Logical Interpolation and Projection onto State in the Duration Calculus

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Abstract

We generalise an interval-related interpolation theorem about abstract-time Interval Temporal Logic (ITL, [Mos85, Dut95]), which was first obtained in [Gue01]. The generalisation is based on the abstract-time variant of a projection operator in the Duration Calculus (DC, [ZCR91, HC97, **?**]), which was introduced in [Hun99] and later studied extensively in [GH02]. We propose a way to understand interpolation in the context of formal verification. We give an example showing that, unlike abstract-time ITL, DC does not have the Craig interpolation property in general, and establish a special form of Craig interpolation for abstract-time DC. Explicit definability after Beth is known to be strongly related to Craig interpolation in general. We show a limitation of a different kind to the scope of Beth definability in ITL by a counterexample too. We call the generalisation of interval-related interpolation that we present *projection-related* interpolation. The DC-specific restrictions apply to it too. We show that both Craig and projection-related interpolation hold about the $\lceil P \rceil$-subset of DC without such restrictions. Our proofs of these theorems for the $\lceil P \rceil$-subset entail algorithms for the construction of the interpolants.

Keywords: duration calculus, projection, Craig interpolation.

1 Introduction

The classical interpolation theorem of Craig (cf. e.g. [CK73]) states that if $\varphi \Rightarrow \psi$ is a valid first-order predicate logic formula, then there exists a first-order formula θ built using only non-logical symbols occurring in both φ and ψ and, possibly, equality, such that the formulas $\varphi \Rightarrow \theta$ and $\theta \Rightarrow \psi$ are valid too. A formula θ with this property is called an *interpolant* between φ and ψ. Similar statements apply to numerous non-classical and modal logics. Many examples can be found in the literature. A book is forthcoming [GM03].

Abstract-time interval temporal logics admit interpolation theorems whose form is specific to their temporal semantics and differs from that of Craig's theorem. Some such theorems were proposed in [Gue01]. In these theorems the parts φ and ψ are written using a shared vocabulary of rigid symbols and disjoint copies of the same

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vocabulary of flexible symbols. Instead of having shared flexible symbols between φ and ψ, it is required that the pairs of corresponding flexible symbols from the two copies of the vocabulary used to write φ and ψ evaluate to the same constants, functions and relations *within some specified subinterval of time*. Given that, it is shown that interpolants between φ and ψ can be restricted to specify properties of the considered subinterval of time only. The interpretation proposed in Section 7 may give some intuition on this setting.

Let the languages built using the two copies of the flexible symbol vocabulary mentioned above be called $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively. Let, given a formula α from $\mathbf{L}_1$, the result of replacing its flexible symbols except the flexible constant ℓ (see the definition of ITL in Section 2) by their counterparts from $\mathbf{L}_2$ be denoted by α'. Let Φ be a finite set of formulas from $\mathbf{L}_1$. Let φ and ψ be in $\mathbf{L}_1$ too. Let c_0, c_1 and c_2 be rigid constants, which are shared by $\mathbf{L}_1$ and $\mathbf{L}_2$. Then in the case of ITL the interpolated formula has the form

$$\big(\ell = c_1; \underbrace{\bigwedge_{\chi\in\Phi} \Box\forall(\chi \Leftrightarrow \chi'); \ell = c_2}_{A}\big) \wedge \ell = c_0 \Rightarrow (\varphi \Rightarrow \psi'), \tag{1}$$

where $\forall$ denotes the universal closure of its argument formula. The antecedent of the implication (1) is to express that the formulas from Φ are equivalent to their counterparts from $\mathbf{L}_2$ within the interval defined using c_0, c_1 and c_2. In particular, Φ can be chosen to consist of atomic formulas so that the above antecedent would express the equality of the interpretations of the corresponding pairs of flexible symbols from $\mathbf{L}_1$ and $\mathbf{L}_2$ within this interval. According to a theorem from [Gue01], if (1) is valid in ITL, then there exists a formula θ in $\mathbf{L}_1$ such that the formulas

$$(\ell = c_0 \wedge c_1 + c_2 \leq c_0) \Rightarrow (\varphi \Rightarrow (\ell = c_1; \theta; \ell = c_2))$$

and

$$(\ell = c_1; \theta; \ell = c_2) \wedge \ell = c_0 \Rightarrow \psi$$

are valid too. Note that here the interpolant θ occurs in the same context as A from (1) which can be used to express the equality of interpretations of pairs of corresponding flexible symbols. This way θ is restricted to specify a property of the subinterval of the reference interval where this equality is supposed to hold. It can be additionally restricted to contain only non-logical symbols occurring in both φ and ψ. The theorem as in [Gue01] does not include this restriction, but it can be included using almost the same proofs. Thus the Craig and the interval-related conditions get combined.

Obviously, conjunctions of contexts of the kind $(\ell = c_1; [\,]; \ell = c_2) \wedge \ell = c_0$ can form more complex antecedents

$$\bigwedge_{j\in J} \big(\ell = c_{1,j}; \bigwedge_{\chi\in\Phi_j} \Box\forall(\chi \Leftrightarrow \chi'); \ell = c_{2,j}\big) \wedge \ell = c_0 \Rightarrow \ldots$$

in the interpolated formulas, and, following the pattern of the proofs from [Gue01], it can be shown that implications with such antecedents can be interpolated by appropriate vector interpolants $\langle\theta_j : j \in J\rangle$, with a component formula θ_j to occur in each of the contexts $(\ell = c_{1,j}; [\,]; \ell = c_{2,j}) \wedge \ell = c_0$.

In this paper we present a more flexible generalisation of interval-related interpolation. In this new form the subintervals where pairs of corresponding symbols from the two copies of the considered vocabulary are required to have identical interpretations are specified by a means of the interpretation of a distinguished flexible symbol, the positions of these intervals can vary according to the interpretation of this symbol and the interpolant is a single formula. Thus it is achieved that the involvement of the $2 \times |J|$ parameter constants $c_{1,j}, c_{2,j}$, $j \in J$, if needed for this purpose only, can be avoided.

The kind of flexible non-logical symbol that we use to specify the subintervals in question is a Duration Calculus (DC, [ZCR91]) *state variable*. State variables are the basic ingredient used to extend ITL to DC. They are interpreted as boolean functions on *time points*, unlike the other flexible non-logical ITL symbols, which depend on *intervals* for their interpretations. State variables are restricted to evaluate to functions which have the *finite variability property*. This means that, given a state variable, every bounded interval of time can be partitioned into finitely many subintervals within each of which the state variable evaluates to a constant. This makes state variables appropriate to specify finite sequences of subintervals. DC allows the formation of boolean combinations of state variables called *state expressions*.

In [Hun99], DC was extended by a *projection* operator which, given a formula φ and a state expression S, returns the truth value of φ at the interval obtained by gluing the parts of the reference interval where S holds, with the interpretations of the non-logical symbols occurring in φ transferred from that original reference interval. In this setting φ is called the *projected formula* and the interval at which it is evaluated is called the *projected interval*. This kind of projection in DC can be viewed as analogous to the operator Π which was introduced to discrete-time ITL in [HM83]. It greatly facilitates the specification of interleaving. An extensive study of projection onto state in DC can be found in [GH02]. Another group of projection operators for ITL and DC which take formulas instead of state expressions to define their projected intervals have been introduced and studied in [Mos86, Mos95, Jif99, BT03, Gue04]. We call the projection operators from [HM83] and [Hun99] projection *onto state* to distinguish them from those other ones. Formulas in the scope of a projection operator can obviously specify only properties of the parts of the reference interval which participate in the construction of the projected interval. That is why we use projection as a context to restrict interpolants in generalised interval-related interpolation.

In case the Craig interpolant for an implication $\varphi \Rightarrow \psi$ can be chosen depending only on φ, the respective logic is said to have the *uniform* interpolation property. Classical propositional logic has uniform interpolation, but first-order predicate logic does not. We prove both uniform Craig interpolation in its standard form and uniform projection-related interpolation for the $\lceil P \rceil$-subset of the extension DC^* of DC by *iteration*. We show that interpolants can be found constructively in this subset.

Structure of the paper. We first give brief formal definitions of abstract-time ITL and abstract-time DC and projection onto state in DC. We only define projection on the subset of DC where state variables are the only kind of flexible non-logical symbols. We establish a specialised form of Craig interpolation for DC by translating

DC into a suitable ω-theory in some appropriately extended ITL vocabulary. This form is weaker than the standard one for Craig interpolation. We show that Craig interpolation does not hold for DC in its stronger form.

We prove projection-related interpolation for DC by reducing it to this specialised form of Craig interpolation. Then we prove Craig interpolation and projection-related interpolation for the $\lceil P \rceil$-subset of DC^* without the weakening from the predicate case. Finally we give some comments on the connection between explicit definability after Beth and interval-related interpolation, propose a possible interpretation of the main results, point to related work on hybrid logics (see [HyL]), and make some concluding remarks.

2 Preliminaries

2.1 Interval temporal logic

Interval Temporal Logic (ITL) was first introduced for the case of discrete time in [HM83, Mos85]. It is a classical first-order modal logic with a fixed domain and one normal binary modality $(.;.)$, known as *chop*. Results in this paper apply to its abstract-time variant [Dut95]. The preliminaries below are about this system.

2.1.1 Languages

Given a first-order vocabulary of *constant symbols* c, d, ..., *function symbols* f, g, ..., *relation symbols* R, ..., and countably many *individual variables* x, y, ..., terms t and formulas φ in the corresponding ITL language are defined by the BNFs:

$t ::= c \mid x \mid f(t, \dots, t)$

$\varphi ::= \bot \mid R(t, \dots, t) \mid \varphi \Rightarrow \varphi \mid (\varphi;\varphi) \mid \exists x\varphi$

Non-logical symbols are either *rigid* or *flexible*, depending on the type of their interpretation, as it becomes clear below. Formulas built using only rigid symbols are called rigid too. Every ITL vocabulary contains the rigid constant 0, the flexible constant ℓ, the rigid binary function symbol $+$ and equality $=$.

2.1.2 Frames, models and satisfaction

A *time domain* is a linearly ordered set. Given a time domain $\langle T, \leq \rangle$, we denote the set $\{[\tau_1, \tau_2] : \tau_1, \tau_2 \in T, \tau_1 \leq \tau_2\}$ by $\mathbf{I}(T)$. Given $\sigma_1, \sigma_2 \in \mathbf{I}(T)$ such that $\max \sigma_1 = \min \sigma_2$, we denote $\sigma_1 \cup \sigma_2$ by $\sigma_1;\sigma_2$. A *duration domain* is a system of the type $\langle D, 0^{(0)}, +^{(2)} \rangle$ which satisfies the following axioms:

$(D1)$ $x + (y + z) = (x + y) + z$

$(D2)$ $x + 0 = x, \quad 0 + x = x$

$(D3)$ $x + y = x + z \Rightarrow y = z, \quad x + z = y + z \Rightarrow x = y$

$(D4)$ $x + y = 0 \Rightarrow x = 0$

$(D5)$ $\exists z(x + z = y \vee y + z = x), \quad \exists z(z + x = y \vee z + y = x)$

Given a time domain $\langle T, \leq \rangle$ and a duration domain $\langle D, 0, + \rangle$, a function $m : \mathbf{I}(T) \to D$ is called a *measure function*, if the following properties hold for all $\sigma, \sigma' \in \mathbf{I}(T)$:

$(M1)\quad \min\sigma = \min\sigma' \wedge m(\sigma) = m(\sigma') \Rightarrow \max\sigma = \max\sigma'$
$(M2)\quad m(\sigma;\sigma') = m(\sigma) + m(\sigma')$
$(M3)\quad m(\sigma) = x + y \Rightarrow \exists\tau \in \sigma\; m([\min\sigma,\tau]) = x$

An ITL *frame* is a tuple of the form $\langle\langle T,\leq\rangle,\langle D,0,+\rangle,m\rangle$ where $\langle T,\leq\rangle$ is a time domain, $\langle D,0,+\rangle$ is a duration domain and $m:\mathbf{I}(T)\to D$ is a measure function.

Given an ITL frame F with its components named as above, and an ITL language $\mathbf{L}$, a function I on the vocabulary of $\mathbf{L}$ is an *interpretation of* $\mathbf{L}$ *into* F, if it satisfies the following conditions:

- $I(c)\in D$, $I(f):D^n\to D$, and $I(R):D^n\to\{0,1\}$ for rigid constant symbols c, n-ary rigid function symbols f and relation symbols R
- $I(c):\mathbf{I}(T)\to D$, $I(f):\mathbf{I}(T)\times D^n\to D$, and $I(R):\mathbf{I}(T)\times D^n\to\{0,1\}$ for the corresponding kinds of flexible symbols;
- $I(x)\in D$ for individual variables x;
- $I(=)$ is $=$, $I(0)=0$, $I(+)=+$, and $I(\ell)=m$.

Given an ITL language $\mathbf{L}$, an ITL *model for* $\mathbf{L}$ is a pair of the form $\langle F,I\rangle$ where F is an ITL frame and I is an interpretation of $\mathbf{L}$ into F. Given an ITL language $\mathbf{L}$, a model $M=\langle\langle T,\leq\rangle,\langle D,0,+\rangle,m,I\rangle$ for it, and an interval $\sigma\in\mathbf{I}(T)$, the value $I_\sigma(t)$ of terms t in $\mathbf{L}$ is defined by induction on the construction of t as follows:

$I_\sigma(x)$	$=$	$I(x)$	for individual variables x
$I_\sigma(c)$	$=$	$I(c)$	for rigid constants c
$I_\sigma(c)$	$=$	$I(c)(\sigma)$	for flexible constants c
$I_\sigma(f(t_1,\ldots,t_n))$	$=$	$I(f)(I_\sigma(t_1),\ldots,I_\sigma(t_n))$	for rigid n-place f
$I_\sigma(f(t_1,\ldots,t_n))$	$=$	$I(f)(\sigma,I_\sigma(t_1),\ldots,I_\sigma(t_n))$	for flexible n-place f

Given an interpretation I of an ITL language $\mathbf{L}$ into a frame F, a symbol s from $\mathbf{L}$ and an object a of the type of s in F, we denote the interpretation which assigns a to s and is equal to I for all the other symbols from $\mathbf{L}$ by I_s^a. The relation $M,\sigma\models\varphi$ where $M=\langle F,I\rangle$ is an ITL model for some language $\mathbf{L}$ which contains φ and $\sigma\in\mathbf{I}(T)$ is defined by induction on the construction of φ as follows (we assume that the components of F are named as above):

$M,\sigma\not\models\bot$		
$M,\sigma\models R(t_1,\ldots,t_n)$	iff	$I(R)(I_\sigma(t_1),\ldots,I_\sigma(t_n))=1$, for rigid n-place R
$M,\sigma\models R(t_1,\ldots,t_n)$	iff	$I(R)(\sigma,I_\sigma(t_1),\ldots,I_\sigma(t_n))=1$, for flexible R
$M,\sigma\models\varphi\Rightarrow\psi$	iff	either $M,\sigma\models\psi$ or $M,\sigma\not\models\varphi$
$M,\sigma\models(\varphi;\psi)$	iff	$M,\sigma_1\models\varphi$ and $M,\sigma_2\models\psi$ for some $\sigma_1,\sigma_2\in\mathbf{I}(T)$ such that $\sigma_1;\sigma_2=\sigma$
$M,\sigma\models\exists x\varphi$	iff	$\langle F,I_x^a\rangle,\sigma\models\varphi$ for some $a\in D$

Given a frame F, we denote its components by $\langle T_F,\leq_F\rangle$, $\langle D_F,0_F,+_F\rangle$ and m_F, respectively.

2.1.3 Abbreviations

First-order logic abbreviations and infix notation are used in ITL in the ordinary way. These include $\top$, $\neg$, $\wedge$, $\vee$, $\Leftrightarrow$ and $\forall$. The following abbreviations, are specific to $(.;.)$:

$$\Diamond\varphi \rightleftharpoons (\top;\varphi;\top)\ , \qquad \Box\varphi \rightleftharpoons \neg\Diamond\neg\varphi\ ,$$

$$(\varphi_1;\varphi_2;\ldots;\varphi_n) \rightleftharpoons (\varphi_1;\ldots;(\varphi_{n-1};\varphi_n)\ldots)\ ,$$

$$\varphi^0 \rightleftharpoons \ell = 0\ , \qquad \varphi^{k+1} \rightleftharpoons (\varphi;\varphi^k)\ .$$

Note that we use the abbreviations $\Diamond$ and $\Box$ in the way that is common to the literature on DC, which is different from the convention adopted in the literature on discrete-time ITL. The relation symbol $\leq$ is usually defined in ITL by the clause

$$x \leq y \rightleftharpoons \exists z(x + z = y)\ .$$

We use one more abbreviation in this paper:

$$\{t\}^k \rightleftharpoons (t = 0 \vee t = \ell)^k\ .$$

It is meant to be used mostly with duration terms $\int S$. At intervals of non-zero length, the formula $\{\int S\}^k$ restricts S to change at most $k - 1$ times.

2.1.4 Proof system

The Hilbert-style proof system for ITL below was proved complete for abstract-time ITL in [Dut95].

$(A1)$	$(\varphi;\psi) \wedge \neg(\chi;\psi) \Rightarrow (\varphi \wedge \neg\chi;\psi)$, $\quad (\varphi;\psi) \wedge \neg(\varphi;\chi) \Rightarrow (\varphi;\psi \wedge \neg\chi)$
$(A2)$	$((\varphi;\psi);\chi) \Leftrightarrow (\varphi;(\psi;\chi))$
(B)	$(\exists x\varphi;\psi) \Rightarrow \exists x(\varphi;\psi)$, $\quad (\psi;\exists x\varphi) \Rightarrow \exists x(\psi;\varphi)$, if x is not free in ψ
(R)	$(\varphi;\psi) \Rightarrow \varphi$, $\quad (\psi;\varphi) \Rightarrow \varphi$, if φ is rigid
$(L1)$	$(\ell = x;\varphi) \Rightarrow \neg(\ell = x;\neg\varphi)$, $\quad (\varphi;\ell = x) \Rightarrow \neg(\neg\varphi;\ell = x)$
$(L2)$	$(\ell = x;\ell = y) \Leftrightarrow \ell = x + y$
$(L3)$	$\varphi \Rightarrow (\ell = 0;\varphi)$, $\quad \varphi \Rightarrow (\varphi;\ell = 0)$
(N)	$\dfrac{\varphi}{\neg(\neg\varphi;\psi)}$, $\qquad \dfrac{\varphi}{\neg(\psi;\neg\varphi)}$
$(Mono)$	$\dfrac{\varphi \Rightarrow \psi}{(\varphi;\chi) \Rightarrow (\psi;\chi)}$, $\qquad \dfrac{\varphi \Rightarrow \psi}{(\chi;\varphi) \Rightarrow (\chi;\psi)}$

The system also includes axioms and rules for first-order logic with equality and the axioms $D1$-$D5$ about duration domains. Substitution $[t/x]\varphi$ of individual variable x by term t in formula φ is allowed in axiom and rule instances only if either t is rigid or x does not occur in the scope of $(.;.)$ in φ.

Sequent systems for ITL cannot be cut-free, because the propositional subset of ITL is undecidable [Ras02].

2.2 Abstract-time duration calculus

DC was first introduced for the case of real time in [ZCR91]. A comprehensive survey of DC can be found in [HC97]. Abstract-time DC was introduced in [Gue98].

2.2.1 Languages

A DC vocabulary is an extension of an ITL vocabulary by a set of *state variables* P, Q, ... State variables are used to construct *state expressions* S, which are defined by the BNF:

$$S ::= \mathbf{0} \mid P \mid S \Rightarrow S$$

The syntax of DC formulas is the same as that for ITL formulas, except for the terms occurring in them. The syntax of DC terms t includes *duration terms* $\int S$, which are formed using state expressions:

$$t ::= c \mid x \mid \int S \mid f(t, \ldots, t)$$

We denote the set of state variables occurring in a DC state expression, term or formula E by $SV(E)$.

2.2.2 Frames, models and satisfaction

Abstract-time DC frames are the same as abstract-time ITL frames. Given a frame $\langle\langle T, \leq\rangle, \langle D, 0, +\rangle, m\rangle$ and a language $\mathbf{L}$, a DC interpretation I of the vocabulary of $\mathbf{L}$ into F is like an ITL interpretation on the ITL non-logical symbols in $\mathbf{L}$, and maps every state variable P to a function $I(P) : T \to \{0, 1\}$. $I(P)$ is required to have the following *finite variability property*:

> For every $\sigma \in \mathbf{I}(T)$ there exist an $n < \omega$ and $\sigma_1, \ldots, \sigma_n \in \mathbf{I}(T)$ such that $\sigma = \sigma_1; \ldots; \sigma_n$ and $I(P)$ is constant on $[\min \sigma_i, \max \sigma_i)$, $i = 1, \ldots, n$.

DC models are like ITL models, the only difference being that their second component is a DC interpretation and therefore assigns values to state variables too. The following equalities define the value $I_\tau(S)$ of a state expression S at time $\tau \in T$ under interpretation I:

$$\begin{array}{lcl} I_\tau(\mathbf{0}) & = & 0 \\ I_\tau(P) & = & I(P)(\tau) \text{ for state variables } P \\ I_\tau(S_1 \Rightarrow S_2) & = & \max\{1 - I_\tau(S_1), I_\tau(S_2)\} \end{array}$$

We use the following technical definition to extend I_σ to duration terms:

Definition 1 Let $h : T \to \{0, 1\}$ have the finite variability property and $\sigma \in \mathbf{I}(T)$. Given $\sigma_1, \ldots, \sigma_n \in \mathbf{I}(T)$ such that $\sigma = \sigma_1; \ldots; \sigma_n$ and h is constant on $[\min \sigma_i, \max \sigma_i)$, $i = 1, \ldots, n$, we put

$$\int\limits_{\min \sigma}^{\max \sigma} h(\tau) d\tau = \sum_{i=1,\ldots,n,\ I_{min\sigma_i}(S)=1} m(\sigma_i).$$

The value of $\int\limits_{\min \sigma}^{\max \sigma} h(\tau) d\tau$ defined above obviously does not depend on the particular choice of $\sigma_1, \ldots, \sigma_n$. Finite variability propagates from the interpretations of the state variables to the functions $\lambda\tau.I_\tau(S)$ where S is a state expression. We put

$$I_\sigma(\int S) = \int\limits_{\min \sigma}^{\max \sigma} I(S)(\tau) d\tau.$$

The clauses about I_σ on other kinds of terms and those about $\models$ are like in ITL.

2.2.3 Abbreviations

The connectives $\neg$, $\vee$, $\wedge$ and $\Leftrightarrow$ are used as abbreviations in state expressions in the usual way. The following abbreviations are also frequently used:

$$\mathbf{1} \rightleftharpoons \mathbf{0} \Rightarrow \mathbf{0}, \qquad \lceil S \rceil \rightleftharpoons \int S = \ell \wedge \ell \neq 0.$$

The flexible constant ℓ can be regarded as abbreviation for $\int \mathbf{1}$ in DC.

2.2.4 Proof system

Abstract-time DC is a conservative extension to ITL. An ω-complete proof system for it was first presented in [Gue98]. It was obtained by adding several axioms and an ω-rule to the proof system for ITL known from [Dut95]. The ω-rule can be regarded as an explicit formulation of the intended deductive power of the induction rules from the proof system for real-time DC from [HC92]. Various forms of the rule and the list of axioms can be chosen to obtain such an ω-complete system. In the rest of this paper we employ the following variant, which is convenient to prove our results:

$$(\omega') \qquad \frac{\forall k < \omega\ [\{\int S\}^k / R]\alpha}{[\top / R]\alpha}$$

$(DC0)$ $\quad \ell = 0 \Rightarrow \int S = 0$

$(DC1)$ $\quad \int \mathbf{0} = 0$

$(DC2')$ $\quad \int S_1 = 0 \vee \int S_2 = \ell \Rightarrow \int (S_1 \Rightarrow S_2) = \ell$

$(DC3')$ $\quad \int S_1 = \ell \wedge \int S_2 = 0 \Rightarrow \int (S_1 \Rightarrow S_2) = 0$

$(DC4')$ $\quad (\int S = x; \int S = y) \Rightarrow \int S = x + y$

In (ω') above, R denotes an arbitrary 0-place (flexible) relation symbol. The items marked by $'$ above are different from those in the system from [Gue98].

2.2.5 Real-time DC

Let $\mathbf{R}_+$ be the set of the non-negative reals and $F_{\mathbf{R}}$ be the real-time-based frame

$$\langle \langle \mathbf{R}, \leq_{\mathbf{R}} \rangle, \langle \mathbf{R}_+, 0_{\mathbf{R}}, +_{\mathbf{R}} \rangle, \lambda\sigma \in \mathbf{I}(\mathbf{R}). \max \sigma - \min \sigma \rangle.$$

Real-time DC, as it was originally introduced in [ZCR91], can be regarded as the DC theory of $F_{\mathbf{R}}$. The completeness of a finitary Hilbert-style proof system for real-time DC relative to the ITL theory of $F_{\mathbf{R}}$ was first demonstrated in [HC92].

3 Projection onto state in abstract-time DC

In this paper we present projection-related interpolation for DC vocabularies whose flexible symbols are ℓ and state variables only. The original definition of projection onto state in [Hun99] was restricted to state variables too and applied to the case of real time. The definition in [GH02] applies to real time and vocabularies with all kinds of non-logical symbols. To achieve it as a generalisation of the definition from [Hun99], the introspectivity of projection had to be sacrificed, in order to preserve compositionality. Yet introspectivity is inherent to the intended meaning of our generalisation of interval-related interpolation. To keep projection introspective, we have restored the restriction to state variables in this paper. The definition of projection onto state we adopt here is an abstract-time variant of that from [GH02], simplified considerably thanks to the restored restriction. It can be viewed as an abstract-time variant of the original definition from [Hun99] as well.

DC projection formulas have the form (φ/S) where φ is a formula and S is a state expression. We need some auxiliary notation in order to define $\models$ on the new kind of formulas. Let $\langle F, I\rangle$ be a model for the DC language **L** whose only flexible symbols are ℓ and state variables.

Definition 2 Let $\sigma \in \mathbf{I}(T_F)$, and $h : T_F \to \{0,1\}$ have the finite variability property. We define $\sigma^h \in \mathbf{I}(T_F)$ by the conditions

$$\min\sigma^h = \min\sigma \text{ and } m_F(\sigma^h) = \int\limits_{\min\sigma}^{\max\sigma} h(\tau)d\tau.$$

In words, σ^h is an initial subinterval of σ which has the duration of h in σ as its total duration.

We define the interpretation $I^{\sigma,h}$ on state variables P from **L** into F by the clauses

$I^{\sigma,h}(P)(\tau') = I(P)(\tau)$ for $\tau' \in [\min\sigma^h, \max\sigma^h)$,

if $m([\min\sigma, \tau']) = \int\limits_{\min\sigma}^{\tau} h(\tau'')d\tau''$ and $h(\tau) = 1$;

$I^{\sigma,h}(P)(\max\sigma^h) = I(P)(\max\sigma)$.

We put $I^{\sigma,h}(\ell) = m_F$ and $I^{\sigma,h}(s) = I(s)$ for all rigid symbols s.

We are not interested in defining $I^{\sigma,h}(P)$ outside σ^h, because this would be irrelevant to the definition of the relation $\models$ for projection formulas to follow. The interpretation $I^{\sigma,h}$ on σ^h is obtained by "gluing" the parts of I on the subintervals of σ where h evaluates to 1. An exception is made for the case of $\int\limits_{\min\sigma}^{\max\sigma} h(\tau)d\tau = 0$, which, however, has no effect on the values of duration terms at subintervals of σ^h under $I^{\sigma,h}$. The condition $h(\tau) = 1$ on τ in the last clause of the definition of $I^{\sigma,h}$ is in order to make the choice of τ unambiguous: Bounded maximal subintervals at which h evaluates to 0 have the form $[\tau_1, \tau_2)$. Given such a subinterval $[\tau_1, \tau_2)$, $\int\limits_{\min\sigma}^{\tau} h(\tau'')d\tau''$ is the same for all $\tau \in [\tau_1, \tau_2]$. However, the only $\tau \in [\tau_1, \tau_2]$ such that $h(\tau) = 1$ is τ_2 and the clause about the value of $I^{\sigma,h}(P)(\tau')$ for $\tau' \in [\min\sigma^h, \max\sigma^h)$ selects this time point.

Given a formula φ and a state expression H, we put

$$\langle F, I\rangle, \sigma \models (\varphi/H) \text{ iff } \langle F, I^{\sigma,\lambda\tau.I_\tau(H)}\rangle, \sigma^{\lambda\tau.I_\tau(H)} \models \varphi.$$

This means, that (φ/H) holds at reference interval σ iff φ holds at the interval σ^h obtained by clipping off the parts of σ which do not satisfy H and preserving the values of all state variables in the remaining parts, which satisfy H. Some applications of the operator can be found in [GH02].

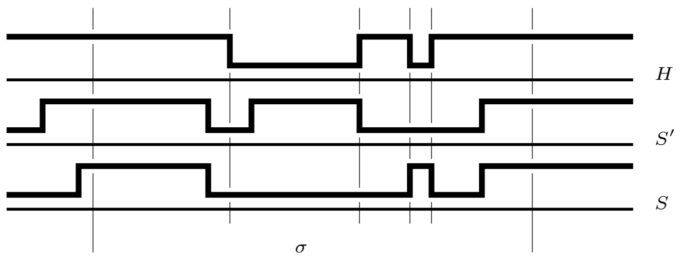

Figure 1: $\sigma \models (\Box\forall x((\int S = x) \Leftrightarrow (\int S' = x))/H)$, or, equivalently, $\sigma \models \ell = \int(H \Rightarrow (S \Leftrightarrow S'))$.

4 Interpolation in abstract-time DC

The main result in this paper is an interpolation theorem for formulas of the kind

$$\Big(\big(\bigwedge_{\chi\in\Phi} \Box\forall(\chi \Leftrightarrow \chi')\big)/H\Big) \Rightarrow (\varphi \Rightarrow \psi') \tag{2}$$

where $(.)'$ stands for the result of replacing the flexible symbols of the vocabulary of φ, ψ and the formulas from Φ, by corresponding ones from a disjoint vocabulary, like in the introduction. To illustrate (2), Figure 4 shows the simultaneous behaviour of the states S and S', which satisfy the instance of the antecedent of (2) for $\Phi = \{\int S = x\}$ at the reference interval σ.

Following the pattern from earlier work given in the introduction, it would be natural to expect that for Φ, H, φ and ψ such that (2) is a valid formula in DC there exists a formula θ of some accordingly restricted form such that

$$\varphi \Rightarrow \theta \ \text{ and } \ \theta \Rightarrow \psi$$

are valid too. For example, let:

$$\begin{array}{l} D \rightleftharpoons (\ell \leq 1; \lceil H\rceil \wedge \ell \geq 1; \top; \lceil H\rceil), \\ A \rightleftharpoons \Box(\int P \geq 1 \Rightarrow (\top; \lceil Q\rceil)) \wedge \Box\neg(\lceil Q\rceil; \lceil\neg Q\rceil), \\ B \rightleftharpoons (\lceil P'\rceil \wedge \ell = 2; \top) \Rightarrow (\top; \lceil Q'\rceil), \\ I \rightleftharpoons (\top; (\int P = 1/H); \top). \end{array}$$

Then

$$\bigwedge_{X\in\{P,Q\}} \Box\forall z(\int X = z \Leftrightarrow \int X' = z/H) \Rightarrow (D \Rightarrow (A \Rightarrow B)) \tag{3}$$

is a valid formula which is interpolated by $D \wedge I$, because the formulas

$$D \wedge A \Rightarrow D \wedge I \text{ and } D \wedge [P'/P]I \Rightarrow B$$

are valid too. Note that the occurrences of P and Q in both the equivalences in (3) and the interpolant $D \wedge I$ are in the scope of $(./H)$.

In this paper we prove an interpolation theorem for implications of the form (2) with the restriction on the interpolants θ to have all the variables $P \in SV(\theta) \setminus SV(H)$ occurring only in the scope of $(./H)$. We obtain the theorem as a corollary of Craig interpolation, which holds for DC in a weak form. Before deriving the theorem about valid implications of the form (2), we prove Craig interpolation for DC in this form. Before doing this, we show that Craig interpolation does not hold for DC in its standard and stronger form known from first-order predicate logic. The counterexample (informally) explains the choice of weakening we make, in order to obtain a sound Craig interpolation theorem. The interpolation theorem for the implications (2) inherits the weaking condition from the weak Craig interpolation theorem which we use to derive it. Despite that, it achieves the increase of flexibility described in the introduction.

Here follows a counterexample to Craig's interpolation property in DC. Let P be a state variable. Consider a DC interpretation I into the real-time-based frame $F_{\mathbf{R}}$ which satisfies $I(1) = 1_{\mathbf{R}}$. Let A denote the conjunction:

$(\ell = 1 \wedge \lceil P\rceil; \top)\wedge$	the reference interval σ has an initial subinterval of length 1 where P holds and
$\Box\neg(\ell = 1 \wedge \lceil P\rceil; \ell = 1 \wedge \neg\lceil\neg P\rceil)\wedge$	if a $\lceil P\rceil$-subinterval of σ of length 1 ends at least 1 time unit earlier than σ itself, then it is followed by a $\lceil\neg P\rceil$-subinterval of σ of length 1 and
$\Box\neg(\ell = 1 \wedge \lceil\neg P\rceil; \ell = 1 \wedge \neg\lceil P\rceil)$	if a $\lceil\neg P\rceil$-subinterval of σ of length 1 ends at least 1 time unit earlier than σ itself, then it is followed by a $\lceil P\rceil$-subinterval of σ of length 1.

Clearly $\langle F_{\mathbf{R}}, I\rangle, \sigma \models A \wedge (\top; \ell = 1 \wedge \lceil\neg P\rceil)$ only if the duration of σ is an even positive integer, that is, $\max\sigma - \min\sigma \in \{2k : k \in \mathbf{N}\}$. Let Q be a state variable that is distinct from P. A direct check shows that:

$$\models_{DC} A \wedge (\top; \ell = 1 \wedge \lceil\neg P\rceil) \Rightarrow ([Q/P]A \Rightarrow (\top; \ell = 1 \wedge \lceil\neg Q\rceil)) \quad (4)$$

To interpolate this valid formula, we can use no other flexible symbol but ℓ. However, no state variable-free DC formula θ satisfies both

$$\models_{DC} A \wedge (\top; \ell = 1 \wedge \lceil\neg P\rceil) \Rightarrow \theta \text{ and } \models_{DC} \theta \wedge A \Rightarrow (\top; \ell = 1 \wedge \lceil\neg P\rceil), \quad (5)$$

nor variants of these written with, e.g., Q instead of P. This is so, because the satisfaction of a DC formula θ built using ℓ only at $\langle F_{\mathbf{R}}, I\rangle, \sigma$ depends only on the duration $\max\sigma - \min\sigma$ of σ, and for every such θ the set

$$\ell(\theta) = \{\max\sigma - \min\sigma : \langle F_{\mathbf{R}}, I\rangle, \sigma \models \theta\}$$

can be represented as the union of finitely many intervals in $\mathbf{R}_+$. Hence obviously $\{2k : k \in \mathbf{N}\}$ cannot be $\ell(\theta)$ for a θ of this form. The possibility to represent $\ell(\theta)$ this way is entailed by the following simple proposition:

Proposition 3 *Let θ have the syntax*

$$t ::= 0 \mid 1 \mid \ell \mid t + t$$

$$\theta ::= \bot \mid t \leq t \mid \theta \Rightarrow \theta \mid (\theta;\theta) \mid \exists x\theta$$

Then θ has a quantifier-free and $(.;.)$-free equivalent over $\langle F_{\mathbf{R}}, I\rangle$.

Proof: Every quantifier-free and $(.;.)$-free formula built using the individual variables from some fixed finite set X and ℓ is equivalent to one of the form

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n_i} \sum_{x\in X\cup\{\ell\}} a_{i,j,x}x \prec \sum_{x\in X\cup\{\ell\}} b_{i,j,x}x \tag{6}$$

where $\prec \in \{\leq, <\}$, $a_{i,j,x}$ and $b_{i,j,x}$ are natural numbers, and ax stands for $\underbrace{x + \cdots + x}_{a \text{ times}}$.

If a formula θ is of the form (6), then finding an equivalent to $\exists x\theta$ in the same form for a $x \in X$ amounts to expressing a condition for the existence of a solution x to the system of linear inequalities (6) with the variables from $x \setminus X$ and ℓ as parameters by another system of linear inequalities on these parameters. To eliminate $(.;.)$ we use that

$$\models_{DC} (\theta_1;\theta_2) \Leftrightarrow \exists x\exists y(\ell = x + y \wedge 0 \leq x \wedge 0 \leq y \wedge [x/\ell]\theta_1 \wedge [y/\ell]\theta_2)\ ,$$

provided that $x, y \notin FV(\theta_1), FV(\theta_2)$ and θ_1 and θ_2 are $(.;.)$ - and $\exists$-free themselves. $\dashv$

Next in this section comes an appropriately restricted form of the Craig interpolation theorem for DC with abstract time. Its proof follows the pattern of the proof of Craig's theorem about first-order classical predicate logic in [CK73] and the proof of interval-related interpolation theorems in [Gue01]. We first translate abstract-time DC into a suitable ω-theory in ITL. The translation is similar to the one involved in the proof of relative completeness of the finitary proof system for real-time DC first presented in [HC92]. The ITL ω-theory we translate into needs only finitely many instances of its non-logical axioms and rule to axiomatise. After presenting the translation we show how the expressibility of projection in the chosen subset of DC enables a projection-related interpolation theorem to be proved as a corollary to Craig interpolation.

4.1 Abstract-time DC as an ω-theory in ITL

Duration terms are essentially a system of flexible constants with constraints on their values that follow from their being representations of the durations of state expressions. The translation of DC into an ITL theory utilises the possibility to do with finitely many such constants for all the duration terms that can be built using a finite set of state variables. The constraints on the constants can be formulated as axioms and rules of a corresponding ITL theory that can be obtained as the translations of the axioms $DC0$, $DC1$, $DC2'$, $DC3'$ and $DC3'$, and the rule ω' about duration terms from the proof system for DC given in Section 2.2. We denote this translation by t. Here follows its definition:

Definition 4 Given a state expression S in a DC language $\mathbf{L}$, we denote the class of the state expressions S' in $\mathbf{L}$ which are propositionally equivalent to S by $[S]$. Let $\ell^{[S]}$

be a fresh flexible constant for every such set of state expressions $[S]$ in $\mathbf{L}$. We denote the ITL language built using the flexible constants $\ell^{[S]}$, and all the non-logical symbols of $\mathbf{L}$, except the state variables, by $\mathbf{L}^{ITL}$.

Given a term t from $\mathbf{L}$, $\mathtt{t}(t)$ is the result of replacing every occurrence of a duration term $\int S$ in t by the corresponding flexible constant $\ell^{[S]}$. Similarly, given a formula φ from $\mathbf{L}$, $\mathtt{t}(\varphi)$ is obtained by replacing the occurrences of duration terms in φ by their corresponding flexible constants.

The translation $\mathtt{t}$ from $\mathbf{L}$ into $\mathbf{L}^{ITL}$ defined above is obviously invertible up to the propositional equivalence of state expressions. Now, given a DC language $\mathbf{L}$, let the theory $ITL_{\mathbf{L}}$ in $\mathbf{L}^{ITL}$ have the $\mathtt{t}$-translations of the axioms $DC0$, $DC1$, $DC2'$, $DC3'$ and $DC4'$ as its axioms:

$(DC0_{ITL})$ $\ell = 0 \Rightarrow \ell^{[S]} = 0$

$(DC1_{ITL})$ $\ell^{[\mathbf{0}]} = 0$

$(DC2'_{ITL})$ $\ell^{[S_1]} = 0 \vee \ell^{[S_2]} = \ell \Rightarrow \ell^{[S_1 \Rightarrow S_2]} = \ell$

$(DC3'_{ITL})$ $\ell^{[S_1]} = \ell \wedge \ell^{[S_2]} = 0 \Rightarrow \ell^{[S_1 \Rightarrow S_2]} = 0$

$(DC4'_{ITL})$ $(\ell^{[S]} = x; \ell^{[S]} = y) \Rightarrow \ell^{[S]} = x + y$

where S, S_1 and S_2 range over the set of state expressions in $\mathbf{L}$. Let $ITL_{\mathbf{L}}$ be also closed under the $\mathtt{t}$-translation of the rule ω' for every state expression S:

$$(\omega'_{ITL}) \quad \frac{\forall k < \omega \; [\{\ell^{[S]}\}^k / R]\alpha}{[\top / R]\alpha}$$

In the sequel we use the following property of $\mathtt{t}$ and $ITL_{\mathbf{L}}$:

Proposition 5 *A set of formulas Γ in $\mathbf{L}$ is satisfiable iff $ITL_{\mathbf{L}} \cup \{\mathtt{t}(\varphi) : \varphi \in \Gamma\}$ is satisfiable.*

Remark: A language $\mathbf{L}$ with only finitely many state variables contains finitely many different state expressions modulo propositional equivalence. Hence the corresponding language $\mathbf{L}^{ITL}$ contains finitely many flexible constants of the form $\ell^{[S]}$ for such $\mathbf{L}$. That is why $ITL_{\mathbf{L}}$ has finitely many instances of its non-logical axioms $DC0_{ITL}$, $DC1_{ITL}$, $DC2'_{ITL}$, $DC3'_{ITL}$ and $DC4'_{ITL}$ and rule ω'_{ITL}.

4.2 Craig interpolation for abstract-time DC

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be DC languages. Let $\mathbf{L}_0$ and $\mathbf{L}_3$ be the DC languages based on the intersection and on the union of the vocabularies of $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively. To obtain Craig interpolation for the abstract-time DC with the interpolated formulas being implications from formulas in $\mathbf{L}_1$ to formulas in $\mathbf{L}_2$ and the interpolant being a formula from $\mathbf{L}_0$, we first formulate and prove a specialised Craig interpolation property about the ω-theories $ITL_{\mathbf{L}_i}$, $i = 0, 1, 2, 3$. In the theorem below we consider interpolation of implications with their antecedent being a formula in $\mathbf{L}_1^{ITL}$ and their succedent being a formula in $\mathbf{L}_2^{ITL}$ by formulas from $\mathbf{L}_0^{ITL}$. We may assume that $\mathbf{L}_1$ and $\mathbf{L}_2$ have finitely many state variables each. Then the sets $L_i = \{\ell^{[S]} : S \text{ is a state expression in } \mathbf{L}_i\}$, are finite by Remark 4.1, and, consequently, the languages $\mathbf{L}_1$ and $\mathbf{L}_2$ have countably many formulas each.

Theorem 6 *Let φ and ψ be in $\mathbf{L}_1^{ITL}$ and $\mathbf{L}_2^{ITL}$, respectively. Let $ITL_{\mathbf{L}_3} \models_{ITL} \varphi \Rightarrow \psi$. Let $k_v < \omega$, for every $v \in L_1 \cup L_2$. Then there exists a formula θ in $\mathbf{L}_0^{ITL}$ such that*

$$ITL_{\mathbf{L}_1} \vdash_{ITL} \varphi \wedge \bigwedge_{v \in L_1} \{v\}^{k_v} \Rightarrow \theta \text{ and } ITL_{\mathbf{L}_2} \vdash_{ITL} \theta \wedge \bigwedge_{v \in L_2} \{v\}^{k_v} \Rightarrow \psi.$$

Lemma 7 *Let $i \in \{1,2\}$, Γ be a set of formulas from $\mathbf{L}_i^{ITL}$. Let $k_v < \omega$, for every $v \in L_i$ and $\{\{v\}^{k_v} : v \in L_i\} \subset \Gamma$. Then $\Gamma \cup ITL_{\mathbf{L}_i}$ is consistent iff*

$$\Delta = \Gamma \cup \{\Box DC0_{ITL}, \Box DC1_{ITL}, \Box DC2'_{ITL}, \Box DC3'_{ITL}, \Box DC4'_{ITL}\}$$

is consistent in ITL.

Proof: The formulas $\Box DC2'_{ITL}$, $\Box DC3'_{ITL}$, $\Box DC4'_{ITL}$ are theorems of $ITL_{\mathbf{L}_i}$. Hence the consistency of $\Gamma \cup ITL_{\mathbf{L}_i}$ entails the consistency of Δ. For the opposite direction, assume that $\Gamma \cup ITL_{\mathbf{L}_i}$ is inconsistent for the sake of contradiction. Then there exists a proof of $\bot$ in $ITL_{\mathbf{L}_i}$ from Γ. We can eliminate the applications of ω'_{ITL} in such a proof. Since

$$\{v\}^{k_v} \Rightarrow ([\{v\}^{k_v}/R]\alpha \Rightarrow [\top/R]\alpha)$$

is an ITL theorem, every application of ω'_{ITL} can be replaced by two applications of MP using $\{v\}^{k_v}$ and the k_v-th premiss $[\{v\}^{k_v}/R]\alpha$ of this application of ω'_{ITL}. This way we obtain an ITL proof of $\bot$ from Δ, which is a contradiction. $\dashv$

Let C be a countable set of rigid constants, none of which occurs in $\mathbf{L}_3$. Let $\mathbf{L}_i^{ITL}(C)$ be the extension of $\mathbf{L}_i^{ITL}$ by the constants from C, $i = 0, 1, 2, 3$.

Definition 8 Let Γ and Γ' be sets of formulas from $\mathbf{L}_1^{ITL}(C)$ and $\mathbf{L}_2^{ITL}(C)$, respectively. A formula θ from $\mathbf{L}_0^{ITL}(C)$ *separates* Γ and Γ', if the sets

$$\Gamma \cup \{\neg\theta\} \cup ITL_{\mathbf{L}_1(C)} \text{ and } \Gamma' \cup \{\theta\} \cup ITL_{\mathbf{L}_2(C)}$$

are both inconsistent. Γ and Γ' are *inseparable*, if no formula θ from $\mathbf{L}_0^{ITL}(C)$ separates them.

Lemma 9 *If Γ and Γ' are inseparable sets of formulas from $\mathbf{L}_1^{ITL}$ and $\mathbf{L}_2^{ITL}$, respectively, $\{\{v\}^{k_v} : v \in L_1\} \subset \Gamma$ and $\{\{v\}^{k_v} : v \in L_2\} \subset \Gamma'$, then $\Gamma \cup \Gamma' \cup ITL_{\mathbf{L}_3(C)}$ is a satisfiable set of formulas from $\mathbf{L}_3^{ITL}(C)$.*

Proof: Let $\langle \varphi_i : i < \omega \rangle$ and $\langle \psi_i : i < \omega \rangle$ be fixed enumerations of the formulas from $\mathbf{L}_1^{ITL}(C)$ and $\mathbf{L}_2^{ITL}(C)$, respectively. We construct two ascending sequences Γ_i and Γ'_i, $i < \omega$, of consistent sets of formulas from $\mathbf{L}_1^{ITL}(C)$ and $\mathbf{L}_2^{ITL}(C)$, respectively, so that Γ_i and Γ'_i are inseparable and contain formulas with only finitely many constants from C occurring in them for every $i < \omega$. Let $\Gamma_0 = \Gamma$ and $\Gamma'_0 = \Gamma'$. Given Γ_k and Γ'_k for some k, we define Γ_{k+1} by considering the cases:

1. $\Gamma_k \cup \{\varphi_k\}$ and Γ'_k are inseparable.

1a. φ_k is $\exists x\alpha$ for some α from $\mathbf{L}(C)$. We choose a $c \in C$ which does not occur in $\Gamma_k \cup \Gamma'_k$. Then $\Gamma_k \cup \{\varphi_k, [c/x]\alpha\}$ and Γ'_k are inseparable too. We put $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k, [c/x]\alpha\}$.

1b. φ_k is not existential. Then $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k\}$.

2. $\Gamma_k \cup \{\varphi_k\}$ and Γ'_k are not inseparable. Then $\Gamma_{k+1} = \Gamma_k$.

Γ'_{k+1} is defined symmetrically, using Γ_{k+1} and Γ'_k. Let

$$\Gamma_\omega = \bigcup_{k<\omega} \Gamma_k \text{ and } \Gamma'_\omega = \bigcup_{k<\omega} \Gamma'_k.$$

Γ_ω and Γ'_ω are inseparable maximal consistent sets in $\mathbf{L}_1^{ITL}(C)$ and $\mathbf{L}_2^{ITL}(C)$, respectively. This follows by a straightforward argument relying on Lemma 7 for the consistency with the ω-rule ω'_{ITL}.

Besides, both Γ_ω and Γ'_ω have witnesses in C. That is, for every individual variable x and formula φ from $\mathbf{L}(C)$ ($\mathbf{L}'(C)$) such that $\exists x\varphi \in \Gamma_\omega$ ($\exists x\varphi \in \Gamma'_\omega$) there exists a $c \in C$ such that $[c/x]\varphi \in \Gamma_\omega$ ($[c/x]\varphi \in \Gamma'_\omega$).

Since Γ_ω and Γ'_ω are inseparable, they contain the same formulas from $\mathbf{L}_0^{ITL}(C)$. This means that $\Gamma_\omega \cup \Gamma'_\omega$ can be used to consistently define a canonical model for $\mathbf{L}_1^{ITL}(C) \cup \mathbf{L}_2^{ITL}(C)$. See [Dut95] or [Gue98] for details on this construction. This model will obviously satisfy $ITL_{\mathbf{L}_1(C)}$ and $ITL_{\mathbf{L}_2(C)}$, because they are subsets of Γ_ω and Γ'_ω, respectively, and can be straightforwardly enriched to satisfy the remaining theorems of $ITL_{\mathbf{L}_3(C)}$ by defining appropriate interpretations of $\ell^{[S]}$ for state expressions S built using state variables from both $\mathbf{L}_1$ and $\mathbf{L}_2$. This entails that $\Gamma \cup \Gamma' \cup ITL_{\mathbf{L}_3(C)}$ is satisfiable. $\dashv$

Proof:[of Theorem 6)] Now let

$$\Gamma = \{\varphi\} \cup \{\{v\}^{k_v} : v \in L_1\} \text{ and } \Gamma' = \{\neg\psi\} \cup \{\{v\}^{k_v} : v \in L_2\}.$$

For the sake of contradiction, assume that no formula θ_0 with the properties stated in the theorem for θ exists in $\mathbf{L}_0^{ITL}(C)$. Then Γ and Γ' are inseparable. Hence $\{\varphi, \neg\psi\} \cup ITL_{\mathbf{L}_3(C)}$ is satisfiable by Lemma 9, which contradicts $ITL_{\mathbf{L}_3} \models_{ITL} \varphi \Rightarrow \psi$. This contradiction means that there exists a formula θ_0 in $\mathbf{L}_0^{ITL}(C)$. Let $c_1, \ldots, c_k$ be the constants from C which occur in θ_0. Let $x_1, \ldots, x_k$ be some individual variables which do not occur in θ_0. Then we can choose θ to be $\forall x_1 \ldots \forall x_k[x_1/c_1, \ldots, x_k/c_k]\theta_0$, which is a formula in $\mathbf{L}_0^{ITL}$.
$\dashv$

Corollary 10 (Craig Interpolation for Abstract Time DC) *Let φ and ψ be in $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively. Let $\models_{DC} \varphi \Rightarrow \psi$. Let $k_P < \omega$ for every state variable P in $\mathbf{L}_3$. Then there exists a formula θ in $\mathbf{L}_0$ such that*

$$\models_{DC} \bigwedge_{P \in SV(\varphi)} \{\textstyle\int P\}^{k_P} \wedge \varphi \Rightarrow \theta \text{ and } \models_{DC} \bigwedge_{P \in SV(\psi)} \{\textstyle\int P\}^{k_P} \wedge \theta \Rightarrow \psi. \tag{7}$$

Proof: Let S be a state expression in $\mathbf{L}_3$. If an interval σ can be split into k_P subintervals so that P is constant in each of them for every $P \in SV(S)$, then there are at most $\sum\limits_{P \in SV(S)} (k_P - 1)$ internal points of σ at which S changes its value. Hence σ can be split into at most $K = \sum\limits_{P \in SV(S)} (k_P - 1) + 1$ subintervals such that S is constant in each of them. Then

$$\models_{DC} \bigwedge_{P \in SV(S)} \{\textstyle\int P\}^{k_P} \Rightarrow \{\textstyle\int S\}^K \tag{8}$$

and the corollary follows from Theorem 6 about $\mathtt{t}(\varphi)$, $\mathtt{t}(\psi)$ and $k_{\ell[S]} = K$, and Proposition 5. $\dashv$

4.3 Projection-related interpolation in abstract-time DC

We reduce projection-related interpolation to Craig interpolation. The reduction relies on the expressibility of projection in DC formulas which are built using only state variables, ℓ and rigid non-logical symbols which was first established in [Hun99]. Every such DC formula can be transformed into an equivalent projection-free formula in the same language using the equivalences:

$$\models_{DC} ([\ell/x_0, \int S_1/x_1, \ldots, \int S_n/x_n]\varphi/H) \Leftrightarrow \tag{9}$$

$$[\int H/x_0, \int(S_1 \wedge H)/x_1, \ldots, \int(S_n \wedge H)/x_n]\varphi \text{ for rigid } \varphi; \tag{10}$$

$$\models_{DC} (\varphi \vee \psi/H) \Leftrightarrow (\varphi/H) \vee (\psi/H); \tag{11}$$

$$\models_{DC} ((\varphi;\psi)/H) \Leftrightarrow ((\varphi/H);(\psi/H)); \tag{12}$$

$$\models_{DC} (\neg\varphi/H) \Leftrightarrow \neg(\varphi/H); \tag{13}$$

$$\models_{DC} (\exists x\varphi/H) \Leftrightarrow \exists x(\varphi/H). \tag{14}$$

Every projection-free formula χ can be represented in the form

$$[\ell/x_0, \int S_1/x_1, \ldots, \int S_n/x_n]\alpha,$$

where α contains only rigid non-logical symbols. A simple induction on the construction of α using the equivalences (10)-(14) shows that

$$\models_{DC} ([\ell/x_0, \int S_1/x_1, \ldots, \int S_n/x_n]\alpha/H) \Leftrightarrow \tag{15}$$

$$[\int H/x_0, \int(S_1 \wedge H)/x_1, \ldots, \int(S_n \wedge H)/x_n]\alpha. \tag{16}$$

This means that the elimination of $(./H)$ from (χ/H) for projection-free χ can be obtained by the replacement (10) of duration terms known to work for rigid formulas. The variables $x_0, \ldots, x_n$ are just place holders in (16). The restriction not to substitute rigid variables by flexible terms is irrelevant here.

Now let $\mathbf{L}$ be the language of the formulas φ and ψ, the state expression H and the formulas χ from Φ in (2). Let L be a finite set of state variables in the vocabulary of $\mathbf{L}$. Let $SV(H) \cap L = \emptyset$ and $SV(\chi) \subseteq L$ for $\chi \in \Phi$. Let P', $P \in L$ be some fresh state variables and the formula ψ' from (2) be $[P'/P : P \in L]\psi$. For the sake of generality, we allow ψ' and φ to have state variables in common out of L. Let (2), which we intend to interpolate, be valid.

The formulas from Φ in (2) can be assumed projection-free, because of the expressibility of projection. Hence we can assume that (2) is of the form

$$\left(\bigwedge_{\alpha\in A} \Box\forall\left(\begin{array}{l}[\ell/x_0, \int S_1/x_1, \ldots, \int S_n/x_n]\alpha \Leftrightarrow \\ {[\ell/x_0, \int S'_1/x_1, \ldots, \int S'_n/x_n]\alpha}\end{array}\right)/H\right) \Rightarrow (\varphi \Rightarrow \psi'),$$

where the formulas $\alpha \in A$ contain no flexible non-logical symbols and $S_1, \ldots, S_n$ are all the state expressions occurring in the formulas from Φ. Now (16) implies that this is equivalent to

$$\bigwedge_{\alpha \in A} \Box\forall\left(\begin{array}{l}[\int H/x_0, \int(S_1 \wedge H)/x_1, \ldots, \int(S_n \wedge H)/x_n]\alpha \Leftrightarrow \\ [\int H/x_0, \int(S_1' \wedge H)/x_1, \ldots, \int(S_n' \wedge H)/x_n]\alpha\end{array}\right) \Rightarrow (\varphi \Rightarrow \psi'),$$

Since

$$\models_{DC} \bigwedge_{P \in SV(S_i)} \Box(\int P \wedge H = \int P' \wedge H) \Rightarrow \Box(\int S_i \wedge H = \int S_i' \wedge H).$$

$i = 1, \ldots, n$, and

$$\begin{array}{l}\models_{DC} \bigwedge_{i=1}^{n} \Box(\int S_i \wedge H = \int S_i' \wedge H) \Rightarrow \\ \qquad\qquad \left(\begin{array}{l}[\int H/x_0, \int(S_1 \wedge H)/x_1, \ldots, \int(S_n \wedge H)/x_n]\alpha \Leftrightarrow \\ [\int H/x_0, \int(S_1' \wedge H)/x_1, \ldots, \int(S_n' \wedge H)/x_n]\alpha\end{array}\right),\end{array}$$

the validity of (2) implies

$$\models_{DC} \bigwedge_{P \in L} \Box(\int P \wedge H = \int P' \wedge H) \Rightarrow (\varphi \Rightarrow \psi').$$

Now let Q_P, $P \in L$, be fresh state variables. Let us introduce the substitution

$$s_0 \rightleftharpoons [(H \wedge Q_P) \vee (\neg H \wedge P)/P, (H \wedge Q_P) \vee (\neg H \wedge P')/P' : P \in L]. \tag{17}$$

A direct check shows that $\models_{DC} s_0(\int P \wedge H = \int P' \wedge H)$. Hence

$$\models_{DC} s_0\varphi \Rightarrow s_0\psi'.$$

Now we can apply Craig interpolation (Corollary 10) to this formula. The only state variables that the formulas $s_0\varphi$ and $s_0\psi'$ have in common are the newly introduced Q_P, $P \in L$, the state variables from $SV(H)$, and the state variables from $(SV(\varphi) \cap SV(\psi)) \setminus L$. Let M be $SV(H) \cup (SV(\varphi) \cap SV(\psi)) \setminus L$. Let

$$L_1 = \{P, Q_P : P \in L\} \cup M \text{ and } L_2 = \{P', Q_P : P \in L\} \cup M.$$

Let $k_Q < \omega$ for every state variable $Q \in L_1 \cup L_2$. Then Corollary 10 entails that there exists a formula θ_0 built using the state variables Q_P, $P \in L$, the state variables from M, and possibly some rigid symbols such that

$$\models_{DC} \bigwedge_{Q \in L_1} \left\{\int Q\right\}^{k_Q} \wedge s_0\varphi \Rightarrow \theta_0 \text{ and } \models_{DC} \bigwedge_{Q \in L_2} \left\{\int Q\right\}^{k_Q} \wedge \theta_0 \Rightarrow s_0\psi'. \tag{18}$$

Let us introduce the substitution

$$s \rightleftharpoons [(H \wedge P)/Q_P : P \in L]. \tag{19}$$

Lemma 11 *Every state expression in $s\theta_0$ is propositionally equivalent to one of the form*

$$H \wedge U \vee \neg H \wedge V,$$

where $SV(V) \subseteq M$.

Proof: Let S be a state expression occurring in θ_0. We may assume that S is in disjunctive normal form and every elementary conjunction in it contains all the variables Q_P, $P \in L$. Let S^+ be the disjunction of all the elementary conjunctions in S which contain at least one positive occurrence of a variable from among Q_P, $P \in L$. Let S^+ be **0**, if there are no such elementary conjunctions. The rest of the elementary conjunctions in S have the form $V \wedge \bigwedge\limits_{P \in L} \neg Q_P$, where $SV(V) \subseteq M$. Hence their disjunction has a propositional equivalent in this form too. This means that S is equivalent to the state expression $S^+ \vee V \wedge \bigwedge\limits_{P \in L} \neg Q_P$ for some appropriate such V.

Consider $s(S^+ \vee V \wedge \bigwedge\limits_{P \in L} \neg Q_P)$. Since s replaces at least one positive occurrence of a variable from among Q_P, $P \in L$, by a corresponding conjunction $H \wedge P$ in each disjunctive member of S^+, sS^+ is equivalent to a state expression of the form $H \wedge Z$. Furthermore sV is V, because $SV(V)$ is disjoint with the domain of s, and $s \bigwedge\limits_{P \in L} \neg Q_P$ is equivalent to $\neg H \vee H \wedge \bigwedge\limits_{P \in L} \neg P$. Hence sS is equivalent to a state expression of the form $H \wedge \big(Z \vee V \wedge \bigwedge\limits_{P \in L} \neg P\big) \vee \neg H \wedge V$, which matches that stated in the lemma. $\dashv$

Lemma 12 *The formula $s\theta_0$ is equivalent to a formula θ with the syntax*

$$\theta ::= \bot \mid R(t, \dots, t) \mid (\alpha/H) \mid \theta \Rightarrow \theta \mid (\theta;\theta) \mid \exists x\theta \tag{20}$$

where α stands for arbitrary formulas, R stands rigid relation symbols, and the terms t satisfy $SV(t) \subseteq M$.

Proof: Let $\int S$ occur in $s\theta_0$. Then S has the form sT, where $\int T$ occurs in θ. Lemma 11 implies that S has an equivalent of the form $H \wedge U \vee \neg H \wedge V$, where $SV(V) \subseteq M$. Obviously $\models_{DC} \int S = \int(H \wedge U) + \int(\neg H \wedge V)$. Hence $s\theta_0$ has an equivalent where every duration term t is either of the form $\int H \wedge U$ or satisfies $SV(t) \subseteq M$. Using the predicate logic equivalences

$$R(t_1, \dots, t_n) \Leftrightarrow \exists x_1 \dots \exists x_n \big(R(x_1, \dots, x_n) \wedge \bigwedge_{i=1}^{n} x_i = t_i\big) \tag{21}$$

and

$$y = f(t_1, \dots, t_n) \Leftrightarrow \exists x_1 \dots \exists x_n \big(y = f(x_1, \dots, x_n) \wedge \bigwedge_{i=0}^{n} x_i = t_i\big), \tag{22}$$

where $x_1, \dots, x_n \notin FV(t_1), \dots, FV(t_n)$, an equivalent to $s\theta_0$ can be obtained where every duration terms occur in atomic formulas of the form $x = \int S$ only. Using that $\models_{DC} x = \int(H \wedge U) \Leftrightarrow (x = \int U/H)$, an equivalent to $s\theta_0$ can be obtained where only state variables from M can appear out of the scope of a projection onto H. This entails the lemma. $\dashv$

Now let us apply the substitution s to the implications (18). A direct check shows that $ss_0\varphi$ is equivalent to φ. Hence

$$\models_{DC} \bigwedge_{Q \in M} \big\{\textstyle\int Q\big\}^{k_Q} \wedge \bigwedge_{P \in L} \big\{\textstyle\int P\big\}^{k_P} \wedge \bigwedge_{P \in L} \big\{\textstyle\int (H \wedge P)\big\}^{k_{Q_P}} \wedge \varphi \Rightarrow s\theta_0. \tag{23}$$

Furthermore, $s[P/P' : P \in L]s_0\psi'$ is equivalent to ψ, and $s[P/P' : P \in L]\theta_0$ is $s\theta_0$, because $P' \notin SV(\theta_0)$, $P \in L$. Hence

$$\models_{DC} \bigwedge_{Q \in M} \{\textstyle\int Q\}^{k_Q} \wedge \bigwedge_{P \in L} \{\textstyle\int P\}^{k_{P'}} \wedge \bigwedge_{P \in L} \{\textstyle\int (H \wedge P)\}^{k_{Q_P}} \wedge s\theta_0 \Rightarrow \psi. \tag{24}$$

The formulas (23) and (24) are valid for arbitrary systems of natural numbers k_Q, $Q \in L_1 \cup L_2$. Let $k_{P'} = k_P$ and $k_{Q_P} = k_P + \sum_{Q \in SV(H)} (k_Q - 1)$, $P \in L$. Then (8) from the proof of Corollary 10 implies that

$$\models_{DC} \bigwedge_{Q \in L \cup M} \{\textstyle\int Q\}^{k_Q} \Rightarrow \{\textstyle\int (H \wedge P)\}^{k_{Q_P}}$$

for $P \in L$, which allows (23) and (24) to be simplified to

$$\models_{DC} \bigwedge_{Q \in L \cup M} \{\textstyle\int Q\}^{k_Q} \wedge \varphi \Rightarrow s\theta_0 \text{ and } \models_{DC} \bigwedge_{Q \in L \cup M} \{\textstyle\int Q\}^{k_Q} \wedge s\theta_0 \Rightarrow \psi, \tag{25}$$

respectively. Having in mind the syntax (20) obtained for $s\theta_0$ in Lemma 12, this entails the following theorem:

Theorem 13 (Projection - related interpolation in abstract - timeDC) *Let (2) be valid. Let $k_P < \omega$ for every state variable $P \in L$. Then there exists a formula θ with the restricted syntax (20) from Lemma 12 which satisfies*

$$\models_{DC} \bigwedge_{Q \in L \cup M} \{\textstyle\int Q\}^{k_Q} \wedge \varphi \Rightarrow \theta \textit{ and } \models_{DC} \bigwedge_{Q \in L \cup M} \{\textstyle\int Q\}^{k_Q} \wedge \theta \Rightarrow \psi.$$

Replacing subformulas of the interpolant θ which occur on the right sides of $\Leftrightarrow$ in the equivalences (10)-(14), (21) and (22) by the corresponding formulas on the left sides of $\Leftrightarrow$ can be used to combine bigger parts of θ into the scope of single projections onto H. Transforming the entire interpolant into the form (β/H) can be prevented by the occurrences of subformulas of the form $x = \int V$, if variables from M occur in V. Such (sub)formulas may have no equivalents of the form (β/H). Interpolants with no such equivalent cannot be avoided in general. An example is the implication

$$(\Box(\lceil P \rceil \Leftrightarrow \lceil P' \rceil)/H) \Rightarrow (\textstyle\int (\neg P) + \int P = x \Rightarrow \int (\neg P') + \int P' = x),$$

in which $(\Box(\lceil P \rceil \Leftrightarrow \lceil P' \rceil)/H) \Rightarrow$ occurs just to match the syntax (2). One interpolant for it which fulfills the conditions of Theorem 13 is the formula $\int (\neg H) + \int H = x$, or simply $\ell = x$. This means that putting the entire interpolant in a single occurrence of the context $([\,]/H)$, which contains $\bigwedge_{\chi \in \Phi} \Box \forall (\chi \Leftrightarrow \chi')$ from (2), is sometimes impossible.

5 Interpolation in the $\lceil P \rceil$-subset of DC^*

In this section we first recall from [Gue00] that Craig interpolation holds for the $\lceil P \rceil$-subset of DC^* without restrictions on the variability and establish as a corollary that projection-related interpolation holds without such restrictions too. The results in this section are constructive.

The unary modality called *iteration* is defined in DC by the clause:

$M, \sigma \models \varphi^*$ iff either $\min\sigma = \max\sigma$, or there exist $\sigma_1, \ldots, \sigma_n \in \mathbf{I}(T_M)$ such that $\sigma_1; \ldots; \sigma_n = \sigma$ and $M, \sigma_i \models \varphi$, $i = 1, \ldots, n$.

Iteration is also the *Kleene star* associated with $(.;.)$, $\vee$ and $\ell = 0$. It was introduced to DC in [HJ96]. DC^* is the extension of DC by this operator. *Positive iteration* $(.)^+$ can be defined in DC^* by putting $\varphi^+ \rightleftharpoons (\varphi; \varphi^*)$.

5.1 The $\lceil P \rceil$-subset of DC^*

The $\lceil P \rceil$-subset of DC^* is defined by the BNF

$$\varphi ::= \bot \mid \ell = 0 \mid \lceil S \rceil \mid \varphi \Rightarrow \varphi \mid (\varphi; \varphi) \mid \varphi^*$$

Validity is decidable in the $\lceil P \rceil$-subset of DC^*. A decision procedure was first presented in [ZCS93]. Every formula in the $\lceil P \rceil$-subset of DC^* is equivalent to one with the syntax

$$\varphi ::= \bot \mid \ell = 0 \mid \lceil S \rceil \mid \varphi \vee \varphi \mid (\varphi; \varphi) \mid \varphi^* \tag{26}$$

This is essentially a corollary to Lemma 9 from [ZCS93]. Given a state expression S, $(\exists k < \omega)(I, \sigma \models \left\{\int S\right\}^k)$, which follows from the finite variability of $\lambda\tau.I_\tau(S)$, can be written using a $\lceil P \rceil$-subset formula as $I, \sigma \models (\lceil S \rceil \vee \lceil \neg S \rceil)^*$.

5.2 Craig interpolation

Let $\langle F, I \rangle, \sigma \models \exists P\varphi$ be defined by the condition

$\langle F, I_P^f \rangle, \sigma \models \varphi$ for some $f : T_F \to \{0, 1\}$ with the finite variability property.

Proposition 14 *Let the formula φ have the syntax (26). Then $\exists P\varphi$ is equivalent to a formula with the same syntax, which can be constructed from φ using the state variables of φ with the exception of P.*

Proof: Induction on the construction of φ, using the DC^* equivalence

$$\models_{DC^*} \exists P \lceil S \rceil \Leftrightarrow (\lceil [\mathbf{0}/P]S \rceil \vee \lceil [\mathbf{1}/P]S \rceil)^+$$

and the distributivity of $\exists P$ over $\vee$ and $(.;.)$. $\dashv$

Theorem 15 (Craig interpolation for the $\lceil P \rceil$-subset of DC^*) *Let φ and ψ be in the $\lceil P \rceil$-subset of DC^*. Let $\models_{DC^*} \varphi \Rightarrow \psi$. Then a formula θ in the $\lceil P \rceil$-subset of DC^* can be constructed using state variables from $SV(\varphi) \cap SV(\psi)$ only, such that*

$\models_{DC^*} \varphi \Rightarrow \theta$ *and* $\models_{DC^*} \theta \Rightarrow \psi$.

Proof: Let $SV(\varphi) \setminus SV(\psi)$ be $\{P_1, \ldots, P_n\}$. Then θ can be chosen to be a quantifier-free formula that is equivalent to $\exists P_1 \ldots \exists P_n \varphi$, which can be constructed from an equivalent of φ in the form (26) by Proposition 14. $\dashv$

5.3 Projection-related interpolation

Let L be a finite set of state variables in the vocabulary of some DC^* language **L**. Let P', $P \in L$, be some fresh state variables. Let φ and ψ be formulas in the $\lceil P \rceil$-subset of **L** and $SV(H) \cap L = \emptyset$. Let M be $SV(H) \cup (SV(\varphi) \cap SV(\psi)) \setminus L$, like in Subsection 4.3. Consider the formula

$$\Big(\big(\bigwedge_{P \in L} \Box(\ell = 0 \vee \lceil P \Leftrightarrow P' \rceil)\big)/H\Big) \Rightarrow (\varphi \Rightarrow \psi'), \tag{27}$$

where ψ' stands for $[P'/P : P \in L]\psi$.

Theorem 16 (Projection-related interpolation for the $\lceil P \rceil$-subset of DC^*) *If (27) is valid in DC^*, then there exists a θ which satisfies*

$$\models_{DC^*} \varphi \Rightarrow \theta \text{ and } \models_{DC^*} \theta \Rightarrow \psi$$

and has the syntax

$$\theta ::= \bot \mid \ell = 0 \mid \lceil H \wedge U \rceil \mid \lceil V \rceil \mid \theta \vee \theta \mid (\theta;\theta) \mid \theta^*$$

where U stands for arbitrary state expressions and V stands for state expressions satisfying $SV(V) \subseteq M$.

The proof of this theorem is like that of Theorem 13, with an application of Theorem 15 instead of Corollary 10. The proof of Theorem 13 is constructive relative to the application of Corollary 10 and Theorem 15 has a constructive proof. Hence projection-related interpolants in the $\lceil P \rceil$-subset of DC^* can be constructed too.

6 Craig interpolation and Beth definability in ITL

There is a strong connection between the Craig interpolation property and the equivalence between implicit and explicit definability after Beth. Both hold for classical first-order logic (cf. e.g. [CK73]), and Beth definability follows from Craig interpolation by a short proof. Craig interpolation implies Beth definability in many modal logics too. In this section we briefly review the situation with ITL and DC. We first recall the relevant definitions.

Definition 17 Let p be an n-ary relation symbol not in the vocabulary of the first-order logic language **L**. Let φ be a formula from the extension $\mathbf{L}(p)$ of **L** by p. Then φ *defines* p *implicitly*, if

$$\models \varphi \wedge [p'/p]\varphi \Rightarrow (p(x_1, \ldots, x_n) \Leftrightarrow p'(x_1, \ldots, x_n))$$

where p' stands for some fresh n-ary relation symbol, and $x_1, \ldots, x_n \notin FV(\varphi)$.

Given a set of sentences T in $\mathbf{L}(p)$, a formula θ from **L** *defines* p *explicitly*, if

$$T \models \forall x_1 \ldots \forall x_n (p(x_1, \ldots, x_n) \Leftrightarrow \theta).$$

Beth's definability theorem states that if a formula φ defines a relation symbol p implicitly, then there exists a formula θ which defines p implicitly for T being the singleton containing the universal closure of φ. In this form, Beth's theorem applies to abstract-time ITL as well. Yet ITL is a modal system and therefore it admits some other natural forms of implicit definability too. For instance, we may be interested in defining flexible relation symbols in ITL. A flexible relation symbol p can be regarded as defined implicitly by an ITL formula φ, if

$$\models_{ITL} \varphi \wedge [p'/p]\varphi \Rightarrow \Box(p(x_1,\ldots,x_n) \Leftrightarrow p'(x_1,\ldots,x_n)). \tag{28}$$

A corresponding notion of explicit definability should require the existence of a p-free formula θ such that

$$T \models_{ITL} \Box(p(x_1,\ldots,x_n) \Leftrightarrow \theta). \tag{29}$$

The following example shows that these two forms of definability are not equivalent in ITL. Let p and q be 0-ary flexible relation symbols. Let

$$\varphi \rightleftharpoons \neg(\top;q;\neg p;\top) \wedge \neg(\neg(\top;q);p;\top)$$

Let $M,\sigma \models \varphi$ and $\sigma' \in \mathbf{I}(T_M)$, $\sigma' \subseteq \sigma$. Then $M,\sigma' \models p$ iff $M,[\tau,\min\sigma'] \models q$ for some $\tau \in [\min\sigma,\min\sigma']$. Hence φ obviously defines p implicitly. In fact, under this definition, p has the meaning $\Diamond_l q$ where $\Diamond_l$ denotes the left neighbourhood modality of neighbourhood logic (*NL*, [CH98, BC00]) yet appropriately restricted within the interval σ. However, p has no explicit definition of the form (29). The reason is that, in order to define $\Diamond_l q$ at some interval σ', one needs to refer to subintervals of σ which are outside σ', which cannot be done in ITL, because its modality $(.;.)$ allows only reference to subintervals.

This situation can be partly remedied by rewriting (28) in the following equivalent form

$$\models_{ITL} \varphi \wedge [p'/p]\varphi \Rightarrow \left(\begin{array}{l} (\ell = z_1; p(x_1,\ldots,x_n); \ell = z_2) \Leftrightarrow \\ (\ell = z_1; p'(x_1,\ldots,x_n); \ell = z_2) \end{array}\right),$$

where the side condition $z_1, z_2 \notin FV(\varphi)$ is added to $x_1,\ldots,x_n \notin FV(\varphi)$. Then a θ can be found such that

$$\models_{ITL} \varphi \Rightarrow (\theta \Leftrightarrow (\ell = z_1; p(x_1,\ldots,x_n); \ell = z_2)).$$

For the case of the example, θ can be chosen to be $(\ell = z_1 \wedge (\top;q);\top;\ell = z_2)$. Yet this formula defines explicitly the entire formula $(\ell = z_1; p(x_1,\ldots,x_n); \ell = z_2)$, and not just p.

7 A possible interpretation of the results

Interpolation can be understood in the context of formal verification as follows. Let $[\![S]\!]$ and $[\![S']\!]$ be formulas which describe the possible behaviours of some simultaneously running parts S and S' of a system in terms of their observable signals. Let α and β

describe properties of the system's runs in terms of the signals of S and S', respectively. Then

$$\models ([\![S]\!] \wedge \alpha) \Rightarrow ([\![S']\!] \Rightarrow \beta)$$

means that the property α of the behaviour of S implies β about the behaviour of S', because of the two components' having signals in common. An interpolant for the above implication can be regarded as an explicit description of the interaction between S and S' which makes α cause β *written in terms of the signals S and S' share*. In the case of interval-related interpolation we can additionally restrict S and S' to share signals within part of the considered runs only, and an interpolant is a description of what S does when α holds to cause β *during that part* of the runs. In projection-related interpolation the part of the runs in question consists of a finite number of intervals specified by means of the state-expression operand of the respective projection. This state expression specifies the times at which the two parts are in contact and therefore their corresponding pairs of signals are equal.

In the case of uniform interpolation the interpolant can be regarded as an explicit description of all the effects one part of a system can have on whatever other parts might be connected to this part. Uniform interpolants generally represent *strongest consequences* under the given restrictions and among the consequences expressible in the considered language. Unlike interpolants, strongest consequences do not need the rather strong assumption that a certain implication is valid in order to exist. That is why they are of greater use, especially if constructible, which is the case in Theorems 15 and 16.

8 Related work

Propositional hybrid logics (see [HyL]) share some technically important features with first-order ITL. Hybrid logics are characterised by the choice to allow names for individual possible worlds, individual tuples in accessibility relations and quantification over these. Such names are known as *nominals* [Bla93] and *constants* [Pas91]. They make it meaningful to consider *Henkin constants* and construct *Henkin theories* in the languages of hybrid logics which are, in other respects, propositional. It was noticed in [Pas91] that combinatory dynamic logic, which has constants, allows a Henkin-style model existence theorem to be obtained using a single maximal Henkin theory of formulas to construct all the necessary possible worlds, instead of using a set consisting of (possibly all) such theories in the respective vocabulary. Named pairs of possible worlds as in [Pas91] can be identified with the pairs of endpoints of intervals in ITL. The flexible constant ℓ of first-order ITL enables the selection of individual time points as witnesses for the satisfaction of formulas built using ITL's chop modality. This brings ITL expressive power similar to that due to nominals in hybrid logics. The model existence theorem for abstract time first-order ITL can be proved using a construction based on a single Henkin theory too. This suggests a strong analogy between first-order ITL and hybrid logics and, for instance combinatory dynamic logic. The constructions from Henkin-style model existence theorems can be recognised in model-theoretic proofs of Craig interpolation (see, e.g., the proof in [CK73]). The proof of interval-related interpolation for ITL in [Gue01] is built after that of Craig interpolation in [CK73]. It is natural to find strong analogies between the proofs of interpolation theorems for propositional hybrid logics and first-order ITL. The purpose of the interpolation theorems in this paper is to show a possibility to restrict interpolants in a way which is different from that in Craig's theorem, more flexible than that in [Gue01] and possibly useful. Analogies between hybrid logic interpolation and interval- and projection-related interpolation can be sought here too. To do this, first note that both the context for interval-related interpolation from [Gue01] and projection onto a fixed state H define modalities with serial and functional accessibility relations. The recent work [BM03] on Craig interpolation in first-order hybrid logic is particularly interesting, because the approach in it is constructive (proof-theoretic). That paper makes special emphasis on the inspiration drawn from [Fit02, AM03].

As pointed before, $(.;.)$ is introspective. Neighbourhood logic has *expanding* modalities to access intervals outside the reference interval and admits a form of interval-related interpolation where a corresponding antecedent specifies that certain formulas are equivalent to their counterparts written using the other copy of the considered vocabulary everywhere in the past relative to a certain time point. Accordingly, interpolants are also restricted to specify properties of the past relative to that time point [Gue01].

9 Concluding remarks

The $\lceil P \rceil^*$-subset of DC has the same expressive power as that of the propositional subset of discrete-time ITL with the truth values of the propositional variables restricted to depend on the beginnings of reference intervals, and to that of regular expressions. Proposition 14 can be reformulated in terms of appropriate homomorphisms on the regular languages involved in the decidability argument for the $\lceil P \rceil^*$-subset in [ZCS93]. That is why the constructivity and uniformity of interpolation here is natural to expect. A careful examination of projection-related interpolation can lead to regular language theory counterparts too. This emphasises the gap between the decidable subsets of DC, where automata-theoretic decision procedures are readily available, and the full first-order systems of ITL and DC, where abstraction can be handled using flexible constant, function and relation symbols.

In the first-order case, interpolation and its failure are interesting for the account of the correspondence between explicit and implicit definability they give, the way these notions are known from first-order predicate logic, yet in the case of the temporal first-order system ITL and its extension DC. The failure of interpolation means that implicit definitions in DC can be more expressive than explicit ones, unless somehow restricted. The counterexample to interpolation can be viewed as a *rigorous* motivation for resorting to the state-variable-binding quantifier $\exists$ in DC as known from [Pan95], because it obviously contributes to the possibility to interpolate and, consequently, define explicitly in DC. Even with this quantifier the correspondence between implicit and explicit definability in ITL still has some limits, as it was shown in Section 6. The formula (4) has the interpolant $(\ell = 2)^+$ in DC^*. The effect of most of the extending operators for ITL and DC on interpolation seems largely unknown. Providing a flexible formulation of a general interval-related interpolation theorem by means of projection is perhaps the most outstanding idea in this paper. Projection is relevant to specification by DC in various other ways too.

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