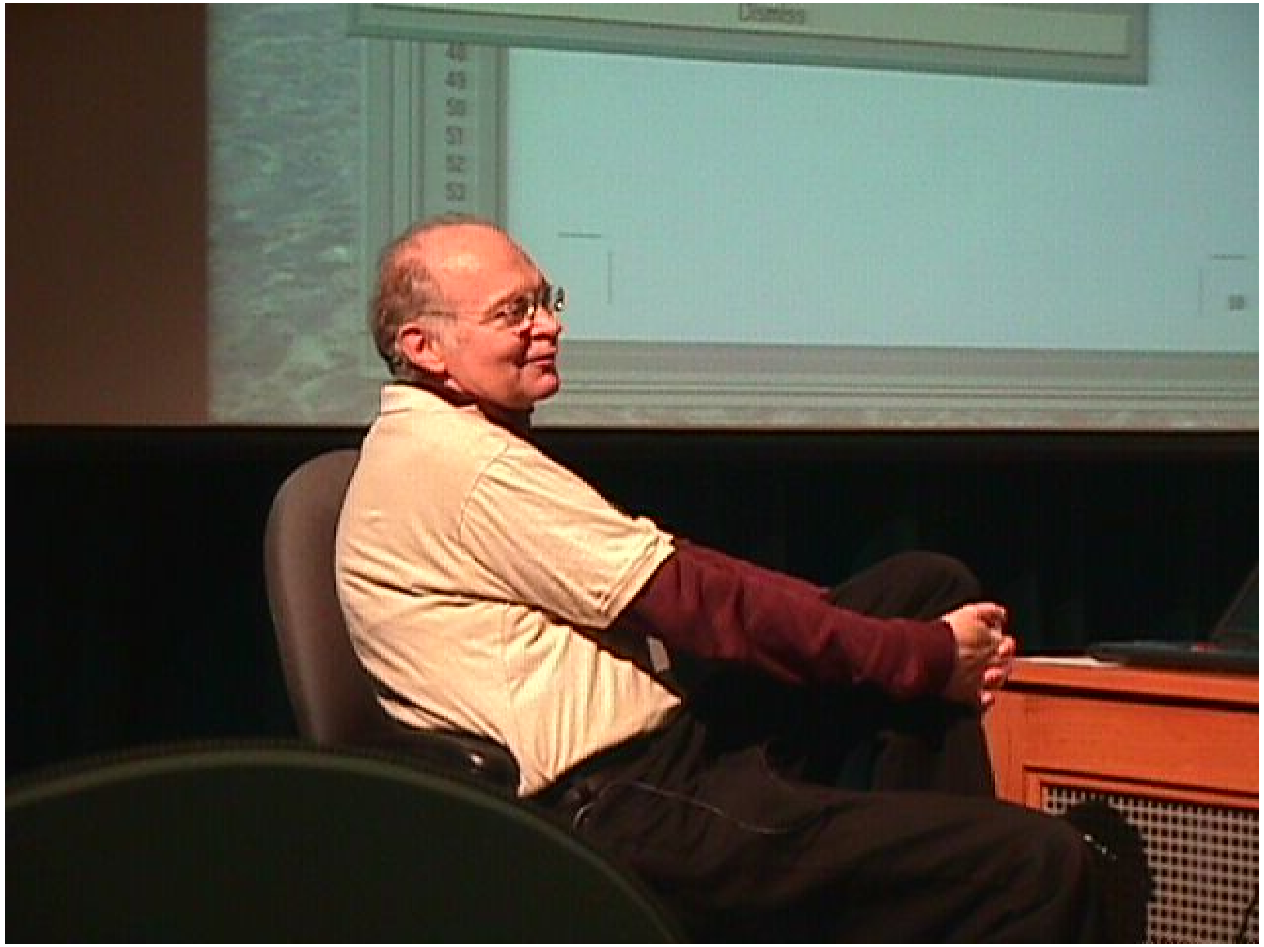
Prof. Don Knuth Inventor of TEX



The Interpolating Polynomial (Monomial basis)

Interpolation condition: For Newton click here

$$P(x_i) = f(x_i), \quad i = 0, \ldots, n$$

$$P_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n$$

$$\underbrace{\begin{bmatrix} 1 & x_0 & \ldots & x_0^{n-1} & x_0^n \\ 1 & x_1 & \ldots & x_1^{n-1} & x_1^n \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & \ldots & x_n^{n-1} & x_n^n \end{bmatrix}}_{V} \underbrace{\begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}}_{\mathbf{a}} = \underbrace{\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}}_{\mathbf{f}}$$

$$\mathbf{m}(x) = [1, x, x^2, \ldots, x^n]^T, \quad \mathbf{a} = [a_0, a_1, \ldots, a_n]^T \quad P_n(x) = \mathbf{a}^T\mathbf{m}(x)$$

Lagrange Interpolation Basis: Lagrange Polynomials

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad i = 0, 1, \ldots, n$$

$$P_n(x) = \sum_{j=0}^{n} f(x_j)\, l_j(x)$$

$$\mathbf{l}(x) = [l_0(x), l_1(x), \ldots, l_n(x)]^T, \qquad \mathbf{f} = [f_0, f_1, \ldots, f_n]^T$$

$$P_n(x) = \mathbf{f}^T \mathbf{l}(x) = \mathbf{a}^T \mathbf{m}(x)$$

Newton Interpolation Basis: Newton polynomials for Lagrange click here

$\pi_k(x) = \prod_{j=0}^{k-1}(x - x_j), \quad k = 0, \ldots, n$

$P_n(x) = d_0\pi_0(x) + d_1\pi_1(x) + \cdots d_n\pi_n(x)$

Interpolation condition: $\sum_{j=0}^{n} d_j\pi_j(x_i) = f(x_i), \quad i = 0, \ldots n.$

Matrix of system is lower triangular $U^T\mathbf{d} = \mathbf{f}$ since $\pi_k(x_j) = 0$, $j < k$

$$\begin{pmatrix} \pi_0(x_0) & \cdots & \pi_n(x_0) \\ \pi_0(x_1) & \cdots & \pi_n(x_1) \\ \vdots & \cdots & \vdots \\ \pi_0(x_n) & \cdots & \pi_n(x_n) \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ 1 & x_1 - x_0 & & & \\ 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - x_1) & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & x_n - x_0 & (x_n - x_0)(x_n - x_1) & \cdots & \prod_{j=0}^{n-1}(x_n - x_j) \end{pmatrix}$$

Solve for $\mathbf{d}$ by forward substitution:

$$d_0 = f(x_0), \quad d_i = \frac{f(x_i) - \sum_{j=0}^{i-1} d_j\pi(x_i)}{\pi_i(x_i)}, \; i = 1, \ldots, n.$$

The source file can be compiled by PDFLaTeX.

If you don't have some style files, download them from:
http://www.dante.de/cgi-bin/ctan-index