Refining Strategic Ability in Alternating-time Temporal Logic☆

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Abstract

We propose extending Alternating-time Temporal Logic (ATL) by a unary operator $\langle i \sqsubseteq \Gamma\rangle$ about distributing the powers of some given agent i to a given set of sub-agents Γ. $\langle i \sqsubseteq \Gamma\rangle\varphi$ means that i's powers can be distributed in a way which satisfies the ATL condition φ on the strategic ability of the coalitions the members of Γ may form, possibly together with others agents. We prove the decidability of model checking of formulas whose $\langle . \sqsubseteq .\rangle$-subformulas have the form $\langle i_1 \sqsubseteq \Gamma_1\rangle \dots \langle i_m \sqsubseteq \Gamma_m\rangle\varphi$, with no further occurrences of $\langle . \sqsubseteq .\rangle$ in φ. We also give some axioms and proof rules about the new operator.

Keywords: Alternating-time temporal logic, refinement, model checking, proof system

Introduction

The basic co-operation modality of Alternating-time Temporal Logics (ATL, [1, 2]) invites perceiving agent coalitions as single agents who enjoy the combined powers of the coalition members. We investigate an operator to reverse this, by addressing the possibility to partition the strategic ability of a single agent among several sub-agents. We write $\langle i \sqsubseteq \Gamma\rangle\varphi$ to denote that the strategic ability of agent i can be partitioned among the members of a set of sub-agents Γ in a way which satisfies φ. Here φ is a formula written in terms of the new agents Γ who assume i's powers, and the other primary agents, except i. The sub-agent names from Γ are supposed to be unrelated to the names of the primary agents such as i, just like bound variables and free variables are in quantified languages. For example, a purchase scenario with the *vendor* represented by salesperson SP and delivery team DT can be described by the formula

$$\begin{array}{l}\langle \mathit{vendor} \sqsubseteq \mathit{SP}, \mathit{DT}\rangle \left(\begin{array}{l}\langle\!\langle \mathit{customer}, \mathit{SP}\rangle\!\rangle\Diamond \mathit{purchase\ agreement} \wedge \\ [\![\mathit{SP}, \mathit{others}]\!]\Box(\mathit{purchase\ agreement} \Rightarrow \langle\!\langle \mathit{DT}, \mathit{customer}\rangle\!\rangle \bigcirc \mathit{delivery})\end{array}\right) \wedge \\ \langle\!\langle\emptyset\rangle\!\rangle\Box(\mathit{purchase\ agreement} \Rightarrow \langle\!\langle \mathit{customer}\rangle\!\rangle(\mathit{purchase\ agreement}\,\mathsf{U}\,\mathit{delivery})).\end{array} \tag{1}$$

Here *others* stands for some third party agents who are different from both *vendor* and *customer*. The formula states that the powers of the *vendor* can be divided between SP and DT so that SP and the *customer* can eventually reach a *purchase agreement* and, once a *purchase agreement* is in place, neither SP, nor third parties can prevent DT from cooperating with the *customer* to achieve *delivery* in one step. The second conjunctive member of the formula has no particular relevance to distributing powers. It just states that, once reached, a *purchase agreement* stays in place until *delivery* happens, unless the *customer* gives up on it. In terms of just *vendor* and *customer* as the agents, a coarser description of the same scenario can be given by the formula

$$\begin{array}{l}\langle\!\langle \mathit{customer}, \mathit{vendor}\rangle\!\rangle\Diamond \mathit{purchase\ agreement} \wedge \\ [\![\mathit{others}]\!]\Box(\mathit{purchase\ agreement} \Rightarrow \langle\!\langle \mathit{vendor}, \mathit{customer}\rangle\!\rangle \bigcirc \mathit{delivery})\wedge \\ \langle\!\langle\emptyset\rangle\!\rangle\Box(\mathit{purchase\ agreement} \Rightarrow \langle\!\langle \mathit{customer}\rangle\!\rangle(\mathit{purchase\ agreement}\,\mathsf{U}\,\mathit{delivery}))\end{array} \tag{2}$$

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According to the meaning we assign to $\langle vendor \sqsubseteq SP, DT\rangle$ below, (2) is a logical consequence of (1). We consider only partitionings such that the combined powers of all of i's sub-agents are always equal to i's powers:

$$\langle\langle\Delta, i\rangle\rangle\varphi \Leftrightarrow [i \sqsubseteq \Gamma]\langle\langle\Delta \cup \Gamma\rangle\rangle\varphi \tag{3}$$

where $[i \sqsubseteq \Gamma]$ stands for $\neg\langle i \sqsubseteq \Gamma\rangle\neg$. Note that the argument formula of $\langle vendor \sqsubseteq SP, DT\rangle$ in (1) mentions coalitions which consist of both sub-agents of i such as SP and DT and primary agents such as *customer* and *others*. Coalitions $\Delta \subseteq \Gamma$ in the scope of $\langle i \sqsubseteq \Gamma\rangle$ can be no stronger than i. Coalitions Δ such that both $\Delta\cap\Gamma$ and $\Delta \setminus \Gamma$ are nonempty may have powers contributed by their members from outside Γ. The realizability of schemes such as the example one generally depends on the basic composition of agents' actions. For instance, it is always possible to distribute the powers of an agent so that no proper subset of the set of the sub-agents receives any substantial strategic ability:

$$\neg\langle\langle\emptyset\rangle\rangle \bigcirc p \Rightarrow \langle i \sqsubseteq \Gamma\rangle \bigwedge_{\Delta\subsetneq\Gamma} \neg\langle\langle\Delta\rangle\rangle \bigcirc p. \tag{4}$$

This formula states that, unless p is to become true in one step inevitably, the powers of i can be distributed among the sub-agents from Γ so that no proper sub-coalition Δ of Γ can achieve p. The possibility that i and, consequently, the whole of Γ can achieve p is not ruled out. Together with (3), (4) expresses the possibility to distribute i's powers so that its sub-agents can only use these powers when acting *unanimously.* After formulating the semantics of $\langle i \sqsubseteq \Gamma\rangle$, we prove that (4) is valid.

It is also always possible to distribute the powers of an agent in a way which allows actions on its behalf to be chosen by *simple majority vote.* The formula below states that, if i can both achieve p, and achieve $\neg p$ in one step, then the powers of i can be partitioned among the sub-agents from Γ so that a sub-coalition $\Delta \subseteq \Gamma$ can choose whether p holds or not in one step iff more than half of the members of Γ are in it:

$$\langle\langle i\rangle\rangle \bigcirc \neg p \wedge \langle\langle i\rangle\rangle \bigcirc p \Rightarrow \langle i \sqsubseteq \Gamma\rangle \bigwedge_{\varphi\in\{p,\neg p\}} \left(\bigwedge_{\Delta\subset\Gamma, |\Delta|\leq|\Gamma\setminus\Delta|} \neg\langle\langle\Delta\rangle\rangle \bigcirc \varphi \wedge \bigwedge_{\Delta\subseteq\Gamma, |\Delta|>|\Gamma\setminus\Delta|} \langle\langle\Delta\rangle\rangle \bigcirc \varphi \right). \tag{5}$$

Below we show that (5) is valid wrt the semantics we propose for $\langle . \sqsubseteq .\rangle$ too. We call the modality $\langle . \sqsubseteq .\rangle$ *refinement* because of its resemblance with the refinement operator $\exists_a$ from [3].

Our main result about ATL with $\langle . \sqsubseteq .\rangle$ in this paper is the decidability of model checking for the subset in which $\langle . \sqsubseteq .\rangle$ is restricted to occur only in subformulas of the form $\langle i_1 \sqsubseteq \Gamma_1\rangle \ldots \langle i_m \sqsubseteq \Gamma_m\rangle\varphi$, with no further occurrences of $\langle . \sqsubseteq .\rangle$ in φ. This is sufficient for the handling of scenarios like the example one above, but with refinements affecting more than one primary agent. We also propose some axioms and proof rules for $\langle . \sqsubseteq .\rangle$ which we do not claim to be sufficient for deriving all the valid formulas in the extension of ATL by $\langle . \sqsubseteq .\rangle$.

Structure of the paper. After brief formal preliminaries on ATL on CGMs, we introduce our proposed operator and briefly discuss related work. Next we present our main result about the decidability of model checking and give some axioms and rules for the extension of ATL by $\langle . \sqsubseteq .\rangle$. We conclude by commenting on the informal meaning of the new operator.

1. Preliminaries

In this section we give the definitions of concurrent game models (CGMs) and ATL on CGMs as known from the literature.

Definition 1 (concurrent game structures and models) A *concurrent game structure* (CGS) for some given set of agents $\Sigma = \{1, \ldots, N\}$ is a tuple of the form $\langle W, \langle Act_i : i \in \Sigma\rangle, o\rangle$ where

W is a nonempty set of *states*;

Act_i is a nonempty set of *actions*, $i \in \Sigma$; given a $\Gamma \subseteq \Sigma$, Act_Γ stands for $\prod_{i\in\Gamma} Act_i$;

$o : W \times Act_\Sigma \to W$ is a *transition (outcome)* function.

A *concurrent game model* (CGM) for Σ and a given set of atomic propositions AP is a tuple of the form

$$\langle W, \langle Act_i : i \in \Sigma\rangle, o, V\rangle \tag{6}$$

where $\langle W, \langle Act_i : i \in \Sigma\rangle, o\rangle$ is a CGS for Σ and $V \subseteq W \times AP$ is a *valuation relation*.

Unless stated otherwise we assume the components of CGMs below to be named like in (6). We always assume Act_i, $i \in \Sigma$, to be pairwise disjoint.

Below we write a_Γ to indicate that $a \in Act_\Gamma$, $\Gamma \subseteq \Sigma$. Given $a_\Delta \in Act_\Delta$ and $\Gamma \subseteq \Delta$, a_Γ also stands for the subvector of a_Δ which consists of the actions from a_Δ for the members of Γ. Given disjoint $\Gamma, \Delta \subseteq \Sigma$, we write $a_\Gamma \cdot b_\Delta$ for the only $c \in Act_{\Gamma\cup\Delta}$ which is determined by putting $c_i = a_i$ for $i \in \Gamma$ and $c_i = b_i$ for $i \in \Delta$.

Definition 2 (ATL **on CGMs)** The syntax of ATL formulas is given by the BNF

$$\varphi, \psi ::= \bot \mid p \mid (\varphi \Rightarrow \psi) \mid \langle\!\langle\Gamma\rangle\!\rangle \bigcirc \varphi \mid \langle\!\langle\Gamma\rangle\!\rangle(\varphi \mathsf{U} \psi) \mid [\![\Gamma]\!](\varphi \mathsf{U} \psi)$$

where p ranges over atomic propositions and Γ ranges over finite sets of agents. Satisfaction of ATL formulas is defined in terms of strategies. Consider a CGM M. A *strategy* for $i \in \Sigma$ in M is a function from W^+ to Act_i. Given a vector of strategies $s_\Gamma = \langle s_i : i \in \Gamma\rangle$ for the members of $\Gamma \subseteq \Sigma$, the *set of the possible outcomes of* Γ *following* s_Γ *from state* w *onwards* is the set of infinite runs

$$\mathrm{out}(w, s_\Gamma) = \{w^0 w^1 \ldots \in W^\omega : w^0 = w, w^{k+1} = o(w^k, a^k), a^0 a^1 \ldots \in Act_\Sigma^\omega,\ a_\Gamma^k = s_\Gamma(w^0 \ldots w^k), k < \omega\}.$$

Assuming a fixed M, we write S_Γ for the set of all vectors of strategies for Γ in M. Satisfaction is defined on CGMs M, states $w \in W$ and formulas φ by the clauses:

$M, w \not\models \bot$		
$M, w \models p$	iff	$V(w, p)$
$M, w \models \varphi \Rightarrow \psi$	iff	either $M, w \models \psi$, or $M, w \not\models \varphi$
$M, w \models \langle\!\langle\Gamma\rangle\!\rangle \bigcirc \varphi$	iff	there exists an $s_\Gamma \in S_\Gamma$ s. t. $w^0 w^1 \ldots \in \mathrm{out}(w, s_\Gamma)$ implies $M, w^1 \models \varphi$
$M, w \models \langle\!\langle\Gamma\rangle\!\rangle(\varphi \mathsf{U} \psi)$	iff	there exists an $s_\Gamma \in S_\Gamma$ s. t. for any $w^0 w^1 \ldots \in \mathrm{out}(w, s_\Gamma)$ there exists a $k < \omega$ s. t. $M, w^0 \models \varphi, \ldots, M, w^{k-1} \models \varphi$ and $M, w^k \models \psi$
$M, w \models [\![\Gamma]\!](\varphi \mathsf{U} \psi)$	iff	for every $s_\Gamma \in S_\Gamma$ there exists a $w^0 w^1 \ldots \in \mathrm{out}(w, s_\Gamma)$ and a $k < \omega$ s. t. $M, w^0 \models \varphi, \ldots, M, w^{k-1} \models \varphi$ and $M, w^k \models \psi$

$\top$, $\neg$, $\vee$, $\wedge$ and $\Leftrightarrow$ and the remaining combinations of $\langle\!\langle.\rangle\!\rangle$ and $[\![.]\!]$ with the temporal connectives $\bigcirc$, $\Diamond$ and $\Box$ are regarded as derived constructs. See, e.g., [2] for the definitions. In the sequel we write $\langle\!\langle\Gamma, \Delta\rangle\!\rangle$ for $\langle\!\langle\Gamma \cup \Delta\rangle\!\rangle$ and $\langle\!\langle\Gamma, i\rangle\!\rangle$ for $\langle\!\langle\Gamma \cup \{i\}\rangle\!\rangle$ for better readability. As usual, $\mathrm{Subf}(\varphi)$ denotes the set of the subformulas of formula φ, including φ itself.

2. Refining Strategic Ability in ATL: ATL$_\sqsubseteq$

In this section we first define a relation between CGMs which, for any given agent i and set of agents $\Gamma \neq \emptyset$, links CGMs with i in its set of agents Σ with CGMs with the same vocabulary of atomic propositions and $(\Sigma \setminus \{i\}) \cup \Gamma$ as the set of agents. This relation represents the partitioning of the strategic abilities of i among the members of Γ. We use this relation to define $\langle i \subseteq \Gamma\rangle$ and then use the definition to prove the validity of the example formulas (4) and (5).

Definition 3 (Γ-to-i homomorphisms of CGMs) Given Σ and AP, an $i \in \Sigma$ and some nonempty finite set of agent names Γ which is disjoint with Σ, consider CGM $M = \langle W, \langle Act_j : j \in \Sigma\rangle, o, V\rangle$ for Σ and AP and CGM $M' = \langle W', \langle Act'_j : j \in \Sigma'\rangle, o', V'\rangle$ for AP as the set of atomic propositions and $\Sigma' = (\Sigma \setminus \{i\}) \cup \Gamma$ as the set of agent names. A mapping $h : \prod\limits_{j\in\Gamma} Act'_j \to Act_i$ is a Γ*-to-*i *homomorphism from* M' *to* M, if

(i) $W' = W$, $V' = V$ and $Act_j = Act'_j$ for $j \in \Sigma \setminus \{i\}$;
(ii) range $h = Act_i$;
(iii) $o'(w, a) = o(w, a_{\Sigma\setminus\{i\}} \cdot h(a_\Gamma))$ for all $w \in W$ and all $a \in Act'_{\Sigma'}$.

Informally, a Γ-to-i homomorphism h links two CGMs M' and M which have the same vocabulary of atomic propositions, the same states and the same valuation relations. The actions for agents $j \in \Sigma \setminus \{i\}$ in M' are the same as in M too, and have the same effect on transitions. In M', the set Act_i of the actions of agent i of M is replaced by a vector $\langle Act'_j : j \in \Gamma\rangle$ of sets of actions for some new agents $j \in \Gamma$ among whom the powers of i are distributed. To exercise the strategic ability of i, every $j \in \Gamma$ chooses an action $a_j \in Act'_j$. These actions translate into an action $h(\langle a_j : j \in \Gamma\rangle)$ for i, which is then used together with the actions chosen by the agents from $\Sigma \setminus \{i\}$ to make transitions in M' in the way determined by the outcome function o of M. The surjectivity of h is required in order to guarantee that anything i can do can also be done by Γ. Next we introduce the operator which is central to this work. Formulas with the new operator as the main connective have the form $\langle i \sqsubseteq \Gamma\rangle\varphi$ where i is an agent name, and Γ is a set of agent names. The occurrences of $j \in \Gamma$ in the subformula φ of $\langle i \sqsubseteq \Gamma\rangle\varphi$ are *bound* in the usual sense. In the sequel we write $\Sigma(\varphi)$ for the set of the agent names which have *free* occurrences in φ. $\Sigma(\varphi)$ is defined by induction on the construction on φ. The clauses for atomic φ and φ with an ATL operator as the main connective are as expected, e.g., $\Sigma(\langle\langle\Gamma\rangle\rangle(\psi \mathsf{U} \chi))$ is $\Sigma(\psi) \cup \Sigma(\chi) \cup \Gamma$. For $\langle . \sqsubseteq . \rangle$ as the main connective we put

$$\Sigma(\langle i \sqsubseteq \Gamma\rangle\psi) \rightleftharpoons (\Sigma(\psi) \setminus \Gamma) \cup \{i\}.$$

Definition 4 (refinement operator) Let M, i and Γ be as above. Let $\Sigma' = (\Sigma \setminus \{i\}) \cup \Gamma$ and $\Sigma(\varphi) \subseteq \Sigma'$. Then

$$M, w \models \langle i \sqsubseteq \Gamma\rangle\varphi$$

iff there exist an M' for Σ' and AP such that $M', w \models \varphi$, and a Γ-to-i homomorphism from M' to M.

Informally, $\langle i \sqsubseteq \Gamma\rangle\varphi$ means that the powers of i can be distributed among the members of Γ so that φ holds about the new set of agents $(\Sigma \setminus \{i\}) \cup \Gamma$. The Γ-to-i homomorphism, which is required to exist for $\langle i \sqsubseteq \Gamma\rangle\varphi$ to be satisfied, can be regarded as encoding *the rules* which determine how the actions of all the members of Γ translate into an action on behalf of i. It is technically convenient to assume that all considered formulas are *clean*, which means that $\Gamma' \cap \Gamma'' = \emptyset$ for any two occurrences $\langle i' \sqsubseteq \Gamma'\rangle$ and $\langle i'' \sqsubseteq \Gamma''\rangle$ of the refinement modality in the considered formula, and, additionally, $i' \notin \Gamma', i'' \notin \Gamma''$.

In the sequel we write $\langle i \sqsubseteq \Gamma, \Delta\rangle$ for $\langle i \sqsubseteq \Gamma \cup \Delta\rangle$ and $\langle i \sqsubseteq \Gamma, j\rangle$ for $\langle i \sqsubseteq \Gamma \cup \{j\}\rangle$ for better readability. We also write Δ_i^Γ for $(\Delta \setminus \{i\}) \cup \Gamma$. E.g., Σ' from Definition 4 can be written as Σ_i^Γ.

Now that the relevant definitions are in place, we can prove the validity of the example formulas (4) and (5).

Proof of the validity of (4): Let $M = \langle W, \langle Act_i : i \in \Sigma\rangle, o, V\rangle$ be a finite CGM. Let $w \in W$ be such that $M, w \models \neg\langle\langle\emptyset\rangle\rangle \bigcirc p$. We need to construct a CGM $M' = \langle W', \langle Act'_j : j \in \Sigma'\rangle, o', V'\rangle$ and a Γ-to-i homomorphism h from M' to M so that $M', w \models \bigwedge\limits_{\Delta \subsetneq \Gamma} \neg\langle\langle\Delta\rangle\rangle \bigcirc p$. Let $Act_i = \{a^0, \ldots, a^{|Act_i|-1}\}$. Let $Act'_j = \{b_j^0, \ldots, b_j^{|Act_i|-1}\}$ for all $j \in \Gamma$. Given $b_\Gamma \in Act'_\Gamma$, let z^j be the indices of $b_j \in Act'_j$, in the enumerations $b_j^0, \ldots, b_j^{|Act_i|-1}$ of Act'_j, $j \in \Gamma$, that is, let $b_\Gamma = \langle b_j^{z^j} : j \in \Gamma\rangle$. Then we put $h(b_\Gamma) = a^k$ where $k = \left(\sum\limits_{j\in\Gamma} z^j\right) \bmod |Act_i|$. The outcome function o' of M' is now determined from Definition 3. Since $M, w \not\models \langle\langle\emptyset\rangle\rangle \bigcirc p$, there exists at least one tuple of actions $a_\Sigma \in Act_\Sigma$ such that $M, o(w, a_\Sigma) \models \neg p$. A direct check shows that if $\Delta \subsetneq \Gamma$, then for any tuple of actions $\langle b_j^{z^j} : j \in \Delta\rangle$ there exists a tuple of actions $\langle b_j^{z^j} : j \in \Gamma \setminus \Delta\rangle$ such that $h(\langle b_j^{z^j} : j \in \Gamma\rangle) = a_i$, which is the action for i from the vector a_Σ which takes M to a non-p state. Hence proper subcoalitions Δ of Γ cannot enforce $\bigcirc p$.

A similar proof works for infinite M, the only difference being the way $h(b_\Gamma)$ is defined. For finite M, h maps $b_\Gamma = \langle b_j^{z^j} : j \in \Gamma\rangle$ to a^k where $k = \left(\sum\limits_{j\in\Gamma} z^j\right) \bmod |Act_i|$. The relevant property of h is that fixing any

$|\Gamma| - 1$ of its arguments gives a surjective function of the remaining single argument. This ensures that no proper subcoalition of Γ can restrict the choice of an action on behalf of Γ. For infinite Act_is, Act'_j, $j \in \Gamma$, can be chosen to be $Act_i \times N_j$ where N_j, $j \in \Gamma$ are some pairwise disjoint infinite subsets of ω, and a h with this property can be defined as follows:

$$h(\langle\langle b^j, k^j\rangle : j \in \Gamma\rangle) = b^{j_0} \text{ where } j_0 \in \Gamma \text{ is determined from the condition } k^{j_0} = \max\{k^j : j \in \Gamma\}.$$

Now if $\Delta \subsetneq \Gamma$ and $k_l = \max\{k^j : j \in \Gamma\} > \max\{k^j : j \in \Delta\}$ for some $l \in \Gamma \setminus \Delta$, then h would return b_l, which can be any action from Act_i. $\dashv$

The construction of h in the proof of the validity of (4), which is meant to prevent proper sub-coalitions from imposing any substantial restriction on the decisions of the whole coaliton, recurs in various forms in other proofs in the rest of the paper too. To avoid repetition in subsequent proofs, we give this construction for the case of finite models only, despite that the respective statements hold for infinite ones as well. The infinite model variant of the construction is always related to the finite one like in the proof for (4) above.

Formula (5) states that i's powers can be distributed so that collective decisions need to be made using a simple majority vote. Expectedly, the proof of the validity of (5) is based on a Γ-to-i homomorphism, which implements simple majority vote.

Proof of the validity of (5): Let M, w and M' be as in the proof about (4). In particular, let Act_i and Act'_j, $j \in \Gamma$, have the enumerations $a_i^0, \dots, a_i^{|Act_i|-1}$, and $b_j^0, \dots, b_j^{|Act_i|-1}$, respectively. We define h, and, consequently, the outcome function o' of M', by putting:

$$h(\langle b_j^{z^j} : j \in \Gamma\rangle) = \begin{cases} a_i^z, & \text{if } z \text{ is such that } |\{j \in \Gamma : z^j = z\}| > |\{j \in \Gamma : z^j \neq z\}|; \\ a_i^k, & \text{where } k = \left(\sum\limits_{j\in\Gamma} z^j\right) \bmod |Act_i|, \text{ in case no } z \text{ with the above property exists.} \end{cases}$$

Obviously there may be at most one z satisfying the condition for $h(\langle b_j^{z^j} : j \in \Gamma\rangle)$ to evaluate to a_i^z above. A direct check shows that if $\Delta \subseteq \Gamma$ and $|\Delta| < |\Gamma \setminus \Delta|$, then for any $a_i \in Act_i$ and any tuple of actions $\langle b_j^{z^j} : j \in \Delta\rangle$ there exists a tuple of actions $\langle b_j^{z^j} : j \in \Gamma \setminus \Delta\rangle$ such that $h(\langle b_j^{z^j} : j \in \Gamma\rangle) = a_i$. Hence minority subcoalitions Δ of Γ in M' cannot prevent any particular action from being chosen on behalf of the primary agent i. In particular, just like in the proof about (4), the existence of a tuple of actions $a_\Sigma \in Act_\Sigma$ such that $M, o(w, a_\Sigma) \models \neg p$, which follows from $M, w \models \langle\langle i\rangle\rangle \bigcirc \neg p$, entails that minority Δs cannot prevent i's component a_i of a_Σ to be chosen on behalf of i and therefore cannot prevent $\neg p$ from being achieved in one step. On the contrary, for any action $a_i \in Act_i$, a majority coalition of sub-agents Δ can achieve $h(b_\Gamma) = a_i$ by choosing $b_j = b_j^z$, $j \in \Delta$, where z is determined from the condition $a_i = a_i^z$. Hence a majority coalition Δ can enforce any action on behalf of i and therefore, if i can achieve $\bigcirc p$ in M, then so do majority sub-coalitions of Γ in M'. The argument about achieving $\bigcirc\neg p$ is symmetrical. $\dashv$

3. Related Work

As mentioned above, there is an analogy between our $\langle . \sqsubseteq . \rangle$ and the refinement quantifier $\exists_a$ of *Refinement Modal Logic* (RML, [3]) and its extensions to special classes of multimodal frames [4]. Given a multimodal Kripke model M with one of the accessibility relations labelled a, $M, w \models \exists_a \varphi$ iff there exists another model M' of similar type and a relation S between the state spaces of M and M' which is refinement wrt the accessibility relations labelled a in the two models, and bisimulation wrt the other pairs of corresponding accessibility relations, and a state w' of M' such that $S(w, w')$ and $M', w' \models \varphi$. In short, in RML, $\exists_a \varphi$ holds iff the accessibility relation for a can be refined so that φ holds.

A notion of *refinement* of alternating transition systems, ATL's original type of models from [1], allowing, unlike [5], the powers of different *sets* of agents to be related, was studied in [6]. The approach of [6] suggests considering a refinement modality of the form $\langle \Delta \sqsubseteq \Gamma\rangle$ with $|\Delta| \geq 1$. The authors of [6] stopped short of extending ATL *syntax* by such an operator. Our model checking algorithm extends to the case of non-singleton coalition-to-coalition refinement as in our CGM-based setting in a straightforward way.

Abstraction techniques with the agents being just *knowers* were studied in [7, 8]. Abstraction involving over- and under-approximation of coalitions to contain the computational cost of model checking was proposed in [9]. A formalization of teaming sub-agents under a scheduler as turn-based simulation was proposed in [10, 11]. Modelling varying the considered set of agents is addressed in *modular interpreted systems* [12, 13]. Distinctively, our setting is about varying the set of agents in a system by just redistributing strategic ability, with the overall activities which the system can accommodate unchanged. Considering actions which are complete with a description of their effect and an additional parameter to the co-operation modality meant to specify the availability of actions to agents as in [14, 15] enables specifying power distribution by dividing the actions among the considered sub-agents. This approach is, broadly speaking, complementary to our work. In CGMs, the effect of actions is defined by means of the transition function. We propose reasoning about distributing the ability to enforce temporal conditions, and *synthesizing* implementations of strategic ability through satisfiability checking.

4. Model checking $\langle . \sqsubseteq . \rangle^*$-flat ATL$_\sqsubseteq$

$\langle . \sqsubseteq . \rangle^*$-*flat* ATL$_\sqsubseteq$ is the subset of ATL$_\sqsubseteq$ in which the $\langle . \sqsubseteq . \rangle$-subformulas, which are the subformulas with $\langle . \sqsubseteq . \rangle$ as the main connective, have the form

$$\langle i_1 \sqsubseteq \Gamma_1 \rangle \dots \langle i_m \sqsubseteq \Gamma_m \rangle \varphi \tag{7}$$

where φ has no further occurrences of $\langle . \sqsubseteq . \rangle$. Note that only occurrences of $\langle . \sqsubseteq . \rangle$ of the same polarity can be chained. E.g., if φ and ψ are $\langle . \sqsubseteq . \rangle$-free, then $\langle\langle i \rangle\rangle \Diamond (\langle i \sqsubseteq \Gamma \rangle \langle j \sqsubseteq \Delta \rangle \varphi \wedge [k \sqsubseteq \Upsilon][l \sqsubseteq \Xi]\psi)$ is $\langle . \sqsubseteq . \rangle^*$-flat, but $[i \sqsubseteq \Gamma]\langle j \sqsubseteq \Delta \rangle \varphi$ and $\langle i \sqsubseteq \Gamma \rangle \langle\langle k \rangle\rangle \Diamond \langle j \sqsubseteq \Delta \rangle \varphi$ are not. Our decision method reduces the model checking problem to satisfiability in the $\langle\langle . \rangle\rangle \bigcirc$-subset of ATL, where $\langle\langle . \rangle\rangle$ can be combined with $\bigcirc$ only, or, equivalently, in Coalition Logic [16], which is known to be decidable. We first do the case of $m = 1$ and φ being a boolean combination of $\langle\langle . \rangle\rangle \bigcirc$-formulas with boolean combinations of atomic propositions as the arguments of $\langle\langle . \rangle\rangle \bigcirc$ in full detail. Then we explain how the technique extends to arbitrary m, and, finally, however inefficiently, to formulas of the form (7) with an $\langle . \sqsubseteq . \rangle$-free φ in which the use of the ATL connectives is unrestricted.

The case of $m = 1$. Consider some formula $\langle i \sqsubseteq \Gamma \rangle \varphi$ with φ restricted as above. Let CGM M be as above and consider a CGM $M' = \langle W, \langle Act'_i : i \in \Sigma' \rangle, o', V \rangle$, $\Sigma' = \Sigma_i^\Gamma$, and a Γ-to-i homomorphism h from M' to M. Let $\langle\langle \Delta \rangle\rangle \bigcirc \chi$ be a subformula of φ. For $M', w \models \langle\langle \Delta \rangle\rangle \bigcirc \chi$ to hold, there should be a vector of actions a_Δ such that, for any $b_{\Gamma \setminus \Delta}$, $a_{\Delta \setminus \Gamma} \cdot h(a_{\Delta \cap \Gamma} \cdot b_{\Gamma \setminus \Delta})$ gives $\Delta \setminus \Gamma \cup \{i\}$ a strategy to achieve $\bigcirc \chi$ in M. For a fixed $a_{\Delta \setminus \Gamma}$ this is equivalent to

$$h(a_{\Delta \cap \Gamma} \cdot b_{\Gamma \setminus \Delta}) \in \{a_i \in Act_i : \forall c_{\Sigma \setminus (\Delta \cup \{i\})} M, o(w, a_{\Delta \setminus \Gamma} \cdot a_i \cdot c_{\Sigma \setminus (\Delta \cup \{i\})}) \models \chi\} \tag{8}$$

Henceforth we write $A_{i, a_{\Delta \setminus \Gamma}, w, \chi}$ for the subset of Act_i in (8). It consists of those actions of i which, together with the actions $a_{\Delta \setminus \Gamma}$, form a strategy for $\{i\} \cup (\Delta \setminus \Gamma)$ to achieve χ in one step from w in M.

Now consider a CGM $\overline{M} = \langle \overline{W}, \langle \overline{Act}_j : j \in \Gamma \rangle, \overline{o}, \overline{V} \rangle$ for Γ as the set of agents, $\overline{AP} = Act_i$ as the set of atomic propositions and $\overline{W} = Act_i \cup \{w^0\}$ as the set of states. Let $\overline{V}(w, a)$ be equivalent to $w = a$ for $a \in Act_i$, thus enabling reference to each individual action of i. The intended meaning of the states of $\overline{M}$ from Act_i is to represent the possible choices of i's actions by the members of Γ; w^0 is a distinguished reference state. We assume that $w^0 \notin Act_i$. Let $\overline{Act}_j = Act'_j$ for $j \in \Gamma$, and let $\overline{o}(w^0, a) = h(a)$ for all $a \in \overline{Act}_\Gamma$. Then

$$\overline{M}, w^0 \models \langle\langle \emptyset \rangle\rangle \bigcirc \bigvee_{a \in Act_i} a \wedge \bigwedge_{a, b \in Act_i, a \neq b} \langle\langle \emptyset \rangle\rangle \bigcirc \neg(a \wedge b) \wedge \bigwedge_{a \in Act_i} \langle\langle \Gamma \rangle\rangle \bigcirc a. \tag{9}$$

The first two conjunctive members of the formula in (9) follow from the fact that all the states where $\overline{o}$ takes w^0 are in Act_i, and each of these states satisfies only its corresponding atomic proposition from $\overline{AP}$. The last conjunctive member holds because of the surjectivity of h. Each of i's actions can be enforced by Γ, which is the grand coalition in $\overline{M}$, and therefore each of the states from Act_i can be reached.

Let the translation t replace subformulas of φ of the form $\langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi$ by their corresponding

$$\bigvee_{a_{\Delta\setminus\Gamma}\in Act_{\Delta\setminus\Gamma}} \langle\!\langle\Delta\cap\Gamma\rangle\!\rangle \bigcirc \bigvee_{a_i\in A_{i,a_{\Delta\setminus\Gamma},w,\chi}} a_i.$$

The formula $\mathsf{t}(\langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi)$ states that there exists a vector of actions $a_{\Delta\setminus\Gamma}$ for $\Delta\setminus\Gamma$ which can be combined with some action on behalf of i to form a strategy for achieving $\bigcirc\chi$, and $\Delta\cap\Gamma$ is capable of making Γ choose such an action. Hence Δ can enforce $\bigcirc\chi$ by $\Delta\setminus\Gamma$ choosing an $a_{\Delta\setminus\Gamma}$ with the above property and $\Delta\cap\Gamma$ making Γ play an action from $A_{i,a_{\Delta\setminus\Gamma},w,\chi}$ on behalf of i. This means that $\overline{M}, w^0 \models \mathsf{t}(\langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi)$ implies $M', w \models \langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi$. For the opposite direction, if $M', w \models \langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi$, then there exists an $a_\Delta \in Act'_\Delta$ which guarantees reaching a χ-state in one step regardless of the actions of the non-members of Δ. This means that the restrictions $a_{\Delta\setminus\Gamma}$ and $a_{\Delta\cap\Gamma}$ of a_Δ satisfy $h(a_{\Delta\cap\Gamma}\cdot b_{\Gamma\setminus\Delta}) \in A_{i,a_{\Delta\setminus\Gamma},w,\chi}$ for all $b_{\Gamma\setminus\Delta} \in Act'_{\Gamma\setminus\Delta}$. The latter entails $\overline{M}, w^0 \models \mathsf{t}(\langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi)$. The equivalence between $\overline{M}, w^0 \models \mathsf{t}(\varphi)$ and $M', w \models \varphi$ holds about φ too, as it is a boolean combination of formulas of the form $\langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi$.

Conversely, let a model $\overline{M} = \langle \overline{W}, \langle \overline{Act}_j : j \in \Gamma\rangle, \overline{o}, \overline{V}\rangle$ exist such that $\overline{M}, w^0 \models \mathsf{t}(\varphi)$ and (9) hold. Then we can define an M' and a Γ-to-i homomorphism h to witness $M, w \models \langle i \sqsubseteq \Gamma\rangle\varphi$ as follows. We put $Act'_j = \overline{Act}_j$, $j \in \Gamma$. For every $a_\Gamma \in \overline{Act}_\Gamma$, we define $h(a_\Gamma)$ as the unique $a_i \in Act_i$ such that $\overline{M}, o(w^0, a_\Gamma) \models a_i$. The identity $o'(w, a) = o(w^0, h(a))$ determines o'. Now a direct check shows that $M, w \models \langle i \sqsubseteq \Gamma\rangle\varphi$.

Hence, the existence of a model $\overline{M}$ which satisfies $\mathsf{t}(\varphi)$ and (9) at some state is equivalent to the satisfaction of φ at the given state w of the given M. Since satisfiability of formulas such as $\mathsf{t}(\varphi)$ and (9) is solvable, this entails the solvability of model checking $\langle . \sqsubseteq . \rangle$-formulas.

The case of $m > 1$. For the sake of simplicity we do the case of $m = 2$. Bigger m are handled analogously. Without loss of generality, we assume that $i_1 = 1$ and $i_2 = 2$. Consider formulas of the form $\langle 1 \sqsubseteq \Gamma_1\rangle\langle 2 \sqsubseteq \Gamma_2\rangle\varphi$. We first revise condition (8), with respect to formulas $\langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi \in \mathrm{Subf}(\varphi)$ in which $\Delta \subseteq \Sigma'$, $\Sigma' = (\Sigma_1^{\Gamma_1})_2^{\Gamma_2}$. The $m = 2$-form of (8) is about sets of *pairs* of actions, for 1 and 2, respectively. Given a fixed $a_{\Delta\setminus(\Gamma_1\cup\Gamma_2)}$, (8) assumes the form

$$\begin{aligned}&\langle h_1(a_{\Delta\cap\Gamma_1}\cdot b_{\Gamma_1\setminus\Delta}), h_2(a_{\Delta\cap\Gamma_2}\cdot b_{\Gamma_2\setminus\Delta})\rangle \in\\ &\qquad \{\langle a_1, a_2\rangle \in Act_1 \times Act_2 : \forall c_{\Sigma\setminus(\Delta\cup\{1,2\})} M, o(w, a_1\cdot a_2\cdot a_{\Delta\setminus(\Gamma_1\cup\Gamma_2)}\cdot c_{\Sigma\setminus(\Delta\cup\{1,2\})}) \models \chi\}\end{aligned}$$

We denote the subset of $Act_1 \times Act_2$ above by $A_{1,2,a_{\Delta\setminus(\Gamma_1\cup\Gamma_2)},w,\chi}$. The ability of Δ to achieve χ in one step from w is equivalent to the ability of each of $\Delta\cap\Gamma_1$ and $\Delta\cap\Gamma_2$ to enforce actions a_1 and a_2 on behalf of 1 and 2, respectively, so that $\langle a_1, a_2\rangle \in A_{1,2,a_{\Delta\setminus(\Gamma_1\cup\Gamma_2)},w,\chi}$ for some appopriate $a_{\Delta\setminus(\Gamma_1\cup\Gamma_2)}$. Therefore we define $\mathsf{t}(\langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi)$ as

$$\bigvee_{a_{\Delta\setminus(\Gamma_1\cup\Gamma_2)}\in Act_{\Delta\setminus(\Gamma_1\cup\Gamma_2)}} \; \bigvee_{A_1\times A_2\subseteq A_{1,2,a_{\Delta\setminus(\Gamma_1\cup\Gamma_2)},w,\chi}} \langle\!\langle\Delta\cap\Gamma_1\rangle\!\rangle \bigcirc \bigvee_{a_1\in A_1} a_1 \wedge \langle\!\langle\Delta\cap\Gamma_2\rangle\!\rangle \bigcirc \bigvee_{a_2\in A_2} a_2.$$

Formulas obtained by the $\langle 1 \sqsubseteq \Gamma_1\rangle\langle 2 \sqsubseteq \Gamma_2\rangle$-form of t are boolean combinations of formulas of the form $\langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi$ where $\Delta \subseteq \Gamma_k$ and χ is a disjunction of members of Act_k, for k being either 1 or 2. In the case of $m = 1$ we are interested in the existence of a satisfying model $\overline{M}$ for $\mathsf{t}(\varphi)$ as the transition function $\overline{o}$ of such a model can be used to determine the homomorphism h we need. For the case of $m = 2$, the part of $\overline{M}$ is played by a pair of models $\overline{M}_k = \langle \underbrace{Act_k \cup \{w_{0,k}\}}_{=\overline{W}_k}, \langle \overline{Act}_{k,j} : j \in \Gamma_k\rangle, \overline{o}_k, \overline{V}_k\rangle$ to represent the ability of coalitions within Γ_k to enforce actions with some desired effect on behalf of agent k, $k = 1, 2$. We are interested in the satisfiability of t-translations at pairs of such models in the following sense. In subformulas of $\mathsf{t}(\varphi)$ of the form $\langle\!\langle\Delta'\rangle\!\rangle \bigcirc \chi'$ either $\Delta' \subseteq \Gamma_1$ and χ' is a boolean combination of atomic propositions from $\overline{AP}_1 = Act_1$, or $\Delta' \subseteq \Gamma_2$ and χ' a boolean combination of atomic propositions from $\overline{AP}_2 = Act_2$. We define $\overline{M}_1, \overline{M}_2, w_{0,1}, w_{0,2} \models \langle\!\langle\Delta'\rangle\!\rangle \bigcirc \chi'$ as $\overline{M}_k, w_{0,k} \models \langle\!\langle\Delta\rangle\!\rangle \bigcirc \chi$ for $\Delta' \subseteq \Gamma_k$ and χ' written in terms of Act_k, $k = 1, 2$. The clauses for the satisfaction of $\perp$ and formulas built using $\Rightarrow$ at pairs of models are as usual.

Satisfiability at pairs of models of the special type of formulas above straightforwardly reduces to the usual satisfiability at single models as soon as $\mathsf{t}(\varphi)$ is given a disjunctive normal form: a $\mathsf{t}(\varphi)$ of this form is satisfiable iff some of its disjunctive members is, and each disjunctive member can be viewed as a conjunction of two formulas ψ_k, ψ_k being a conjunction of formulas of the form $\langle\langle\Delta\rangle\rangle \bigcirc \chi$ with $\Delta \subseteq \Gamma_k$ and χ written in terms of $\overline{AP}_k$, $k = 1, 2$. The satisfiability of $\psi_1 \wedge \psi_2$ is obviously equivalent to the satisfiability of both ψ_1 and ψ_2 in the usual sense at some models $\overline{M}_1$ and $\overline{M}_2$ which satisfy (9) wrt Act_1 and Act_2, respectively.

Formulas (7) with arbitrary $\langle . \sqsubseteq . \rangle$-free φ. Removing the restriction on φs to be in the flat $\langle\langle . \rangle\rangle\bigcirc$-subset of ATL makes it necessary to synthesise an M' and the respective h with conditions such as (the many-dimensional form of) (8) associated with not just one but all the states w of M. To enable this, we first elimitate the use of $(.\mathsf{U}.)$ in φ using that $|W|$ is known.[1] Assuming that φ is $(.\mathsf{U}.)$-free, and that $m = 1$ again, for the sake of simplicity, we consider assignments $\|.\| : \mathrm{Subf}(\varphi) \to 2^W$. We are interested in the existence of an assignment $\|.\|$ such that an M' that admits a Γ-to-i homomorphism h to M exists in which φ holds at the given state w and $\{w' : M', w' \models \psi\} = \|\psi\|$ for all $\psi \in \mathrm{Subf}(\varphi)$. For ψ being either $\perp$, or an atomic proposition p, or with $\Rightarrow$ as the main connective, under such an assignment $\|.\|$, $\|\psi\|$ is unambiguously determined by the identities

$$\|\perp\| = \emptyset,\ \|p\| = \{w' \in W : V(p, w')\} \text{ and } \|\psi' \Rightarrow \psi''\| = W \setminus \|\psi'\| \cup \|\psi''\|. \tag{10}$$

For ψ of the form $\langle\langle\Delta\rangle\rangle \bigcirc \psi'$ where $\Delta \cap \Gamma = \emptyset$, $\|\psi\|$ can be determined from $\|\psi'\|$ using the identity

$$\|\langle\langle\Delta\rangle\rangle \bigcirc \psi'\| = \{w' \in W : \exists a_\Delta \forall b_{\Sigma\setminus\Delta} o(w', a_\Delta \cdot b_{\Sigma\setminus\Delta}) \in \|\psi'\|\}. \tag{11}$$

Similarly, if $\Delta \supseteq \Gamma$, then

$$\|\langle\langle\Delta\rangle\rangle \bigcirc \psi'\| = \{w' \in W : \exists a_{\Delta\cup\{i\}} \forall b_{\Sigma\setminus(\Delta\cup\{i\})} o(w', a_{\Delta\cup\{i\}} \cdot b_{\Sigma\setminus(\Delta\cup\{i\})}) \in \|\psi'\|\}. \tag{12}$$

Therefore every acceptable assignment is determined unambiguously from its values $\|\langle\langle\Delta\rangle\rangle \bigcirc \psi\|$ for $\langle\langle\Delta\rangle\rangle \bigcirc \psi \in \mathrm{Subf}(\varphi)$ such that $\emptyset \neq \Delta \cap \Gamma \neq \Gamma$. These values are constrained by the inclusions

$$\|\langle\langle\Delta \setminus \Gamma\rangle\rangle \bigcirc \psi'\| \subseteq \|\langle\langle\Delta\rangle\rangle \bigcirc \psi\| \subseteq \|\langle\langle\Delta \setminus \Gamma, i\rangle\rangle \bigcirc \psi'\|.$$

The existence of the required M' and h which links M' to M depends on the satisfiability of the conjunction

$$\bigwedge_{\substack{\langle\langle\Delta\rangle\rangle\bigcirc\psi\in\mathrm{Subf}(\varphi)\\ \emptyset\neq\Delta\cap\Gamma\neq\Gamma}} \left(\begin{array}{l} \bigwedge\limits_{w'\in\|\langle\langle\Delta\rangle\rangle\bigcirc\psi\|} \bigvee\limits_{a_{\Delta\setminus\Gamma}\in Act_{\Delta\setminus\Gamma}} \langle\langle\Delta\cap\Gamma\rangle\rangle \bigcirc \bigvee\limits_{a_i\in A_{i,a_{\Delta\setminus\Gamma},w',\|\psi\|}} a_i \\ \wedge \\ \bigwedge\limits_{w'\in W\setminus\|\langle\langle\Delta\rangle\rangle\bigcirc\psi\|} \neg \left(\bigvee\limits_{a_{\Delta\setminus\Gamma}\in Act_{\Delta\setminus\Gamma}} \langle\langle\Delta\cap\Gamma\rangle\rangle \bigcirc \bigvee\limits_{a_i\in A_{i,a_{\Delta\setminus\Gamma},w',\|\psi\|}} a_i \right) \end{array} \right)$$

where

$$A_{i,a_{\Delta\setminus\Gamma},w',X} \rightleftharpoons \{a_i \in Act_i : \forall c_{\Sigma\setminus(\Delta\cup\{i\})}(o(w', a_{\Delta\setminus\Gamma} \cdot a_i \cdot c_{\Sigma\setminus(\Delta\cup\{i\})}) \in X)\}$$

at a model of the type of $\overline{M} = \langle Act_i \cup \{w^0\}, \langle\overline{Act}_j : j \in \Gamma\rangle, \overline{o}, \overline{V}\rangle$ already introduced above. The satisfiability of this conjunction, together with the identities (10), (11) and (12) for the subformulas of φ of the forms considered above guarantees that $\{w' : M', w' \models \langle\langle\Delta\rangle\rangle \bigcirc \psi\} = \|\langle\langle\Delta\rangle\rangle \bigcirc \psi\|$ for $\langle\langle\Delta\rangle\rangle \bigcirc \psi \in \mathrm{Subf}(\varphi)$ such that $\emptyset \neq \Delta \cap \Gamma \neq \Gamma$ in models M' which are linked to M by the Γ-to-i homomorphism $h = \lambda a_\Gamma.\overline{o}(w^0, a_\Gamma)$.

Obviously the algorithm implied by the above argument is only good to conclude decidability in principle because of the forbidding number of $\|.\|$s to be considered.

[1] We use the validity of $\langle\langle\Gamma\rangle\rangle(\alpha\mathsf{U}\beta) \Leftrightarrow \langle\langle\Gamma\rangle\rangle^{|W|}(\alpha\mathsf{U}\beta)$ where $\langle\langle\Gamma\rangle\rangle^0(\alpha\mathsf{U}\beta) \rightleftharpoons \beta$ and $\langle\langle\Gamma\rangle\rangle^{k+1}(\alpha\mathsf{U}\beta) \rightleftharpoons \beta\vee(\alpha\wedge\langle\langle\Gamma\rangle\rangle\bigcirc\langle\langle\Gamma\rangle\rangle^k(\alpha\mathsf{U}\beta))$ on models with a finite W as the set of states. The corresponding equivalence for $[\![\Gamma]\!](\alpha\mathsf{U}\beta)$ is defined using $[\![\Gamma]\!]^0(\alpha\mathsf{U}\beta) \rightleftharpoons \beta$ and $[\![\Gamma]\!]^{k+1}(\alpha\mathsf{U}\beta) \rightleftharpoons \beta \vee (\alpha \wedge [\![\Gamma]\!] \bigcirc [\![\Gamma]\!]^k(\alpha\mathsf{U}\beta))$. This can cause an $O(|W|)$-blowup in the number of the subformulas of the given φ, making it clear that we are after nothing more than decidability in principle.

5. Axioms and proof rules for $\langle . \sqsubseteq . \rangle$

The axioms and rules for $\langle . \sqsubseteq . \rangle$ in this section do not form a complete proof system for $\mathrm{ATL}_{\sqsubseteq}$. However they can still be used to, e.g., find equivalents to $\mathrm{ATL}_{\sqsubseteq}$ formulas in the form (7) and this way broaden the scope of the model checking decision method from Section 4. Soundness proofs for the less trivial axioms are given in Appendix Appendix A.

We divide the axioms and rules below into three groups. The first group is about the properties of $\langle . \sqsubseteq . \rangle$ as a binder. The second group is related to the fact that, for i and Γ fixed, $\langle i \sqsubseteq \Gamma \rangle$ has Kripke semantics. This entails the validity of some standard modal axioms. A comprehensive list of such axioms can be found in [17]. The last group consists of axioms which are specific to $\langle . \sqsubseteq . \rangle$.

Substituting single agents by coalitions.. We use substitution $[\Gamma/i]$ of single agent names i by finite non-empty sets of agent names Γ in order to formulate some of the axioms and rules. The result $[\Gamma/i]\varphi$ of applying $[\Gamma/i]$ to a formula φ is defined by induction on the construction of φ. Most of the defining clauses are as expected:

$$[\Gamma/i]\bot \rightleftharpoons \bot,\ [\Gamma/i]p \rightleftharpoons p,\ [\Gamma/i](\varphi_1 \Rightarrow \varphi_2) \rightleftharpoons ([\Gamma/i]\varphi_1 \Rightarrow [\Gamma/i]\varphi_2),$$

and, given that $i \notin \Delta$ and $\Gamma \cap \Delta = \emptyset$,

$$\begin{array}{l}[\Gamma/i]\langle\!\langle \Delta \rangle\!\rangle\varphi \rightleftharpoons \langle\!\langle \Delta \rangle\!\rangle[\Gamma/i]\varphi,\ [\Gamma/i][\![\Delta]\!]\varphi \rightleftharpoons [\![\Delta]\!][\Gamma/i]\varphi \\ [\Gamma/i]\langle\!\langle \Delta, i \rangle\!\rangle\varphi \rightleftharpoons \langle\!\langle \Delta, \Gamma \rangle\!\rangle[\Gamma/i]\varphi,\ [\Gamma/i][\![\Delta, i]\!]\varphi \rightleftharpoons [\![\Delta, \Gamma]\!][\Gamma/i]\varphi\end{array}$$

where φ is of one of the forms $\bigcirc\psi$ and $(\psi \mathsf{U} \chi)$ and, as expected,

$$[\Gamma/i] \bigcirc \psi \rightleftharpoons \bigcirc[\Gamma/i]\psi \text{ and } [\Gamma/i](\psi \mathsf{U} \chi) \rightleftharpoons ([\Gamma/i]\psi \mathsf{U} [\Gamma/i]\chi).$$

Defining $[\Gamma/i]$ on formulas φ with occurrences of the refinement operator requires i to be *free for the agent names from* Γ *in* φ, which means that no occurrences of i in φ are allowed in the scope of a $\langle j \sqsubseteq \Delta \rangle$ such that $\Gamma \cap \Delta \neq \emptyset$. Given that $j \neq i$ and $\Delta \cap \Gamma = \emptyset$, we put

$$[\Gamma/i]\langle j \sqsubseteq \Delta \rangle\varphi \rightleftharpoons \langle j \sqsubseteq \Delta \rangle[\Gamma/i]\varphi.$$

Defining $[\Gamma/i]\langle i \sqsubseteq \Delta \rangle\varphi$ in a straightforward way would take generalizing $\langle i \sqsubseteq \Gamma \rangle$ to an operator $\langle \Gamma \sqsubseteq \Delta \rangle$, about the possibility to redistribute the combined strategic abilities of the members of Γ among the members of Δ, with its semantics defined in terms Δ-to-Γ homomorphisms of type $\prod\limits_{j \in \Delta} Act'_j \to \prod\limits_{i \in \Gamma} Act_i$. This is similar to the refinement relation studied in [6], which links the powers of two groups of agents in *alternating transition systems*, the [1] type of models for ATL. In this paper we stop short of considering this general form of $\langle . \sqsubseteq . \rangle$. Instead, wherever substitutions of the form $[\Gamma/i]$ appear in our axioms and rules, we assume that the argument formulas are $\langle . \sqsubseteq . \rangle$-free.

$\langle . \sqsubseteq . \rangle$ *as a binder*

Let Δ be such that $|\Delta| = |\Gamma|$ and $\Delta \cap \Sigma(\varphi) = \emptyset$. Then, for any bijection $\sigma : \Gamma \to \Delta$, we have

$$\textbf{(ren)} \quad \langle i \sqsubseteq \Gamma \rangle\varphi \Leftrightarrow \langle i \sqsubseteq \Delta \rangle[\sigma(j)/j : j \in \Gamma]\varphi$$

where $[\sigma(j)/j : j \in \Gamma]\varphi$ is the substitution of $j \in \Gamma$ by their respective $\sigma(j) \in \Delta$ in φ.

$\langle i \sqsubseteq \Gamma \rangle$ has no effect on φ such that $\Sigma(\varphi) \cap \Gamma = \emptyset$:

$$\langle i \sqsubseteq \Gamma \rangle(\varphi \wedge \psi) \Leftrightarrow \varphi \wedge \langle i \sqsubseteq \Gamma \rangle\psi, \text{ if } \Sigma(\varphi) \cap \Gamma = \emptyset.$$

$\langle i \sqsubseteq \Gamma \rangle$ *as a modality*

In the classical case, Kripke accessibility relations bind pointed models which are identical except for the designated reference world. Γ-to-i homomorphisms relate models with different frame parts. Nonetheless, for Γ and i fixed, this still counts as Kripke accessibility and the following axioms are valid:

(K) $[i \sqsubseteq \Gamma](\varphi \Rightarrow \psi) \Rightarrow ([i \sqsubseteq \Gamma]\varphi \Rightarrow [i \sqsubseteq \Gamma]\psi)$

(D) $[i \sqsubseteq \Gamma]\varphi \Rightarrow \langle i \sqsubseteq \Gamma \rangle \varphi$

(G1) $\langle i \sqsubseteq \Gamma \rangle [j \sqsubseteq \Delta]\varphi \Rightarrow [j \sqsubseteq \Delta]\langle i \sqsubseteq \Gamma \rangle \varphi$ for $i \notin \Delta, j \notin \Gamma, i \neq j, \Gamma \cap \Delta = \emptyset$

(N) $\dfrac{\varphi}{[i \sqsubseteq \Gamma]\varphi}$

The soundness of **K** is obvious. N is a special case of the rule $\langle . \sqsubseteq . \rangle_L$ below, which enables the introduction of *negative* occurrences of $\langle . \sqsubseteq . \rangle$ in the way known about the classical existential quantifier.

$$(\langle . \sqsubseteq . \rangle_L) \quad \frac{\varphi \Rightarrow [\Gamma/i]\psi}{\langle i \sqsubseteq \Gamma \rangle \varphi \Rightarrow \psi}$$

N can be derived from this rule by negating φ and choosing ψ to be $\bot$. **D** follows from the fact that for any M with Σ as the set of agents, $i \in \Sigma$ and any $\Gamma \neq \emptyset$, $\Gamma \cap \Sigma = \emptyset$, an M' with Σ_i^Γ as the set of agents and at least one Γ-to-i homomorphism from M' to M exist. The soundness of **G1** is proved in Appendix Appendix A.

Substitutivity does not hold in $\mathrm{ATL}_{\sqsubseteq}$ in general. Precautions must be made to avoid agent name capture upon applying the substitution $[\psi/p]$ on valid formulas φ:

$$(\mathit{Subst}) \quad \frac{\varphi}{[\psi/p]\varphi}, \text{ if agent names from } \Sigma(\psi) \text{ have no bound occurrences in } \varphi.$$

For this reason here we write axiom *schemata* instead of axioms in terms of propositional variables and the soundness proofs in Appendix Appendix A include induction on the construction on the formulas which may appear in axiom instances.

The validity of **K** and N entail that $\langle . \sqsubseteq . \rangle$ is monotonic:

$$\frac{\varphi \Rightarrow \psi}{\langle i \sqsubseteq \Gamma \rangle \varphi \Rightarrow \langle i \sqsubseteq \Gamma \rangle \psi},$$

and, thanks to the monotonicity of the temporal connectives of the underlying logic ATL, equivalent subformulas can be substituted for each other in $\mathrm{ATL}_{\sqsubseteq}$:

$$\frac{\varphi \Leftrightarrow \psi}{[\varphi/p]\chi \Leftrightarrow [\psi/p]\chi}.$$

Writing down the appropriate form of axiom **T** requires taking in account that Γ-to-i homomorphisms link models for different sets of agents. Let none of the agent names from Γ occur in φ. Then we have

(T) $\langle i \sqsubseteq \Gamma \rangle [\Gamma/i]\varphi \Rightarrow \varphi$

$\langle . \sqsubseteq . \rangle$*-specific axioms*

Successive occurrences of $\langle . \sqsubseteq . \rangle$ can be reordered or merged:

(comm1) $\langle i \sqsubseteq \Gamma \rangle \langle j \sqsubseteq \Delta \rangle \varphi \Leftrightarrow \langle j \sqsubseteq \Delta \rangle \langle i \sqsubseteq \Gamma \rangle \varphi$ for $i \notin \Delta, j \notin \Gamma, \Gamma \cap \Delta = \emptyset$

(flatten) $\langle i \sqsubseteq \Gamma, j \rangle \langle j \sqsubseteq \Delta \rangle \varphi \Leftrightarrow \langle i \sqsubseteq \Gamma, \Delta \rangle \varphi$ for $j \notin \Gamma, \Gamma \cap \Delta = \emptyset$

$\langle . \sqsubseteq . \rangle$ commutes with $\langle\langle . \rangle\rangle \bigcirc$ and distributes over $\langle\langle . \rangle\rangle(.\mathsf{U}.)$. Let $i \notin \Gamma \cup \Delta$ and $\Gamma \cap \Delta = \emptyset$. Then:

(comm2) $\langle i \sqsubseteq \Gamma \rangle \langle\langle \Gamma, \Delta \rangle\rangle \bigcirc \varphi \Leftrightarrow \langle\langle \Delta, i \rangle\rangle \bigcirc \langle i \sqsubseteq \Gamma \rangle \varphi$

(comm3) $\langle i \sqsubseteq \Gamma \rangle [\![\Gamma, \Delta]\!] \bigcirc \varphi \Rightarrow [\![\Delta, i]\!] \bigcirc \langle i \sqsubseteq \Gamma \rangle \varphi$

(distr1) $\langle i \sqsubseteq \Gamma \rangle \langle\langle \Delta, \Gamma \rangle\rangle (\varphi \mathsf{U} \psi) \Rightarrow \langle\langle \Delta, i \rangle\rangle (\langle i \sqsubseteq \Gamma \rangle \varphi \mathsf{U} \langle i \sqsubseteq \Gamma \rangle \psi)$

(distr2) $\langle i \sqsubseteq \Gamma \rangle [\![\Delta, \Gamma]\!] (\varphi \mathsf{U} \psi) \Rightarrow [\![\Delta, i]\!] (\langle i \sqsubseteq \Gamma \rangle \varphi \mathsf{U} \langle i \sqsubseteq \Gamma \rangle \psi)$

Using **K** one can establish that $\langle i \sqsubseteq \Gamma\rangle$ distributes over $\vee$ for i and Γ fixed. Since the agent names in Γ are bound, a single $\langle i \sqsubseteq \Gamma\rangle$-formula can be written for the disjunction of $\langle i \sqsubseteq \Gamma\rangle\varphi$ and $\langle i \sqsubseteq \Gamma'\rangle\psi$ for $\Gamma' \neq \Gamma$ too, provided that $|\Gamma'| = |\Gamma|$, by virtue of **ren**. In case $|\Gamma'| < |\Gamma|$, one can still equalize the number of sub-agents in $\langle i \sqsubseteq \Gamma\rangle\varphi$ and $\langle i \sqsubseteq \Gamma'\rangle\psi$ thanks to the validity of the equivalence $\langle i \sqsubseteq \Gamma'\rangle\psi \Leftrightarrow \langle i \sqsubseteq \Gamma'^{\Delta}_{j}\rangle[\Delta/j]\psi$ for any $j \in \Gamma'$ and Δ such that $\Sigma(\psi) \cap \Delta = \emptyset$ and $|\Delta| = |\Gamma| - |\Gamma'| + 1$:

$$(\vee_{\sqsubseteq}) \quad \langle i \sqsubseteq \Gamma\rangle\varphi \vee \langle i \sqsubseteq \Gamma'\rangle\psi \Leftrightarrow \langle i \sqsubseteq \Gamma'^{\Delta}_{j}\rangle([\Gamma'^{\Delta}_{j}/\Gamma]\varphi \vee [\Delta/j]\psi)$$

where $[\Gamma'^{\Delta}_{j}/\Gamma]$ denotes some bijection between Γ'^{Δ}_{j} and Γ. The following axiom provides a way to write conjunctions of $\langle i \sqsubseteq \Gamma\rangle$-formulas with the same i and possibly different Γs as single $\langle . \sqsubseteq .\rangle$-formulas:

$$(\wedge_{\sqsubseteq}) \quad \langle i \sqsubseteq \Gamma_1\rangle\varphi_1 \wedge \langle i \sqsubseteq \Gamma_2\rangle\varphi_2 \Leftrightarrow \langle i \sqsubseteq \Gamma_1 \times \Gamma_2\rangle([\{j\} \times \Gamma_2/j : j \in \Gamma_1]\varphi_1 \wedge [\Gamma_1 \times \{j\}/j : j \in \Gamma_2]\varphi_2)$$

This axiom states the equivalence between the possibility to partition the powers of i among the sub-agents from Γ_k so that φ_k holds, $k = 1, 2$, and the possibility to partition the powers of i among the $|\Gamma_1| \times |\Gamma_2|$ many sub-agents, whom we name using the elements of $\Gamma_1 \times \Gamma_2$, with the following property. The sub-agents can either form $|\Gamma_1|$ many groups of the form $\{j\} \times \Gamma_2$, $j \in \Gamma_1$, each group consisting of $|\Gamma_2|$ many agents and having the powers of the respective sub-agent j from a partitioning of the powers of i among the sub-agents from Γ_1 which satisfies φ_1, or form $|\Gamma_2|$ many groups of the form $\Gamma_1 \times \{j\}$, $j \in \Gamma_2$, each group with the powers of the respective sub-agent j from a partitioning of the powers of i among the sub-agents from Γ_2 which satisfies φ_2. The substitutions applied to the occurrences of φ_1 and φ_2 on the righthand side of $\Leftrightarrow$ in $\wedge_{\sqsubseteq}$ indicate that each sub-agent j from Γ_1 in φ_1, resp. from Γ_2 in φ_2, is replaced by its corresponing group $\{j\} \times \Gamma_2$, resp. $\Gamma_1 \times \{j\}$. The substitution instances of φ_1 and φ_2 hold because there exists a partitioning of the powers of i among the sub-agents $\Gamma_1 \times \Gamma_2$ such that the substituting groups of agents of these forms have the same powers as the corresponding agents j under some partitionings among the sub-agents from Γ_1 and Γ_2, respectively, whose existence is expressed on the left-hand side of the equivalence.

The right-to-left implications

$$\langle i \sqsubseteq \Gamma_1 \times \Gamma_2\rangle([\{j\} \times \Gamma_2/j : j \in \Gamma_1]\varphi_1 \wedge [\Gamma_1 \times \{j\}/j : j \in \Gamma_2]\varphi_2) \Rightarrow \langle i \sqsubseteq \Gamma_k\rangle\varphi_k,\ k = 1, 2,$$

from $\wedge_{\sqsubseteq}$ are easy to derive using **T** and **flatten** as follows. Let Γ_1 be $\{j^0, \ldots, j^{|\Gamma_1|-1}\}$. Then $|\Gamma_1|$ applications of **flatten** imply that $\langle i \sqsubseteq \Gamma_1 \times \Gamma_2\rangle[\{j\} \times \Gamma_2/j : j \in \Gamma_1]\varphi_1$ is equivalent to

$$\langle i \sqsubseteq \Gamma_1\rangle\langle j^0 \sqsubseteq \{j^0\} \times \Gamma_2\rangle \ldots \langle j^{|\Gamma_1|-1} \sqsubseteq \{j^{|\Gamma_1|-1}\} \times \Gamma_2\rangle[\{j\} \times \Gamma_2/j : j \in \Gamma_1]\varphi_1.$$

Now $\langle i \sqsubseteq \Gamma_1 \times \Gamma_2\rangle[\{j\} \times \Gamma_2/j : j \in \Gamma_1]\varphi_1 \Rightarrow \langle i \sqsubseteq \Gamma_1\rangle\varphi_1$ can be obtained by rewriting the substitution $[\{j\} \times \Gamma_2/j : j \in \Gamma_1]$ as $[\{j^{|\Gamma_1|-1}\} \times \Gamma_2/j^{|\Gamma_1|-1}] \ldots [\{j^0\} \times \Gamma_2/j^0]$ and applying **T** $|\Gamma_1|$ times to

$$\langle i \sqsubseteq \Gamma_1\rangle\langle j^0 \sqsubseteq \{j^0\} \times \Gamma_2\rangle \ldots \langle j^{|\Gamma_1|-1} \sqsubseteq \{j^{|\Gamma_1|-1}\} \times \Gamma_2\rangle[\{j^{|\Gamma_1|-1}\} \times \Gamma_2/j^{|\Gamma_1|-1}] \ldots [\{j^0\} \times \Gamma_2/j^0]\varphi_1.$$

The proof about $k = 2$ is similar. A detailed proof of the soundness of the left-to-right implication in $\wedge_{\sqsubseteq}$ for finite CGMs is given in Appendix Appendix A. It extends to infinite CGMs by modifying the constructions of the relevant homomorphisms like in the proof of the validity of (4).

If agent i can contribute to the achievement of an objective, then its powers can be partitioned so that each of the sub-agents can achieve the objective. Similarly, if φ cannot be prevented without the contribution of i, then i's powers can be distributed so that proper subcoalitions of the sub-agents cannot help prevent φ either: Let $\mathcal{D} \subseteq \mathcal{P}(\Delta) \setminus \emptyset$ and let $\mathcal{D}$ be upward closed wrt inclusion. Let $i \notin \Gamma$. Let φ be either of the form $\bigcirc\psi$ or of the form $(\psi_1 \mathsf{U} \psi_2)$. Then

$$(\mathbf{sub}) \quad \langle\!\langle \Gamma, i\rangle\!\rangle\varphi \wedge \neg\langle\!\langle \Gamma\rangle\!\rangle\varphi \Rightarrow \langle i \sqsubseteq \Delta\rangle \left(\bigwedge_{\Delta' \in \mathcal{D}} \langle\!\langle \Gamma, \Delta'\rangle\!\rangle[\Delta'/i]\varphi \wedge \bigwedge_{\Delta' \in \mathcal{P}(\Delta)\setminus\mathcal{D}} \neg\langle\!\langle \Gamma, \Delta'\rangle\!\rangle[\Delta'/i]\varphi \right).$$

This axiom generalizes (4) and (5). Its soundness is established by an argument similar to that for the validity of (5).

Concluding Remarks

The reduction of model checking $\langle . \sqsubseteq . \rangle^*$-flat $\mathrm{ATL}_\sqsubseteq$ properties of CGMs to validity in the $\langle\langle . \rangle\rangle\bigcirc$-subset of ATL, which we use to demonstrate the decidability of this problem, indicates that the operator $\langle i \sqsubseteq \Gamma \rangle$ can be used to express that agent i is indeed a collective body consisting of its sub-agents $j \in \Gamma$. The members of Γ exercise their powers by competing for influence on the decisions on behalf of i inside this collective body. Model checking a formula of the form $\langle i \sqsubseteq \Gamma \rangle \varphi$ requires the construction of a model where the members of Γ appear as primary agents and a mapping h from Γ's actions in that model to the actions of i in the given one. The mapping h, which we call Γ-to-i homomorphism for the way it relates the outcome functions of CGMs, can be viewed as the rules which determine how the members of Γ reach collective decisions on what action to take on behalf of i. Therefore establishing that $\langle i \sqsubseteq \Gamma \rangle \varphi$ holds at a state of the given model amounts to *synthesizing* the rules for Γ so that their powers, possibly combined with those of other primary agents from the given model, satisfy φ. The example formulas (4) and (5) and the proofs of their validity show that the rules can always be chosen to be *unanimity* or *simple majority vote*. More general forms of voting can be handled using axiom (**sub**).

In this paper we have chosen to consider Γ-to-i homomorphisms h which, for a given vector of actions a'_Γ for the set of sub-agents Γ, produce the same action $h(a'_\Gamma)$ for agent i in all states. Furthermore, we only consider homomorphisms which link models with the same statespace. Considering more general types of homomorphisms h such as $Act'_\Gamma \times W \rightarrow Act_i$ corresponds to the synthesis problem for rules for Γ in which the reference state can immediately affect the powers of sub-agents. Allowing this would not change the scope of our decision method for model checking, but would change validity in $\mathrm{ATL}_\sqsubseteq$. For instance, axioms **distr1** and **distr2** would be possible to reformulate as equivalences.

Further generalization of Γ-to-i homomorphisms can be achieved by allowing models M' from Definition 3 to have bigger statespaces than target models M. This can be interpreted as giving sub-agents additional *memory* and is relevant to partitioning agent powers in systems with *incomplete information*. Modelling additional memory can be made even more conveniently if *interpreted systems* as in [18] are used as the type of model instead of CGMs. Statespaces are compartmented in interpreted systems: global states are vectors of local states, each local state ranging over the corresponding agent's local statespace. This facilitates allowing M's to have bigger statespaces for just the agents from Γ and the same statespaces for the primary agents, except i. Formulating these more general forms of the semantics of $\langle . \sqsubseteq . \rangle$ is increasingly complex and could turn more manageable if explicit strategy languages such as strategy logics [19, 20] are taken as the basis.

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References

[1] R. Alur, T. Henzinger, O. Kupferman, Alternating-time Temporal Logic, in: Proceedings of FCS'97, 1997, pp. 100–109. doi: 10.1007/3-540-49213-5_2.

[2] R. Alur, T. Henzinger, O. Kupferman, Alternating-time temporal logic, Journal of the ACM 49 (2002) 1–42.

[3] L. Bozzelli, H. van Ditmarsch, T. French, J. Hales, S. Pinchinat, Refinement modal logic, Inf. Comput. 239 (2014) 303–339.

[4] J. Hales, T. French, R. Davies, Refinement quantified logics of knowledge and belief for multiple agents, in: T. Bolander, T. Braüner, S. Ghilardi, L. S. Moss (Eds.), Advances in Modal Logic, College Publications, 2012, pp. 317–338.

[5] R. Alur, T. A. Henzinger, O. Kupferman, M. Y. Vardi, Alternating refinement relations, in: D. Sangiorgi, R. de Simone (Eds.), CONCUR, volume 1466 of *LNCS*, Springer, 1998, pp. 163–178.

[6] M. Ryan, P.-Y. Schobbens, Agents and roles: Refinement in alternating-time temporal logic, in: J.-J. C. Meyer, M. Tambe (Eds.), ATAL, volume 2333 of *LNCS*, Springer, 2001, pp. 100–114. doi: 10.1007/3-540-45448-9_8.

[7] C. Enea, C. Dima, Abstractions of multi-agent systems, in: H.-D. Burkhard, G. Lindemann, R. Verbrugge, L. Z. Varga (Eds.), CEEMAS, volume 4696 of *LNCS*, Springer, 2007, pp. 11–21. doi: 10.1007/978-3-540-75254-7_2.

[8] M. Cohen, M. Dam, A. Lomuscio, F. Russo, Abstraction in model checking multi-agent systems, in: C. Sierra, C. Castelfranchi, K. S. Decker, J. S. Sichman (Eds.), AAMAS (2), IFAAMAS, 2009, pp. 945–952. doi: 10.1145/1558109.1558144.

[9] M. Köster, P. Lohmann, Abstraction for Model Checking Modular Interpreted Systems over ATL, in: L. A. Dennis, O. Boissier, R. H. Bordini (Eds.), ProMAS, volume 7217 of *LNCS*, Springer, 2011, pp. 95–113. doi: 10.1007/978-3-642-31915-0_6.

[10] G. D. Giacomo, P. Felli, Agent composition synthesis based on ATL, in: W. van der Hoek, G. A. Kaminka, Y. Lespérance, M. Luck, S. Sen (Eds.), AAMAS, IFAAMAS, 2010, pp. 499–506. doi: 10.1145/1838206.1838277.

[11] G. D. Giacomo, F. Patrizi, S. Sardiña, Automatic behavior composition synthesis, Artif. Intell. 196 (2013) 106–142.

[12] W. Jamroga, T. Ågotnes, Modular interpreted systems, in: E. H. Durfee, M. Yokoo, M. N. Huhns, O. Shehory (Eds.), AAMAS, IFAAMAS, 2007, p. 131. doi: 10.1145/1329125.1329286.

[13] W. Jamroga, A. Meski, M. Szreter, Modularity and openness in modeling multi-agent systems, in: G. Puppis, T. Villa (Eds.), GandALF, volume 119 of *EPTCS*, 2013, pp. 224–239. doi: 10.4204/EPTCS.119.19.

[14] A. Herzig, E. Lorini, D. Walther, Reasoning about actions meets strategic logics, in: D. Grossi, O. Roy, H. Huang (Eds.), LORI, volume 8196 of *LNCS*, Springer, 2013, pp. 162–175. doi: 10.1007/978-3-642-40948-6_13.

[15] A. Herzig, Private communication, 2014.

[16] M. Pauly, A modal logic for coalitional power in games, Journal of Logic and Computation 12 (2002) 149–166.

[17] G. E. Hughes, M. J. Cresswell, A Companion to Modal Logic, Methuen, London, 1984.

[18] A. Lomuscio, F. Raimondi, Model checking knowledge, strategies, and games in multi-agent systems, in: Proceedings of AAMAS'06, ACM Press, 2006, pp. 161–168. doi: 10.1145/1160633.1160660.

[19] K. Chatterjee, T. A. Henzinger, N. Piterman, Strategy Logic, in: L. Caires, V. T. Vasconcelos (Eds.), CONCUR, volume 4703 of *LNCS*, Springer, 2007, pp. 59–73. doi: 10.1007/978-3-540-74407-8_5.

[20] F. Mogavero, A. Murano, M. Y. Vardi, Reasoning About Strategies, in: K. Lodaya, M. Mahajan (Eds.), FSTTCS, volume 8 of *LIPIcs*, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2010, pp. 133–144. doi: 10.4230/LIPIcs.FSTTCS.2010.133.

Appendix A. Soundness proofs

Proof of the soundness of $\langle . \sqsubseteq . \rangle_L$:. Consider an arbitrary CGM $M = \langle W, \langle Act_j : j \in \Sigma \rangle, o, V \rangle$. Assume that $M, w \models \neg\psi \wedge \langle i \sqsubseteq \Gamma \rangle \varphi$ for the sake of contradiction. Then there exists a CGM M' and a Γ-to-i homomorphism h from M' to M such that $M', w \models \varphi$. To reach contradiction, we need to prove that $M', w \not\models [\Gamma/i]\psi$. We prove that $M', w \models [\Gamma/i]\psi$ is equivalent to $M, w \models \psi$ by induction on the construction of ψ. The only non-trivial cases to be considered are that of ψ of the form $\langle\!\langle \Delta, i \rangle\!\rangle \bigcirc \chi$, $\langle\!\langle \Delta, i \rangle\!\rangle (\chi_1 \mathsf{U} \chi_2)$ and $[\![\Delta, i]\!](\chi_1 \mathsf{U} \chi_2)$. We only do $\langle\!\langle \Delta, i \rangle\!\rangle \bigcirc \chi$ here; the other two cases are similar. Let $a_{\Delta \cup \{i\}} \in Act_{\Delta \cup \{i\}}$ be such that $M, o(w, a_{\Delta \cup \{i\}} \cdot b_{\Sigma \setminus (\Delta \cup \{i\})}) \models \chi$ for all $b_{\Sigma \setminus (\Delta \cup \{i\})} \in Act_{\Sigma \setminus (\Delta \cup \{i\})}$. Let $a'_\Gamma \in Act'_\Gamma$ be such that $h(a'_\Gamma) = a_i$. Then $o'(w, a_\Delta \cdot a'_\Gamma \cdot b_{\Sigma \setminus (\Delta \cup \{i\})}) = o(w, a_{\Delta \cup \{i\}} \cdot b_{\Sigma \setminus (\Delta \cup \{i\})})$ for all $b_{\Sigma \setminus (\Delta \cup \{i\})}$ and, by the inductive hypothesis, $M', o'(w, a_\Delta \cdot a'_\Gamma \cdot b_{\Sigma \setminus (\Delta \cup \{i\})}) \models [\Gamma/i]\chi$ for all $b_{\Sigma \setminus (\Delta \cup \{i\})}$. This entails that $M', w \models \langle\!\langle \Delta, \Gamma \rangle\!\rangle \bigcirc [\Gamma/i]\chi$. For the converse direction, if there exist $a'_\Gamma \in Act'_\Gamma$ and $a_\Delta \in Act'_\Delta = Act_\Delta$ such that $M', o'(w, a_\Delta \cdot a'_\Gamma \cdot b_{\Sigma \setminus (\Delta \cup \{i\})}) \models [\Gamma/i]\chi$ for all $b_{\Sigma \setminus (\Delta \cup \{i\})}$, which is the satisfaction condition for $\langle\!\langle \Delta, \Gamma \rangle\!\rangle \bigcirc [\Gamma/i]\chi$ at w in M', then, by the inductive hypothesis, $M, o(w, a_\Delta \cdot h(a'_\Gamma) \cdot b_{\Sigma \setminus (\Delta \cup \{i\})}) \models \chi$ for all $b_{\Sigma \setminus (\Delta \cup \{i\})}$, and therefore $M, w \models \langle\!\langle \Delta, i \rangle\!\rangle \bigcirc \chi$.

This argument about ψ of the form $\langle\!\langle \Delta, i \rangle\!\rangle \bigcirc \chi$ exploits the correspondence between the vector of actions a'_Γ, which, together with an appropriate a_Δ, forms a one-step strategy for $\Gamma \cup \Delta$ to achieve χ and the action $h(a'_\Gamma)$, which, together with the same a_Δ, forms a similar strategy one-step strategy for $\{i\} \cup \Delta$. The cases of ψ having one of the forms $\langle\!\langle \Delta, i \rangle\!\rangle (\chi_1 \mathsf{U} \chi_2)$ and $[\![\Delta, i]\!](\chi_1 \mathsf{U} \chi_2)$ are dealt with by using the same correspondence to link strategies which allow for arbitrary numbers of steps. $\dashv$

Proof of the soundness of G1:. Assume that there exist an M' for $\Sigma \setminus \{i\} \cup \Gamma$ and a Γ-to-i homomorphism h' from M' to M such that for every M'' for $\Sigma \setminus \{i, j\} \cup \Gamma \cup \Delta$ and every Δ-to-j homomorphism h'' from M'' to M' we have $M'', w \models \varphi$. Now let M''' be a CGM for $\Sigma \setminus \{j\} \cup \Delta$ and let h''' be a Δ-to-j homomorphism from M''' to M. We need to prove that an M^{iv} for $\Sigma \setminus \{i, j\} \cup \Gamma \cup \Delta$ and a Γ-to-i homomorphism h^{iv} from M^{iv} to M''' exist such that $M^{iv}, w \models \varphi$. We define $M^{iv} = \langle W, \langle Act^{iv}_k : k \in \Sigma \setminus \{i, j\} \cup \Gamma \cup \Delta \rangle, o^{iv}, V \rangle$ by putting $Act^{iv}_k = Act''_k$ for $k \in \Gamma$ and $o^{iv}(w, a) = o'''(w, a_{(\Sigma \setminus \{i,j\}) \cup \Delta} \cdot h'(a_\Gamma))$ for all $w \in W$ and all $a \in Act^{iv}_{(\Sigma \setminus \{i,j\}) \cup \Delta \cup \Gamma}$. Defining o^{iv} this way obviously entails that h', which was first introduced as a Γ-to-i homomorphism from M' to M, is also a Γ-to-i homomorphism from M^{iv} to M'''. Furthermore, $o^{iv}(w, a) = o'''(w, a_{(\Sigma \setminus \{i,j\}) \cup \Delta} \cdot h'(a_\Gamma)) = o(w, a_{\Sigma \setminus \{i,j\}} \cdot h'(a_\Gamma) \cdot h'''(a_\Delta)) = o'(w, a_{\Sigma \setminus \{i,j\} \cup \Gamma} \cdot h'''(a_\Delta))$. Hence h''' is a Δ-to-j homomorphism from M^{iv} to M'. This concludes the soundness proof for **G1** because every CGM for $\Sigma \setminus \{i, j\} \cup \Gamma \cup \Delta$ which admits a Δ-to-j to M' satisfies φ at w. $\dashv$

Proof of the soundness of T:. Consider a CGM M' for $(\Sigma \setminus \{i\}) \cup \Gamma$ and a Γ-to-i homomorphism h from M' to M. We use induction on the construction of formulas φ to prove that if $w \in W$ then $M', w \models [\Gamma/i]\varphi$ iff $M, w \models \varphi$. We omit the cases of φ being $\perp$, an atomic proposition or a formula with $\Rightarrow$ as the main connective. Let φ be $\langle\!\langle \Delta, i \rangle\!\rangle \bigcirc \psi$ where $\Delta \subseteq \Sigma \setminus \{i\}$. Then $M', w \models [\Gamma/i]\varphi$ means that there exists a vector of actions $a_{\Delta \cup \Gamma} \in Act'_{\Delta \cup \Gamma}$ such that $M', o'(w, a_{\Delta \cup \Gamma} \cdot b_{\Sigma \setminus (\{i\} \cup \Delta)}) \models [\Gamma/i]\psi$ for all $b_{\Sigma \setminus (\{i\} \cup \Delta)} \in Act_{\Sigma \setminus (\{i\} \cup \Delta)}$. By the inductive hypothesis and the definition of o', this is equivalent to $M, o(w, a_\Gamma \cdot h(a_\Gamma) \cdot b_{\Sigma \setminus (\{i\} \cup \Delta)}) \models \psi$ for all such $b_{\Sigma \setminus (\{i\} \cup \Delta)} \in Act_{\Sigma \setminus (\{i\} \cup \Delta)}$. Hence $M, w \models \langle\!\langle \Delta, i \rangle\!\rangle \bigcirc \psi$. Let $a_{\Delta \cup \{i\}} \in Act_{\Delta \cup \{i\}}$ be such that $M, o(w, a_{\Delta \cup \{i\}} \cdot b_{\Sigma \setminus (\{i\} \cup \Delta)}) \models \psi$ for all $b_{\Sigma \setminus (\{i\} \cup \Delta)} \in Act_{\Sigma \setminus (\{i\} \cup \Delta)}$. Then, by the surjectivity of h, there exists an $a'_\Gamma \in Act'_\Gamma$ such that $a_i = h(a'_\Gamma)$, whence $M, o'(w, a_\Delta \cdot a'_\Gamma \cdot b_{\Sigma \setminus (\{i\} \cup \Delta)}) \models \psi$ for all $b_{\Sigma \setminus (\{i\} \cup \Delta)} \in Act_{\Sigma \setminus (\{i\} \cup \Delta)}$. By the inductive hypothesis, this is equivalent to $M', o'(w, a_\Delta \cdot a'_\Gamma \cdot b_{\Sigma \setminus (\{i\} \cup \Delta)}) \models [\Gamma/i]\psi$ for all such $b_{\Sigma \setminus (\{i\} \cup \Delta)}$. This entails that $M', w \models \langle\!\langle \Delta, \Gamma \rangle\!\rangle \bigcirc [\Gamma/i]\psi$. $\dashv$

Proof of the soundness of $\wedge_\sqsubseteq$:. Let M be a CGM, with its components written as usual, and let $w \in W$. Let $M, w \models \langle i \sqsubseteq \Gamma_1 \rangle \varphi_1 \wedge \langle i \sqsubseteq \Gamma_2 \rangle \varphi_2$. This means that there exist some CGMs $M^k = \langle W^k, \langle Act^k_j : j \in \Sigma^k \rangle, o^k, V^k \rangle$ with $\Sigma^k = (\Sigma \setminus \{i\}) \cup \Gamma_k$ as the set of agents, and Γ_k-to-i homomorphisms $h^k : Act^k_{\Gamma_k} \to Act_i$ be from M^k to M such that $M^k, w \models \varphi_k$, $k = 1, 2$. We need to construct a CGM $M^{1,2} = \langle W^{1,2}, \langle Act^{1,2}_j : j \in \Sigma^{1,2} \rangle, o^{1,2}, V^{1,2} \rangle$ with $\Sigma^{1,2} = (\Sigma \setminus \{i\}) \cup \Gamma_1 \times \Gamma_2$ as the set of the agents and a $\Gamma_1 \times \Gamma_2$-to-i homomorphism

$h^{1,2}$ from $M^{1,2}$ to M such that $M, w \models [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\varphi_1 \wedge [\Gamma_1 \times \{j\}/j : j \in \Gamma_2]\varphi_2$. As expected, we put

$$W^{1,2} = W^1 = W^2 = W,\ V^{1,2} = V^1 = v^2 = V \text{ and } Act_j^{1,2} = Act_j^1 = Act_j^2 = Act_j \text{ for } j \in \Sigma \setminus \{i\}.$$

Without loss of generality we assume that the sets Act_j^k, $j \in \Gamma_k$, $k = 1,2$ are pairwise disjoint.

Let $Act_i = \{a_{i,0}, \ldots, a_{i,|Act_i|-1}\}$. Let $Act_j^k = \{a_{j,0}^k, \ldots, a_{j,|Act_j^k|-1}^k\}$, $j \in \Gamma_k$, $k = 1,2$.

Let $Z = \max\{|Act_i|, \max\limits_{j \in \Gamma_k, k=1,2} |Act_j^k|\}$. We put

$$Act_{j_1,j_2}^{1,2} = (Act_{j_1}^1 \cup Act_{j_2}^2) \times \{0, \ldots, Z-1\}.$$

Given $a_{\Gamma_1 \times \Gamma_2} = \langle\langle b_{j_1,j_2}, k_{j_1,j_2}\rangle : \langle j_1, j_2\rangle \in \Gamma_1 \times \Gamma_2\rangle \in Act_{\Gamma_1 \times \Gamma_2}^{1,2}$, we define $h^{1,2}(a_{\Gamma_1 \times \Gamma_2})$ by considering the following three cases:

Case 1. There exists a $j_1 \in \Gamma_1$ and a $b \in Act_{j_1}^1$ such that $b = b_{j_1,j_2}$ for all $j_2 \in \Gamma_2$. Then we define the vector of actions $c_{\Gamma_1} \in Act_{\Gamma_1}^1$ as follows. For $j_1 \in \Gamma_1$ such that a $b_{j_1,j_2} \in Act_j^1$ and b_{j_1,j_2} is the same for all $j_2 \in \Gamma_2$, we put $c_{j_1} = b_{j_1,j_2}$ (for some arbitrary $j_2 \in \Gamma_2$.) For j_1 which do not satisfy this condition, we put $c_{j_1} = a_{j_1,z}^1$ where $z = \left(\sum\limits_{j_2 \in \Gamma_2} k_{j_1,j_2}\right) \bmod |Act_{j_1}^1|$. Having defined c_{Γ_1}, we put $h^{1,2}(a_{\Gamma_1 \times \Gamma_2}) = h^1(c_{\Gamma_1})$.

Case 2. There exists a $j_2 \in \Gamma_2$ and a $b \in Act_{j_2}^2$ such that $b = b_{j_1,j_2}$ for all $j_1 \in \Gamma_1$. This case is symmetrical to Case 1. We define the vector of actions $c_{\Gamma_2} \in Act_{\Gamma_2}^2$ as follows. For $j_2 \in \Gamma_2$ such that a $b_{j_1,j_2} \in Act_j^2$ and b_{j_1,j_2} is the same for all $j_1 \in \Gamma_1$, we put $c_{j_2} = b_{j_1,j_2}$ (for some arbitrary $j_1 \in \Gamma_1$.) For j_2 which do not satisfy this condition, we put $c_{j_2} = a_{j_2,z}^2$ where $z = \left(\sum\limits_{j_1 \in \Gamma_1} k_{j_1,j_2}\right) \bmod |Act_{j_2}^2|$. Having defined c_{Γ_2}, we put $h^{1,2}(a_{\Gamma_1 \times \Gamma_2}) = h^2(c_{\Gamma_2})$.

Our assumption that Act_j^k, $j \in \Gamma_k$, $k = 1,2$ are pairwise disjoint entails that Cases 1 and 2 are mutually exclusive: it is impossible to have both a whole row a_{j_1,j_2}, $j_2 \in \Gamma_2$, of actions with their first components in $Act_{j_1}^1$ and a whole column a_{j_1,j_2}, $j_2 \in \Gamma_2$ of actions with their first components in $Act_{j_2}^2$.

Case 3. If neither of Cases 1 and 2 applies, we put $h^{1,2}(a_{\Gamma_1 \times \Gamma_2}) = a_{i,z}$ where $z = \left(\sum\limits_{j_k \in \Gamma_k, k=1,2} k_{j_1,j_2}\right) \bmod |Act_i|$.

Informally, $h^{1,2}$ works as follows. The sub-agents from $\Gamma_1 \times \Gamma_2$ can choose to either operate in groups of the form $\{j_1\} \times \Gamma_2$, each such group assuming the powers of the respective sub-agent j_1 from Γ_1, or operate in groups of the form $\Gamma_1 \times \{j_2\}$, which can assume the powers of the respective sub-agents from Γ_2, or not coalesce at all. By acting in one of the first two ways, the sub-agents can implement whatever strategies are relevant to the satisfaction of $[\{j\} \times \Gamma_2/j : j \in \Gamma_1]\varphi_1$ and $[\Gamma_1 \times \{j\}/j : j \in \Gamma_2]\varphi_2$, respectively. In the third case the sub-agents act *erratically*: the action $a_{i,z} = h^{1,2}(a_{\Gamma_1 \times \Gamma_2})$ is chosen so that no substantial restriction on the choice of $a_{i,z}$ can be inferred from the actions of the members of any proper sub-coalition of $\Gamma_1 \times \Gamma_2$. Note that it is sufficient that *some* of the sub-agents from $\Gamma_1 \times \Gamma_2$ choose to act unanimously as a group of the form $\{j_1\} \times \Gamma_2$ for Case 1 of the definition to apply. If at least one such group exists, then the agents from $\Gamma_1 \times \Gamma_2$ who do not belong to similar groups are assumed to be acting erratically too and the actions $a_{j_1,z}^1$ which are submitted on their behalf to h^1 are chosen like in Case 3. The same holds about Case 2. The rest of the proof is about showing how this works in detail.

We use induction on the construction of formulas φ_1 written with $\Sigma^1 = \Sigma \setminus \{i\} \cup \Gamma_1$ as the set of agents to show that $M^1, w \models \varphi_1$ is equivalent to $M^{1,2}, w \models [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\varphi_1$ for all $w \in W$. The proof about formulas φ_2 with $\Sigma^2 = \Sigma \setminus \{i\} \cup \Gamma_2$ as the set of agents is similar. The cases about φ_1 being $\perp$, an atomic proposition or a formula with $\Rightarrow$ as the main connective are trivial and we omit them. So is the case of φ_1 of the form $\langle\langle \Delta \rangle\rangle \bigcirc \chi_1$ where $\Delta \cap \Gamma_1 = \emptyset$. Let φ_1 be $\langle\langle \Delta' \cup \Delta'' \rangle\rangle \bigcirc \chi_1$ where $\Delta' \subseteq \Sigma \setminus \{i\}$, $\Delta'' \subseteq \Gamma_1$ and χ_1 is written with Σ^1 as the set of agents. Assume that $M^{1,2}, w \models \langle\langle \Delta' \cup \Delta'' \times \Gamma_2 \rangle\rangle \bigcirc [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$. Then there exist two vectors of actions $a_{\Delta'} \in Act_{\Delta'}^{1,2} = Act_{\Delta'}^1 = Act_{\Delta'}$ and $b_{\Delta'' \times \Gamma_2} \in Act_{\Delta'' \times \Gamma_2}^{1,2}$ such that for any two vectors of actions $c_{\Sigma \setminus (\{i\} \cup \Delta')} \in Act_{\Sigma \setminus (\{i\} \cup \Delta')}$ and $d_{(\Gamma_1 \setminus \Delta'') \times \Gamma_2} \in Act_{(\Gamma_1 \setminus \Delta'') \times \Gamma_2}^{1,2}$ we have $M^{1,2}, o^{1,2}(w, a_{\Delta'} \cdot b_{\Delta'' \times \Gamma_2} \cdot c_{\Sigma \setminus (\{i\} \cup \Delta')} \cdot d_{(\Gamma_1 \setminus \Delta'') \times \Gamma_2}) \models [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$. By the inductive hypothesis, this is equivalent to $M^1, o^{1,2}(w, a_{\Delta'} \cdot b_{\Delta''\times\Gamma_2} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}) \models \chi_1$. By definition $o^{1,2}(w, a_{\Delta'} \cdot b_{\Delta''\times\Gamma_2} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}) = o(w, a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot h^{1,2}(b_{\Delta''\times\Gamma_2} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}))$. We do the cases of $\Delta'' = \Gamma_1$ and $\Delta'' \subsetneq \Gamma_1$ separately. If $\Delta'' = \Gamma_1$, then $d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}$ is the empty vector, and, by the surjectivity of h^1, there exists a vector of actions $b^1_{\Gamma_1} \in Act^1_{\Gamma_1}$ in M^1 such that $h^1(b^1_{\Gamma_1}) = h^{1,2}(b_{\Delta''\times\Gamma_2})$. Then $o(w, a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot h^{1,2}(b_{\Delta''\times\Gamma_2})) = o(w, a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot h^1(b^1_{\Gamma_1}))$, whence for the chosen $a_{\Delta'}$ and $b^1_{\Gamma_1}$ and any $c_{\Sigma\setminus(\{i\}\cup\Delta')} \in Act_{\Sigma\setminus(\{i\}\cup\Delta')}$ we have $M^1, o^1(w, a_{\Delta'} \cdot b^1_{\Gamma_1} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')}) \models \chi_1$. This means that $M^1, w \models \langle\langle\Delta' \cup \Delta''\rangle\rangle \bigcirc \chi_1$. Now let $\Delta'' \subsetneq \Gamma_1$. If there exists no $j_1 \in \Delta''$ such that the first component of b_{j_1,j_2} is the same member of $Act^1_{j_1}$ for all $j_2 \in \Gamma_2$, then, by just varying the actions $d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}$ of the sub-agents from $(\Gamma_1 \setminus \Delta'') \times \Gamma_2$, the sub-agents from $\Delta'' \times \Gamma_2$ can be prevented from achieving any substantial restriction on $h^{1,2}(b_{\Delta''\times\Gamma_2} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2})$, because the sub-agents from $(\Gamma_1 \setminus \Delta'') \times \Gamma_2$ can choose their actions in a way which would cause $h^{1,2}(b_{\Delta''\times\Gamma_2} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2})$ to be determined according to Case 3 of the definition of $h^{1,2}$. This means that indeed $M^{1,2}, w \models \langle\langle\Delta'\rangle\rangle \bigcirc [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$ and not just $M^{1,2}, w \models \langle\langle\Delta' \cup \Delta'' \times \Gamma_2\rangle\rangle \bigcirc [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$, i.e., the sub-agents from $\Delta'' \times \Gamma_2$ make no substantial contribution to achieving $\bigcirc[\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$ at the next step. As we claimed above, the equivalence between $M^{1,2}, w \models \langle\langle\Delta'\rangle\rangle \bigcirc [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$ and $M^1, w \models \langle\langle\Delta'\rangle\rangle \bigcirc \chi_1$ is trivial. Hence, in this case $M^{1,2}, w \models \langle\langle\Delta'\rangle\rangle \bigcirc [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$ entails $M^1, w \models \langle\langle\Delta' \cup \Delta''\rangle\rangle \bigcirc \chi_1$

Now assume that there exists a $j_1 \in \Delta''$ such that the first component of b_{j_1,j_2} is the same member of $Act^1_{j_1}$ for all $j_2 \in \Gamma_2$. Then $h^{1,2}(b_{\Delta''\times\Gamma_2} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2})$ is determined according to Case 1 of the definition of $h^{1,2}$. According to this definition, $h^{1,2}(b_{\Delta''\times\Gamma_2} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}) = h^1(e_{\Gamma_1})$ for some $e_{\Gamma_1} \in Act^1_{\Gamma_1}$ such that the restriction $e_{\Delta''}$ of e_{Γ_1} to Δ'' is determined from $b_{\Delta''\times\Gamma_2}$ and the restriction $e_{\Gamma_1\setminus\Delta''}$ of e_{Γ_1} to $\Gamma_1 \setminus \Delta''$ is determined from $d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}$, and $e_{\Gamma_1\setminus\Delta''}$ can be any member of $Act^1_{\Gamma_1\setminus\Delta''}$ when $d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}$ ranges over $Act^{1,2}_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}$. Furthermore, $o^{1,2}(w, a_{\Delta'} \cdot b_{\Delta''\times\Gamma_2} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}) = o(w, a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot h^{1,2}(b_{\Delta''\times\Gamma_2} \cdot d_{(\Gamma_1\setminus\Delta'')\times\Gamma_2})) = o(w, a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot h^1(e_{\Delta''} \cdot e_{\Gamma_1\setminus\Delta''})) = o^1(w, a_{\Delta'} \cdot e_{\Delta''} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot e_{\Gamma_1\setminus\Delta''})$.

Hence, $a_{\Delta'} \cdot e_{\Delta''} \in Act^1_{\Delta'\cup\Delta''}$ is such that for any $c_{\Sigma\setminus(\{i\}\cup\Delta')} \in Act_{\Sigma\setminus(\{i\}\cup\Delta')}$ and $e_{\Gamma_1\setminus\Delta''} \in Act^1_{\Gamma_1\setminus\Delta''}$ $M^1, o^1(w, a_{\Delta'} \cdot e_{\Delta''} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot e_{\Gamma_1\setminus\Delta''}) \models \chi_1$. This means that $M^1, w \models \langle\langle\Delta' \cup \Delta''\rangle\rangle \bigcirc \chi_1$.

So far we have established that $M^{1,2}, w \models \langle\langle\Delta' \cup \Delta''\rangle\rangle \bigcirc [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$ entails $M^1, w \models \langle\langle\Delta' \cup \Delta''\rangle\rangle \bigcirc \chi_1$. It remains to be shown that the converse holds too. Let $a_{\Delta'} \in Act^1_{\Delta'} = Act_{\Delta'}$ and $b_{\Delta''} \in Act^1_{\Delta''}$ be such that $M^1, o^1(w, a_{\Delta'} \cdot b_{\Delta''} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot d_{\Gamma_1\setminus\Delta''}) \models \chi_1$. Consider the vector of actions $e_{\Delta''\times\Gamma_2} \in Act^{1,2}_{\Delta''\times\Gamma_2}$ with its actions being $e_{j_1,j_2} = \langle b_{j_1}, 0\rangle \in\in Act^{1,2}_{\langle j_1,j_2\rangle}$, $j_1 \in \Delta''$, $j_2 \in \Gamma_2$. By the definition of $h^{1,2}$, for any $f_{(\Gamma_1\setminus\Delta'')\times\Gamma_2} \in Act^{1,2}_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}$, $h^{1,2}(e_{\Delta''\times\Gamma_2} \cdot f_{(\Gamma_1\setminus\Delta'')\times\Gamma_2})$ has the form $h^1(b_{\Delta''} \cdot g_{\Gamma_1\setminus\Delta''})$ where $g_{\Gamma_1\setminus\Delta''}$ is determined from $f_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}$. Hence, $o^{1,2}(w, a_{\Delta'} \cdot e_{\Delta''\times\Gamma_2} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot f_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}) = o(w, a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot h^{1,2}(e_{\Delta''\times\Gamma_2} \cdot f_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}))$ has the form $o^1(w, a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot b_{\Delta''} \cdot g_{\Gamma_1\setminus\Delta''})$, and, for such $a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot b_{\Delta''} \cdot g_{\Gamma_1\setminus\Delta''}$, we have $M^1, o^1(w, a_{\Delta'} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot b_{\Delta''} \cdot g_{\Gamma_1\setminus\Delta''}) \models \chi_1$. By the inductive hypothesis, this entails that $M^{1,2}, o^{1,2}(w, a_{\Delta'} \cdot e_{\Delta''\times\Gamma_2} \cdot c_{\Sigma\setminus(\{i\}\cup\Delta')} \cdot f_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}) \models [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$. This holds for all $c_{\Sigma\setminus(\{i\}\cup\Delta')}$ and all $f_{(\Gamma_1\setminus\Delta'')\times\Gamma_2}$. Hence $M^{1,2}, w \models \langle\langle\Delta' \cup \Delta''\rangle\rangle \bigcirc [\{j\} \times \Gamma_2/j : j \in \Gamma_1]\chi_1$.

The proof is similar for φ_1 being of the forms $\langle\langle\Delta' \cup \Delta''\rangle\rangle(\chi'_1 \mathsf{U} \chi''_1)$ and $[\![\Delta', \Delta'']\!](\chi'_1 \mathsf{U} \chi''_1)$, except that the equivalence between the strategic abilities of Δ'' in M^1 and $\Delta'' \times \Gamma_2$ in $M^{1,2}$ must be established about strategies that may take more than one step to carry out. $\dashv$