An Alternating-time Temporal Logic with Knowledge, Perfect Recall and Past: Axiomatisation and Model-checking

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Abstract

We present a variant of *ATL* with incomplete information which includes the distributed knowledge operators corresponding to synchronous action and perfect recall. The cooperation modalities assume the use the distributed knowledge of coalitions and accordingly refer to perfect recall incomplete information strategies. We propose a model-checking algorithm for the logic. It is based on techniques for games with imperfect information and partially observable objectives, and involves deciding emptiness for automata on infinite trees. We also propose an axiomatic system and prove its completeness for a rather expressive subset. As for the constructs left outside this completely axiomatised subset, we present axioms by which they can be defined in the subset on the class of models in which every state has finitely many successors and give a complete axiomatisation for a "flat" subset of the logic with these constructs included.

Keywords: epistemic alternating-time temporal logic, axiomatisation, model-checking.

Introduction

Alternating time temporal logic (*ATL*, [AHK97, AHK02]) was introduced as a reasoning tool for the analysis of strategic abilities of coalitions in infinite multiplayer games with temporal winning conditions. Several variants of *ATL* have been proposed in the literature. The main differences arise from various restrictions on the considered games such as the players' information on the game state, which may be either *complete* or *incomplete* (*imperfect*), and their ability to keep complete record of the past, which is known as *perfect recall* [JvdH04, Sch04]. A classification of the variants of epistemic linear- and branching-time *temporal* logics (without the game-theoretic modalities) with *synchrony* taken in account too can be found in [vdMK03, HvdMV04]. Some subtle issues related to the commitment to fixed strategies are reflected in the semantics of the cooperation modalities of the system of *ATL* from [ÅGJ07]. The awareness of coalitions of the existence of winning strategies is another aspect of *ATL* semantics which is specific to the case of incomplete information, has become a source of ramification too, and is important to our study.

The completeness of a proof system and the decidability of validity for *ATL* with complete information was demonstrated in [GvD06]. As known from a personal communication of Mihalis Yannakakis to the authors of [AHK02], model-checking is undecidable for *ATL* with incomplete information and perfect recall. (The case of complete information admits a polynomial time algorithm, and a self-contained proof of the undecidability can be found in [Dc11]). This undecidability has stimulated the introduction of several systems [Sch04, vOJ05, JvdH06] with restrictions leading to more feasible model-checking. An extensive study of the complexity of the model-checking problem for the variants of *ATL* which arise from allowing imperfect information and/or perfect recall was done in [DJ08].

The formal analysis of multi-agent systems has justified combining *ATL* with modal logics of knowledge [vdHW03, JvdH04]. Such combinations can be viewed as extending temporal logics of knowledge (cf. e.g

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[HFMV95]) in the way *ATL* extends computational tree logic *CTL*. Epistemic goals require a semantics with partial information (*views*) for agents to interpret. Variants of the cooperation modalities which correspond to different forms of coordination within coalitions were proposed in [JvdH04]. The recent work [JÅ07] proposes a combination of *ATL* with the epistemic modalities for collective knowledge. In that system formulas are interpreted at *sets* of states and the existence of strategies which are winning for all the epistemically indiscernible states can be expressed by combining epistemic and cooperation modalities. Such strategies are called *uniform* with respect to the corresponding form of collective knowledge.

Along with the alternating transition systems proposed in [AHK02], *ATL* has been given semantics on *interpreted systems*, which are known from the study of knowledge-based programs [HFMV95], and other structures, some of which have been shown to be equivalent [GJ04]. Most of the proposed extensions of *ATL* and other temporal logics by epistemic modalities include only the future temporal operators and the indiscernibility relations which are needed for the semantics of the **S5** epistemic modalities are either defined as the equality of current local states of the corresponding agents or assumed to be given explicitly in the respective structures and required to respect equality of local state [LR06a, LR06b]. The axiomatisation of knowledge in the presence of past temporal operators has been studied in [FvdMR05], where indiscernibility is defined as equality of local state again. *ATL* with epistemic modalities extends epistemic *CTL*. Model-checking an extension of *LTL* by epistemic modalities, including common knowledge, with perfect recall semantics, but no past temporal operators in the language, has been studied in [vdMS99]. Model-checking algorithms for a variant of *CTL* with knowledge (but no collective forms of knowledge) have been proposed in [Dim08, SG02]. Extensions of *CTL* by modalities to reason about indiscernibility with respect to path observations in the past have been proposed in [AČC07]. The model-checking problem for a corresponding more expressive system of μ-calculus has been found to be undecidable.

In this paper we continue the study of a variant of *ATL* with epistemic modalities proposed in our previous work [GD08, DEG10]. In our system of *ATL*, along with the future temporal connectives, which are allowed to appear in combination with the cooperation modality, we allow the unrestricted use of the past connectives. This greatly facilitates the formulation of epistemic goals. We demonstrate the use of the past connectives on the example of expressing $(.\mathsf{U}.)$-objectives in a subset of the language with $\Diamond$ as the only iterative future temporal operator. We show that, in models with finitely many successors to every state, goals of the form $(\varphi \mathsf{U} \psi)$ are equivalent to goals which amount to the coalition eventually learning that the reference run went through a sequence of states satisfying the considered $(.\mathsf{U}.)$-formula. This form of epistemic goal can be written using just $\Diamond$, the epistemic operator and past temporal connectives. We assume incomplete information and perfect recall and a semantics for the cooperation modalities of *ATL* which renders the model-checking problem decidable. According to our semantics, strategies are supposed to be uniform with respect to the distributed knowledge of the coalition. Furthermore strategies are functions on the combined local history of all the members of the coalition. This corresponds to the unrestricted sharing of information within the coalition in order to follow the strategy in a coordinated way. With strategies that are uniform with respect to distributed knowledge and allow the agents to act using their combined knowledge a coalition can be viewed as a single player whose abilities and information are the sum of those of all the members.

We propose a Hilbert-style proof system for our logic and demonstrate its completeness on the subset in which the cooperation modalities can be used only in formulas of the forms $\langle\!\langle \Gamma \rangle\!\rangle \circ \varphi$, $\langle\!\langle \Gamma \rangle\!\rangle \Box \varphi$ and $\langle\!\langle \Gamma \rangle\!\rangle \Diamond \mathsf{D}_\Gamma \varphi$, where φ is unrestricted. Our completeness proof yields the finite model property of the considered subset and builds on techniques from temporal logic (cf. e.g. [LP00]) and Goranko and Drimmelen's work [GvD06] on *ATL* with complete information. As mentioned above, goals of the form $\langle\!\langle \Gamma \rangle\!\rangle (\varphi \mathsf{U} \psi)$, which are left outside the scope of the subset, can be expressed in it on the class of models with finitely many successors to every state. Unfortunately we do not know whether the expressing axiom schemata have sufficient deductive power to provide completeness for the whole logic when added to the proof system that we show complete for the subset.

We also give a model-checking algorithm for our variant of *ATL*. A rather ad-hoc model-checking algorithm was already given in [GD08], where the past temporal connectives were allowed only in the scope of the epistemic modality. The algorithm in this paper is an extension of that from [DEG10], where the past connectives were excluded and the logic was defined on *game arenas.* It is based on transition system manipulation techniques known from the theory of games with partial information [CDHR06] and involves automata on infinite trees. As mentioned above, the assumption that coalition strategies are coordinated is crucial for achieving decidable model-checking. According to a private communication of Yannakakis to the authors of [AHK97], model-checking is undecidable in the case of uncoordinated strategies. A self-contained proof of this result was obtained in [Dc11], where it was shown that two-player coalitions are sufficient for the undecidability. It is not difficult to see that this result corresponds to the undecidability of solving two-player games with both players having partial observations and a non-observable winning condition. Some results that are relevant to model-checking ATL^* can be found in [PR90] too.

Structure of the paper After brief preliminaries on ATL and its semantics on interpreted systems we introduce our extension of ATL by a modality for distributed knowledge and our form of the semantics of the cooperation modalities. In the subsequent sections we present our axiomatisation of the logic and our model-checking algorithm. We conclude by discussing some open problems for future work.

1 ATL with incomplete information, perfect recall, communicating coalitions and past (ATL_{iR}^{DP})

Here follows a BNF for the class of ATL_{iR}^{DP} formulas, and some informal reading for each of the main connectives:

$\varphi, \psi ::=$	$\bot$	|	logical falsehood;
	p	|	atomic proposition p holds (*now*);
	$(\varphi \Rightarrow \psi)$	|	if φ, then ψ;
	$\ominus\varphi$	|	φ held one step ago;
	$(\varphi \mathsf{S} \psi)$	|	either ψ holds now, or ψ held sometime before, and φ has been true ever since ψ held last;
	$\mathsf{D}_\Gamma \varphi$	|	Γ know φ (from their combined knowledge);
	$\langle\langle\Gamma\rangle\rangle \circ \varphi$	|	Γ can enforce φ in one step;
	$\langle\langle\Gamma\rangle\rangle(\varphi \mathsf{U} \psi)$	|	Γ can enforce reaching a state satisfying ψ with φ being true all the way to that state;
	$[\![\Gamma]\!](\varphi \mathsf{U} \psi)$		Γ cannot prevent reaching a state satisfying ψ, with φ being true all the way to that state.

Here Γ ranges over finite sets of agents, and p ranges over propositional variables. We write $\mathrm{Var}(\varphi)$ for the set of the propositional variables which occur in φ.

In this paper we define ATL_{iR}^{DP} on *interpreted systems*. An *interpreted system* is defined with respect to some given finite set $\Sigma = \{1, \ldots, N\}$ of *agents*, and a set of *propositional variables* (*atomic propositions*) AP. There is also an *environment* $e \notin \Sigma$; in the sequel we write Σ_e for $\Sigma \cup \{e\}$.

Definition 1 (interpreted systems) An *interpreted system* for Σ and AP is a tuple of the form $\langle\langle L_i : i \in \Sigma_e\rangle, I, \langle Act_i : i \in \Sigma_e\rangle, t, V\rangle$ where:

L_i, $i \in \Sigma_e$, are nonempty sets of *local states*; L_Γ stands for $\prod\limits_{i\in\Gamma} L_i$, $\Gamma \subseteq \Sigma_e$;

elements of L_{Σ_e} are called *global states*;

$I \subseteq L_{\Sigma_e}$ is a nonempty set of *initial global states*;

Act_i, $i \in \Sigma_e$, are nonempty sets of *actions*; Act_Γ stands for $\prod\limits_{i\in\Gamma} Act_i$;

$t : L_{\Sigma_e} \times Act_{\Sigma_e} \to L_{\Sigma_e}$ is a *transition* function;

$V \subseteq L_{\Sigma_e} \times AP$ is a valuation of the atomic propositions.

For every $i \in \Sigma_e$ and $l', l'' \in L_{\Sigma_e}$ such that $l'_i = l''_i$ and $l'_e = l''_e$ the function t satisfies $(t(l', a))_i = (t(l'', a))_i$.

In the literature an interpreted system also includes a *protocol* $P_i : L_i \to \mathcal{P}(Act_i)$ for every $i \in \Sigma_e$. $P_i(l)$ is the set of actions which are available to i when its local state is l. Protocols are not essential to our study here. Setting the effect of all the currently prohibited actions to that of some fixed permitted action (which is always supposed to exist) allows a system with arbitrary protocols to be transformed into an equivalent one in which all actions are always permitted. Our variant of interpreted systems also has a technically convenient feature which we borrowed from [LR06a], where a system of *ATL without* the epistemic operators was intoduced, and is not present in [HFMV95], nor in the model-checker MCMAS [LQR]: every agent's next local state can be directlty affected by the local state of the environment through the transition function. The logic admits equivalent semantics on other types of models of infinite games as well. See [GJ04] for a comparative study for the variants of such structures for the case of complete information. Our model-checking algorithm works for finite interpreted systems. We also present some axioms which link the meaning of (.U.)-goals to that of a special form of $\Diamond$-goals on the class of the interpreted systems with finite sets of actions and finite sets of initial states, which entails the countability of the respective generated submodels.

Definition 2 (global runs) Given an $n < \omega$, a *run of length* n is a sequence

$$r = l^0 a^0 \ldots a^{n-1} l^n \in L_{\Sigma_e}(Act_{\Sigma_e} L_{\Sigma_e})^n$$

such that $l^0 \in I$ and $l^{j+1} = t(l^j, a^j)$ for all $j < n$. We denote the set of all runs of length n by $R^n(IS)$. We denote $\bigcup\limits_{n<\omega} R^n(IS)$ by $R^{fin}(IS)$. An *infinite run* is a sequence

$$r = l^0 a^0 \ldots a^{n-1} l^n a^n \ldots \in (L_{\Sigma_e} Act_{\Sigma_e})^\omega$$

such that $l^0 \in I$ and $l^{j+1} = t(l^j, a^j)$ for all $j < \omega$. We denote the set of all infinite runs by $R^\omega(IS)$. We denote the length of run r by $|r|$. We put $|r| = \omega$ for $r \in R^\omega(IS)$. We write $R(IS)$ for $R^{fin}(IS) \cup R^\omega(IS)$.

Given $i, j < \omega$ and $r = l^0 a^0 \ldots a^{n-1} l^n \in R^n(IS)$ such that $i \le j \le n$, we write $r[i..j]$ for $l^i a^i \ldots a^{j-1} l^j$.

Definition 3 (local runs) Given $r = l^0 a^0 \ldots a^{n-1} l^n \in L_{\Sigma_e}(Act_{\Sigma_e} L_{\Sigma_e})^n$ and $\Gamma \subseteq \Sigma$, we write r_Γ for the corresponding *local run*

$$l^0_\Gamma a^0_\Gamma \ldots a^{n-1}_\Gamma l^n_\Gamma \in L_\Gamma(Act_\Gamma L_\Gamma)^n$$

of Γ in which $l^j_\Gamma = \langle l^j_i : i \in \Gamma \rangle$ and $a^j_\Gamma = \langle a^j_i : i \in \Gamma \rangle$.

Definition 4 (indiscernibility) Given $r', r'' \in R(IS)$ and $i \le |r'|, |r''|$, we write $r' \sim_{\Gamma,i} r''$ if $r'[0..i]_\Gamma = r''[0..i]_\Gamma$. We write $r' \sim_\Gamma r''$ for the conjunction of $r' \sim_{\Gamma,|r'|} r''$ and $|r'| = |r''|$.

Runs of length $n < \omega$ are indeed sequences of $2n + 1$ states and actions. The definitions of $r[i..j]$ and r_Γ for infinite r are similar. Sequences of the form $r_\emptyset$ consist of $\langle\rangle$s, and, consequently, $[r]_\emptyset$ is the class of all the runs of length $|r|$. Obviously $\sim_{\Gamma,n}$ and $\sim_\Gamma$ are equivalence relations on $R(IS)$.

Definition 5 We denote $\{r' \in R(IS) : r' \sim_\Gamma r\}$ by $[r]_\Gamma$.

Our semantics for ATL^{DP}_{iR} is based on a coordinated form of strategy.

Definition 6 (coordinated strategies and outcomes in ATL^{DP}_{iR}) Given a $\Gamma \subseteq \Sigma$, a *strategy for* Γ in IS is a function of type

$$\{r_\Gamma : r \in R^{fin}(IS)\} \to Act_\Gamma.$$

We write $S(\Gamma, IS)$ for the set of all the strategies for Γ in the considered interpreted system IS. Given $s \in S(\Gamma, IS)$ and $r \in R^{fin}(IS)$, we write $\mathrm{out}(r, s)$ for the set

$$\{r' = l^0 a^0 \ldots a^{n-1} l^n \ldots \in R^\omega(IS) : r'[0..|r|] = r, a^j_\Gamma = s(r[0..j]_\Gamma) \text{ for all } j \ge |r|\}.$$

of the *outcomes* of r when Γ sticks to s from time $|r|$ on. Given an $X \subseteq R^{fin}(IS)$, $\mathrm{out}(X, s)$ is $\bigcup\limits_{r \in X} \mathrm{out}(r, s)$.

Strategies, as defined above, are determined by the local view of the considered coalition and are therefore *uniform.*

Definition 7 (modelling relation of ATL^{DP}_{iR}) The relation $IS, r \models \varphi$ is defined for $r \in R^{fin}(IS)$ and formulas φ by the clauses:

$IS, r \not\models \bot$;

$IS, l^0 a^0 \dots a^{n-1} l^n \models p$ iff $V(l^n, p)$ for atomic propositions p;

$IS, r \models \varphi \Rightarrow \psi$ iff either $IS, r \not\models \varphi$ or $IS, r \models \psi$;

$IS, r \models \mathsf{D}_\Gamma \varphi$ iff $IS, r' \models \varphi$ for all $r' \in [r]_\Gamma$;

$IS, r \models \langle\langle\Gamma\rangle\rangle \circ \varphi$ iff there exists an $s \in S(\Gamma, S)$ such that $IS, r'[0..|r|+1] \models \varphi$ for all $r' \in \text{out}([r]_\Gamma, s)$;

$IS, r \models \langle\langle\Gamma\rangle\rangle(\varphi \mathsf{U} \psi)$ iff there exists an $s \in S(\Gamma, S)$ such that for every $r' \in \text{out}([r]_\Gamma, s)$ there exists a $k < \omega$ such that $IS, r'[0..|r|+i] \models \varphi$ for all $i < k$ and $IS, r'[0..|r|+k] \models \psi$;

$IS, r \models [\![\Gamma]\!](\varphi \mathsf{U} \psi)$ iff for every $s \in S(\Gamma, S)$ there exists an $r' \in \text{out}([r]_\Gamma, s)$ and a $k < \omega$ such that $IS, r'[0..|r|+i] \models \varphi$ for all $i < k$ and $IS, r'[0..|r|+k] \models \psi$;

$IS, r \models \ominus\varphi$ iff $|r| > 0$ and $IS, r[0..|r|-1] \models \varphi$;

$IS, r \models (\varphi \mathsf{S} \psi)$ iff there exists a $k \leq |r|$ such that $IS, r[0..n-i] \models \varphi$ for all $i < k$ and $IS, r[0..n-k] \models \psi$.

Validity of formulas in entire interpreted systems and on the class of all interpreted systems, that is, in the logic ATL^{DP}_{iR}, is defined as satisfaction at all 0-length runs in the considered interpreted system, and at all the 0-length runs in all the systems in the considered class, respectively.

As mentioned in the introduction, coordinated strategies render $\models_{ATL^{DP}_{iR}}$ decidable for finite interpreted systems, as opposed to the weaker form of coalition strategy, which is established in the literature.

Definition 8 (uncoordinated coalition strategies) An *uncoordinated strategy* for Γ is a vector $s = \langle s_i : i \in \Gamma \rangle$ of functions s_i of type $\{r_i : r \in R^{fin}(IS)\} \to Act_i$.

Obviously uncoordinated strategies can be viewed as a special case of coordinated strategies. Interestingly, our completeness result for a subset of ATL^{DP}_{iR} below applies to the semantics based on uncoordinated strategies as well. Moreover, it applies even in case we allow only *constant* strategies, that is, strategies s which satisfy $s(r) = a$ for some fixed (vector of) actions a and all r. All these classes of coalition strategies produce the same set of valid formulas, that is, the same logic.

Abbreviations $\top$, $\neg$, $\vee$, $\wedge$ and $\Leftrightarrow$ have their usual meanings. To keep the use of (and) down, we assume that $\neg$ and the unary modalities $\ominus, \dots$, including the derived ones which we introduce below, bind the strongest, the binary modalities $\langle\langle\Gamma\rangle\rangle(.\mathsf{U}.)$ and $[\![\Gamma]\!](.\mathsf{U}.)$, and the derived ones below, bind the weakest, and their parentheses are never omitted, and the binary boolean connectives come in the middle, in decreasing order of their binding power as follows: $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$. In formulas, coalitions can be enumerated without the { and }. E.g., the shortest way to write $\langle\langle\{1\}\rangle\rangle(((p \Rightarrow q) \wedge \mathsf{P}_{\{1\}} r)\mathsf{U}\mathsf{D}_{\{2,3\}}(r \vee q)))$ is $\langle\langle 1\rangle\rangle((p \Rightarrow q) \wedge \mathsf{P}_1 r \mathsf{U} \mathsf{D}_{2,3}(r \vee q))$.

The temporal connectives $\diamondminus$ and $\boxminus$ and the temporal constant I, which identifies 0-length runs, are defined by the clauses

$$\diamondminus\varphi \rightleftharpoons (\top \mathsf{S} \varphi),\ \boxminus\varphi \rightleftharpoons \neg\diamondminus\neg\varphi \text{ and } \mathsf{I} \rightleftharpoons \neg\ominus\top.$$

We abbreviate $\underbrace{\ominus \dots \ominus}_{n \text{ times}} \varphi$ to $\ominus^n \varphi$. We write P for the dual of D:

$$\mathsf{P}_\Gamma \varphi \rightleftharpoons \neg \mathsf{D}_\Gamma \neg \varphi.$$

The rest of the combinations of the cooperation modality and future temporal connectives are defined by the clauses

$$\begin{array}{ll}
\langle\langle\Gamma\rangle\rangle\Diamond\varphi \rightleftharpoons \langle\langle\Gamma\rangle\rangle(\top\mathsf{U}\varphi) & [\![\Gamma]\!]\Diamond\varphi \rightleftharpoons [\![\Gamma]\!](\top\mathsf{U}\varphi)\\
\langle\langle\Gamma\rangle\rangle\Box\varphi \rightleftharpoons \neg[\![\Gamma]\!]\Diamond\neg\varphi & [\![\Gamma]\!]\Box\varphi \rightleftharpoons \neg\langle\langle\Gamma\rangle\rangle\Diamond\neg\varphi\\
\langle\langle\Gamma\rangle\rangle(\varphi\mathsf{W}\psi) \rightleftharpoons \neg[\![\Gamma]\!](\neg\psi\mathsf{U}\neg\psi\wedge\neg\varphi) & [\![\Gamma]\!](\varphi\mathsf{W}\psi) \rightleftharpoons \neg\langle\langle\Gamma\rangle\rangle(\neg\psi\mathsf{U}\neg\psi\wedge\neg\varphi)
\end{array}$$

In our model-checking algorithm, to facilitate the presentation, we adopt $\langle\langle\Gamma\rangle\rangle(\varphi\mathsf{W}\psi)$ as basic instead of $[\![\Gamma]\!](\varphi\mathsf{U}\psi)$, which can be defined as $\neg\langle\langle\Gamma\rangle\rangle(\neg\psi\mathsf{W}\neg\psi\wedge\neg\varphi)$.

2 Finite branching, a reduction of $(.\mathsf{U}.)$-goals to epistemic $\Diamond$-goals and the subset $ATL_{iR}^{\Diamond}$

Given a formula ξ, we write $\mathrm{level}_\Gamma\xi$ for the formula

$$\mathsf{D}_\Gamma(\xi\wedge\neg\ominus\diamondminus\xi)\wedge\langle\langle\emptyset\rangle\rangle\circ\langle\langle\emptyset\rangle\rangle\Box\neg\xi.$$

Proposition 9 *Let $r\in R^{fin}(IS)$ and let $IS,r\models\mathrm{level}_\Gamma\xi$. Then, for all $r'\in R^\omega(IS)$ such that $r'[0..|r|]\sim_\Gamma r$, $IS,r'[0..k]\models\xi$ is equivalent to $k=|r|$.*

Furthermore, $\mathrm{level}_\Gamma\xi$ is equivalent to $\mathsf{D}_\Gamma\mathrm{level}_\Gamma\xi$ in ATL_{iR}^{DP}. Given an $r\in R^{fin}(IS)$ such that $IS,r\models\mathrm{level}_\Gamma\xi$, the knowledge of $\mathrm{level}_\Gamma\xi$ can be used by Γ to realise that the actual run is of the same length as r. To obtain $IS,r\models\mathrm{level}_\Gamma\xi$, one can always choose ξ to be $\ominus^{|r|}\mathsf{I}$. Formulas η which change their truth value at most once along every run can sometimes be used to define a ξ that identifies the point of change by putting $\xi\rightleftharpoons\eta\wedge\neg\ominus\eta$. Examples include $\diamondminus p$ and $\mathsf{D}_\Delta\diamondminus(\mathsf{I}\wedge p)$.

Proposition 10 *The formula*

$$\mathrm{level}_\Gamma\xi\Rightarrow(\langle\langle\Gamma\rangle\rangle(\varphi\mathsf{U}\psi)\Leftrightarrow\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\diamondminus(\psi\wedge(\ominus\varphi\mathsf{S}\xi)))\qquad(1)$$

is valid on the class of interpreted systems with finitely many initial states and finitely many successor states to every given state.

This formula states that if Γ is capable of enforcing $(\varphi\mathsf{U}\psi)$, then it is also capable of enforcing a development which gives Γ sufficient evidence that $(\varphi\mathsf{U}\psi)$ was realised. **Proof:** Let us abbreviate $\psi\wedge(\ominus\varphi\mathsf{S}\xi)$ by Φ. Assume that $r\in R^{fin}(IS)$ and $IS,r\models\mathrm{level}_\Gamma\xi\wedge\neg\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\diamondminus\Phi$. Then for every $s\in S(\Gamma,IS)$ there exists an $r'\in\mathrm{out}([r]_\Gamma,s)$ such that for all $k<\omega$ there exists an $r''\in R^{|r|+k}(IS)$ such that $r''\sim_{\Gamma,|r|+k}r'$ and $IS,r''\models\neg\diamondminus\Phi$. Let $\langle T,\prec\rangle$ be the forest in which

$$T=\{r'[0..|r|+k]:r'\in\mathrm{out}([r]_\Gamma,s),k<\omega\}$$

and $r'\prec r''$ iff $|r''|>0$ and $r'=r''[0..|r''|-1]$. Since Act_{Σ_e} is finite, $\langle T,\prec\rangle$ is a finitely branching forest. Since IS has finitely many initial states, $\langle T,\prec\rangle$ is the union of finitely many trees. According to our assumption, $\langle T,\prec\rangle$ contains chains $r_0\prec\ldots\prec r_k$ of arbitrarily big lengths such that $IS,r^n\models\neg\Phi$ for all $n=|r|,\ldots,k$. Then, by König's Lemma, $\langle T,\prec\rangle$ has an infinite sequence of nodes $r_0\prec r_1\prec\ldots$ such that $IS,r_k\models\neg\Phi$ for all $k<\omega$. A direct check shows that if $r'\in(L_{\Sigma_e}Act_{\Sigma_e})^\omega$ is determined by the conditions $r'[0..k]=r_k$, $k<\omega$, then $r'\in\mathrm{out}([r]_\Gamma,s)$ and $IS,r'[0..k]\not\models\Phi$ for all $k\geq|r|$. Since $IS,r'[0..|r|]\models\mathrm{level}_\Gamma\xi$, $IS,r'[0..k]\models\xi$ holds for $k=|r|$ and no other k, by Proposition 9. Now $IS,r'[0..k]\not\models\Phi$ for all $k\geq|r|$ implies that r' does not satisfy the objective $(\varphi\mathsf{U}\psi)$ from time $|r|$ on. Since this holds about any $s\in S(\Gamma,IS)$, $IS,r\not\models\langle\langle\Gamma\rangle\rangle(\varphi\mathsf{U}\psi)$.

Now assume that $IS,r\models\mathrm{level}_\Gamma\xi\wedge\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\diamondminus\Phi$. Then there exists an $s\in S(\Gamma,IS)$ such that for every $r'\in\mathrm{out}([r]_\Gamma,s)$ there exists a $k\geq|r|$ such that $IS,r'[0..k]\models\Phi$. The latter means that $IS,r'[0..k]\models\psi$, there exists an $m\leq k$ such that $IS,r'[0..m]\models\xi$, and $IS,r'[0..n]\models\varphi$ for all $n=m,\ldots,k-1$. The only possible choice for m is $m=|r|$, by Proposition 9. Hence r' satisfies $(\varphi\mathsf{U}\psi)$ from time $|r|$ on, and therefore the existence of an s as above entails $IS, r \models \langle\langle\Gamma\rangle\rangle(\varphi \mathsf{U} \psi)$. $\dashv$ Similarly, in interpreted systems with finite degree of branching we have

$$\mathrm{level}_\Gamma \xi \Rightarrow ([\![\Gamma]\!](\varphi \mathsf{U} \psi) \Leftrightarrow [\![\Gamma]\!]\Diamond \mathsf{P}_\Gamma \diamondminus(\psi \wedge (\ominus\varphi \mathsf{S} \xi))). \quad (2)$$

A counterexample for the case of infinite degree of branching can easily be obtained for, e.g, $[\![1]\!](\top \mathsf{U} p)$, by bunching together ω many runs, with run i reaching a p-state at step i for the first time in a system in which p is undetectable to agent 1. (Agent 1 can be prevented from detecting p by, e.g., choosing 1's local state space to be a singleton.)

The formulas (1) and (2) show that goals of the form $(\varphi \mathsf{U} \psi)$ can be reduced to goals of the forms $\Diamond \mathsf{D}_\Gamma \theta$ and $\Box \mathsf{D}_\Gamma \theta$ in such models, yet with the choice of θ in the latter goal form depending on the particular finite run of the model, and not only on the given φ and ψ. In other words, $(.\mathsf{U}.)$-goals are locally expressible by epistemic $\Diamond$-goals. The relevant epistemic $\Diamond$-goals are about eventually learning that the original $(.\mathsf{U}.)$-goal has been achieved. Goals of the forms $\circ\varphi$ and $\Box\varphi$ can be similarly reformulated as epistemic goals because

$$\models_{ATL_{iR}^{DP}} \langle\langle\Gamma\rangle\rangle \circ \varphi \Leftrightarrow \langle\langle\Gamma\rangle\rangle \circ \mathsf{D}_\Gamma \varphi \text{ and } \models_{ATL_{iR}^{DP}} \langle\langle\Gamma\rangle\rangle \Box \varphi \Leftrightarrow \langle\langle\Gamma\rangle\rangle \Box \mathsf{D}_\Gamma \varphi. \quad (3)$$

In these latter cases no auxiliary formula is involved to make the corresponding epistemic goal depend on the reference finite run.

The formulas (1), (2) and (3) show that ATL_{iR}^{DP} can be regarded as a theory in a subset of its in which goals are restricted to have the forms $\circ \mathsf{D}_\Gamma \theta$, $\Box \mathsf{D}_\Gamma \theta$ and $\Diamond \mathsf{D}_\Gamma \theta$. We call this subset $ATL_{iR}^{\Diamond}$ to indicate that $\Diamond$ (and its dual $\Box$) are the only fixpoint temporal operators allowed to combine with the cooperation modalities. The syntax of $ATL_{iR}^{\Diamond}$ formulas can be defined by the BNF

$$\varphi ::= \bot \mid p \mid (\varphi \Rightarrow \varphi) \mid \ominus\varphi \mid (\varphi \mathsf{S} \varphi) \mid \mathsf{D}_\Gamma \varphi \mid \langle\langle\Gamma\rangle\rangle \circ \varphi \mid \langle\langle\Gamma\rangle\rangle \Box \varphi \mid \langle\langle\Gamma\rangle\rangle \Diamond \mathsf{D}_\Gamma \varphi$$

and its semantics on interpreted systems is as that of ATL_{iR}^{DP}. We keep the double occurrence of Γ in formulas of the form $\langle\langle\Gamma\rangle\rangle \Diamond \mathsf{D}_\Gamma \varphi$ for the sake of compatibility with the more general syntax of ATL_{iR}^{DP}.

In the sequel we present a complete axiomatisation of $ATL_{iR}^{\Diamond}$ with respect to the class of its finite models. The axiomatisation is sound on the class of the interpreted systems with finitely many initial states and finitely many successors to every state, already considered in Section 2. The completeness entails that validity of $ATL_{iR}^{\Diamond}$ formulas is decidable on this class of systems as a corollary to the (strong) finite model property. Both the completeness and the decidability of validity apply to the case of uncoordinated strategies as well.

3 A complete proof system for $ATL_{iR}^{\Diamond}$

The system consists of the set of all propositional tautologies, the rule *Modus Ponens* (MP) and the following axioms and rules:

Axioms and rules about the epistemic operator $\mathsf{D}_.$

$(\mathbf{K}_{\mathsf{D}})$	$\mathsf{D}_\Gamma(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{D}_\Gamma\varphi \Rightarrow \mathsf{D}_\Gamma\psi)$
$(\mathbf{T}_{\mathsf{D}})$	$\mathsf{D}_\Gamma\psi \Rightarrow \psi$
$(\mathbf{4}_{\mathsf{D}})$	$\mathsf{D}_\Gamma\psi \Rightarrow \mathsf{D}_\Gamma\mathsf{D}_\Gamma\psi$
$(\mathbf{5}_{\mathsf{D}})$	$\neg\mathsf{D}_\Gamma\psi \Rightarrow \mathsf{D}_\Gamma\neg\mathsf{D}_\Gamma\psi$
$(\mathit{Mono}_{\mathsf{D}})$	$\mathsf{D}_\Gamma\psi \Rightarrow \mathsf{D}_{\Gamma\cup\Delta}\psi$
(N_{D})	$\dfrac{\varphi}{\mathsf{D}_\Gamma\varphi}$
(INT)	$\dfrac{\chi \Rightarrow \mathsf{P}_\Gamma(p \wedge \psi) \vee \mathsf{P}_\Delta(\neg p \wedge \psi)}{\chi \Rightarrow \mathsf{P}_{\Gamma\cup\Delta}\psi}$ $\quad p \notin \mathrm{Var}(\psi) \cup \mathrm{Var}(\chi)$

Axioms and rules about $\ominus$ *and* $(.\mathsf{S}.)$

$(\ominus\bot)$	$\neg\ominus\bot$
$(\mathbf{K}_\ominus)$	$\ominus(\varphi \Rightarrow \psi) \Leftrightarrow (\ominus\varphi \Rightarrow \ominus\psi)$
$(\mathit{FP}_{(.\mathsf{S}.)})$	$(\varphi\mathsf{S}\psi) \Leftrightarrow \psi \vee (\varphi \wedge \ominus(\varphi\mathsf{S}\psi))$
$(\mathit{Mono}_\ominus)$	$\dfrac{\varphi \Rightarrow \psi}{\ominus\varphi \Rightarrow \ominus\psi}$
$(N_\boxminus)$	$\dfrac{\varphi}{\boxminus\varphi}$

General ATL axioms and rules

$(\langle\langle.\rangle\rangle \circ \bot)$	$\neg\langle\langle\Gamma\rangle\rangle \circ \bot$
$(\langle\langle.\rangle\rangle \circ \top)$	$\langle\langle\Gamma\rangle\rangle \circ \top$
(S)	$\langle\langle\Gamma\rangle\rangle \circ \varphi \wedge \langle\langle\Delta \setminus \Gamma\rangle\rangle \circ \psi \Rightarrow \langle\langle\Gamma \cup \Delta\rangle\rangle \circ (\varphi \wedge \psi)$
$(\mathit{Mono}_{\langle\langle.\rangle\rangle\circ})$	$\dfrac{\varphi \Rightarrow \psi}{\langle\langle\Gamma\rangle\rangle \circ \varphi \Rightarrow \langle\langle\Gamma\rangle\rangle \circ \psi}$
$(N_{\langle\langle\emptyset\rangle\rangle\Box})$	$\dfrac{\varphi}{\langle\langle\emptyset\rangle\rangle\Box\varphi}$

Axioms about the interaction between the temporal, the cooperation and the epistemic modalities

$(\mathsf{D}_\circ)$	$\langle\langle\Gamma\rangle\rangle \circ \varphi \Leftrightarrow \mathsf{D}_\Gamma\langle\langle\Gamma\rangle\rangle \circ \varphi, \qquad \langle\langle\Gamma\rangle\rangle \circ \varphi \Leftrightarrow \langle\langle\Gamma\rangle\rangle \circ \mathsf{D}_\Gamma\varphi$
$(\mathsf{D}_\Box)$	$\langle\langle\Gamma\rangle\rangle\Box\varphi \Leftrightarrow \mathsf{D}_\Gamma\langle\langle\Gamma\rangle\rangle\Box\varphi, \qquad \langle\langle\Gamma\rangle\rangle\Box\varphi \Leftrightarrow \langle\langle\Gamma\rangle\rangle\Box\mathsf{D}_\Gamma\varphi$
(PR)	$\ominus\mathsf{D}_\Gamma\varphi \Rightarrow \mathsf{D}_\Gamma\ominus\varphi$
$(\langle\langle.\rangle\rangle \circ \ominus)$	$\langle\langle\Gamma\rangle\rangle \circ (\ominus\varphi \wedge \psi) \Leftrightarrow \mathsf{D}_\Gamma\varphi \wedge \langle\langle\Gamma\rangle\rangle \circ \psi, \qquad \ominus\langle\langle\emptyset\rangle\rangle \circ \varphi \Rightarrow \langle\langle\emptyset\rangle\rangle \circ \ominus\varphi$
(D_I)	$\mathsf{D}_\Gamma\mathsf{I} \vee \mathsf{D}_\Gamma\neg\mathsf{I}$
$(\mathit{FP}_{\Diamond\mathsf{D}})$	$\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \Leftrightarrow \mathsf{D}_\Gamma\psi \vee \langle\langle\Gamma\rangle\rangle \circ \langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi$
$(\mathit{LFP}_{\Diamond\mathsf{D}})$	$\mathsf{P}_\Gamma\psi \wedge \langle\langle\emptyset\rangle\rangle\Box(\mathsf{P}_\Gamma\psi \Rightarrow [\![\Gamma]\!] \circ \mathsf{P}_\Gamma\psi) \Rightarrow [\![\Gamma]\!]\Box\mathsf{P}_\Gamma\psi$
$(\mathit{FP}_\Box)$	$\langle\langle\Gamma\rangle\rangle\Box\psi \Leftrightarrow \mathsf{D}_\Gamma\psi \wedge \langle\langle\Gamma\rangle\rangle \circ \langle\langle\Gamma\rangle\rangle\Box\psi$

The condition of finite degree of branching from Section 2 is relevant to the soundness of $\mathit{LFP}_{\Diamond\mathsf{D}}$, which can be shown unsound on the class of all interpreted systems. Establishing the soundness of *INT* and $\mathit{LFP}_{\Diamond\mathsf{D}}$ requires some non-trivial steps, which we give in detail next.

INT: This rule is easier to prove correct by first reformulating it in terms of $\mathsf{D}_.$, by means of duality and some De Morgan transformations:

$$(\mathit{INT}') \qquad \frac{\mathsf{D}_\Gamma(p \Rightarrow \psi) \wedge \mathsf{D}_\Delta(\neg p \Rightarrow \psi) \Rightarrow \chi}{\mathsf{D}_{\Gamma\cup\Delta}\psi \Rightarrow \chi} \qquad p \notin \mathrm{Var}(\psi), \mathrm{Var}(\chi).$$

This formulation enables the informal reading *if a condition p can be identified such that Γ know ψ in case p holds, and Δ know ψ otherwise, then $\Gamma \cup \Delta$ know ψ unconditionally, and vice versa, and therefore the same logical consequences χ can be obtained from both premises.* In a language with quantification over propositional variables, which evaluate to sets of states, it would be possible to replace *INT* by the axiom

$$\exists p(\mathsf{D}_\Gamma(p \Rightarrow \psi) \wedge \mathsf{D}_\Delta(\neg p \Rightarrow \psi)) \Leftrightarrow \mathsf{D}_{\Gamma\cup\Delta}\psi.$$

In order to enable the construction of suitable valuations for p, which we need for establishing the soundness of *INT*, we use the fact that the satisfaction of formulas at an arbitrary interpreted system IS is preserved under the *unravelling* of IS into a corresponding *forest-like* interpreted system IS^T, which is defined as follows:

Definition 11 The *unravelling of interpreted system*

$$IS = \langle\langle L_i : i \in \Sigma_e\rangle, I, \langle Act_i : i \in \Sigma_e\rangle, t, V\rangle,$$

is the interpreted system

$$IS^T = \langle\langle L_1, \dots, L_{|\Sigma|}, L_e^T\rangle, I^T, \langle Act_i : i \in \Sigma_e\rangle, t^T, V^T\rangle,$$

for the same vocabulary AP and set of agents Σ, where:

$L_e^T = R^{fin}(IS)$;
$I^T = \{\langle l_\Sigma, \langle l_\Sigma, l_e\rangle\rangle : \langle l_\Sigma, l_e\rangle \in I\}$;
$t^T(\langle l_\Sigma, r\rangle, a) = \langle l'_\Sigma, ral'\rangle$, where $l' = t(\langle l_\Sigma, l_e\rangle, a)$, in case l is the last state in r;
$V^T(\langle l_\Sigma, r\rangle, p)$ iff $V(l, p)$ in case l is the last state in r.

A direct check shows that all the reachable states in IS^T have the form $\langle l_\Sigma, r\rangle$ where l_Σ is the vector of the last local states of the agents in r. This renders defining V^T and t^T on states which do not have this form irrelevant. Furthermore, for all $n < \omega$, $r = l^0a^0l^1a^1 \dots a^{n-2}l^{n-1}a^{n-1}l^n \in R^n(IS)$ and formulas φ, $IS, r \models \varphi$ is equivalent to $IS^T, r^T \models \varphi$ where

$$r^T = \langle l^0_\Gamma, r[0..0]\rangle a^0 \langle l^0_\Gamma, r[0..1]\rangle a^1 \dots a^{n-2}\langle l^n_\Gamma, r[0..n-1]\rangle a^{n-1}\langle l^n_\Gamma, r\rangle.$$

Hence, IS^T and IS satisfy the same formulas. Furthermore, for every $r \in R^{fin}(IS)$ there exists a unique run $r^T \in R^{fin}(IS^T)$ such that the last environment local state in r^T is r, i.e., t^T defines a *forest*, with one tree rooted at each initial state from I^T. This makes it possible to use the satisfaction of an auxiliary propositional variable $p \notin AP$ as the membership condition for an arbitrary set of *finite runs* in IS^T.

Now let $IS, r \models \mathsf{D}_{\Gamma\cup\Delta}\psi \wedge \neg\chi$ for some $r \in R^{fin}(IS)$. Then the same formula is satisfied at r^T in IS^T. According to the side condition of INT', we can assume that the variable p from the premiss of the rule is not in AP. Let IS_p^T be an interpreted system exactly like IS^T, except for having p included in the domain of its valuation relation, which we denote by V_p^T. Let $V_p^T(\langle l_\Sigma, r'\rangle, p)$ be equivalent to $r' \in R^{fin}(IS) \setminus [r]_\Gamma$. Then a direct check shows that $IS_p^T, r^T \models \mathsf{D}_\Gamma(p \Rightarrow \psi) \wedge \mathsf{D}_\Delta(\neg p \Rightarrow \psi)$. Since IS_p^T is the same as IS^T for formulas written in AP, $IS_p^T, r^T \models \neg\chi$. Hence we have shown that, unless the conclusion of INT' is valid, the premiss is not valid either.

INT is a special case of a rule which was proposed in [BV01] as part of an axiomatisation of propositional dynamic logic with *intersection*, which is the operation needed to define the indistinguishability relations of distributed knowledge: $[r]_\Gamma = \bigcap\limits_{i\in\Gamma} [r]_i$. The unravelling step in our soundness argument for *INT* is needed in order to handle our perfect recall semantics. An alternative approach to the axiomatisation of distributed knowledge was taken in [vdHM96]. That work suggests that *INT* might be expendable, but we found the technique of driving models to conform with the standard semantics of D_Γ by a sequence of specialising tranformations from that work too heavy to adapt to our setting as it has many other non-trivial features.
$LFP_{\Diamond\mathsf{D}}$: Let $s \in S(\Gamma, IS)$ and $r \in R^{fin}(IS)$. Then

$$IS, r \models \langle\langle\emptyset\rangle\rangle\Box(\mathsf{P}_\Gamma\psi \Rightarrow [\![\Gamma]\!] \circ \mathsf{P}_\Gamma\psi)$$

means that for all $r' \in R^{inf}(IS)$ and all $k \geq |r|$, the existence of an $r'' \in [r'[0..k]]_\Gamma$ such that $IS, r'' \models \psi$ implies the existence of an $r''' \in \text{out}([r'[0..k]]_\Gamma, s)$ such that $IS, r'''[0..k+1] \models \psi$. Together with $IS, r \models \mathsf{P}_\Gamma\psi$, this entails the existence of an infinite sequence of finite runs $r'_{|r|}, r'_{|r|+1}, \dots$ such that $r'_{|r|} \in [r]_\Gamma$, and for every $k \geq |r|$ there exists an $r''_{k+1} \in \text{out}([r'_k]_\Gamma, s)$ such that $r'_{k+1} \in [r''_{k+1}[0..k+1]]_\Gamma$, and $IS, r'_k \models \psi$ for all $k < \omega$. Since $r'_k[0..n] \in [r_n]_\Gamma$ for all $n \in \{|r|, \dots, k\}$, $IS, r'_k[0..n] \models \mathsf{P}_\Gamma\psi$ for all $k \geq |r|$ and all $n \in \{|r|, \dots, k\}$.

By König's Lemma, this entails the existence of an infinite $r_\omega \in \mathrm{out}([r]_\Gamma, s)$ such that $IS, r_\omega[0..k] \models \mathsf{P}_\Gamma\psi$ for all $k \geq |r|$. Since no restriction was imposed on the choice of $s \in S(\Gamma, IS)$, we have $IS, r \models [\![\Gamma]\!]\Box\mathsf{P}_\Gamma\psi$. The soundness of the rest of the axioms and rules can be established by direct checks.

The soundness of the axioms $(FP_{\Diamond\mathsf{D}})$ and $(LFP_{\Diamond\mathsf{D}})$ shows that $\langle\!\langle\Gamma\rangle\!\rangle\Diamond\mathsf{D}_\Gamma(.)$ admits a fixpoint characterisation, which is crucial for the possibility to adapt standard techniques for demonstrating the completeness of the system. No such characterisation can be achieved for general (.U.)-goals, because of the validity of the equivalence $\langle\!\langle\Gamma\rangle\!\rangle(\varphi\mathsf{U}\psi) \Leftrightarrow \mathsf{D}_\Gamma\langle\!\langle\Gamma\rangle\!\rangle(\varphi\mathsf{U}\psi)$ in which D_Γ, if considered separately, has a greatest fixpoint characterisation, whereas (.U.) is a least fixpoint. In practical terms the difficulty arises from the fact that achieving $(\varphi\mathsf{U}\psi)$ need not become known to the considered coalition immediately, that is, as soon as a state which satisfies ψ is reached for the first time, and, in the case of infinite sets of available actions, which we have ruled out, may as well never become known, despite that the right strategy is being followed.

4 Completeness of the proof system for $ATL_{iR}^{\Diamond}$

In this section we fix an arbitrary formula φ which is consistent in our proof system and construct a finite interpreted system which satisfies it. By a standard argument this entails the completeness of our proof system. For the sake of simplicity we assume that φ is consistent with I and construct an IS which satisfies φ at an initial state (0-length run.) The satisfiability of φ at any finite run is equivalent to the satisfiability of $[\![\emptyset]\!]\Diamond\varphi$ at a 0-length one. The satisfying interpreted system is built for a fixed set of agents $\Sigma = \{1, \ldots, N\}$, which is assumed to include all the agents occurring in φ, and possibly others. In complete information ATL, with no environment, satisfiability may depend on whether all the members of Σ occur in the considered formula [GS09]. In our setting, there is no such difference because the environment can simulate any number of agents and yet cannot be described as being part of any coalition in the logic.

4.1 Auxiliary propositional variables

The vocabulary AP for the interpreted system to be constructed is $\mathrm{Var}(\varphi)$. The construction involves derivability from formulas with occurrences of some fresh auxiliary variables, which we introduce next. Given $i \in \Sigma$ and $\Gamma \subseteq \Sigma$, we write $\Gamma^{<i}$ for $\Gamma \cap \{1, \ldots, i-1\}$, for the sake of brevity. The auxiliary variables that we are about to use are

$q_{\psi,i,\Gamma}$ for all formulas ψ written in $AP, \Gamma \subseteq \Sigma$ and $i \in \Gamma^{<\max\Gamma}$.

We use these variables to construct the formulas

$$p_{\psi,i,\Gamma} \rightleftharpoons q_{\psi,i,\Gamma} \wedge \bigwedge_{j \in \Gamma^{<i}} \neg q_{\psi,j,\Gamma},\ i \in \Gamma^{<\max\Gamma},\ \text{and } p_{\psi,\max\Gamma,\Gamma} \rightleftharpoons \bigwedge_{j \in \Gamma^{<\max\Gamma}} \neg q_{\psi,j,\Gamma}.$$

Obviously these formulas satisfy $\vdash \bigvee_{i\in\Gamma} p_{\psi,i,\Gamma}$ and $\vdash \neg(p_{\psi,i,\Gamma} \wedge p_{\psi,j,\Gamma})$ for $i \neq j$. We put $p_{\psi,\max\Gamma,\Gamma} \rightleftharpoons \top$ in case $|\Gamma| = 1$. We use $p_{\psi,i,\Gamma}$ to construct the formulas

$$\mathsf{D}_{i,\Gamma}\psi \rightleftharpoons \mathsf{D}_i(p_{\psi,i,\Gamma} \Rightarrow \psi),\ i \in \Gamma.$$

Informally, $p_{\psi,i,\Gamma}$ can be regarded as a condition which, together with i's other knowledge, is sufficient for i to infer ψ.

Given a set of formulas x written in AP, we write $\overline{x}$ for the set

$$x \cup \{\mathsf{D}_{i,\Gamma}\psi : \mathrm{Var}(\psi) \subseteq AP, \vdash \bigwedge x \Rightarrow \mathsf{D}_\Gamma\psi, i \in \Gamma, \Gamma \subseteq \Sigma\}.$$

Lemma 12 *Let x be a consistent set of formulas written in AP. Then $\overline{x}$ is consistent too.*

Proof: Let ψ and $\Gamma \subseteq \Sigma$ be such that $\mathsf{D}_\Gamma\psi \in x$. Assume that $\overline{x}_0$ is an inconsistent finite subset of $\overline{x}$ for the sake of contradiction. Then $\overline{x}_0 \cup \{\mathsf{D}_j(p_{\psi,j,\Gamma} \Rightarrow \psi) : j \in \Gamma\}$ is still a subset of $\overline{x}$, and obviously inconsistent too. The only formulas in the latter set which have occurrences of the propositional variables $q_{\psi,j,\Gamma}$, $j \in \Gamma^{<\max\Gamma}$ are those explicitly listed above.

Given an arbitrary set of formulas y with no occurrences of $q_{\psi,i,\Gamma}$, $i \in \Gamma \setminus \min\Gamma$, let

$$y_i = y \cup \{\mathsf{D}_j(p_{\psi,j,\Gamma} \Rightarrow \psi) : j \in \Gamma^{<i}\} \cup \{\mathsf{D}_{\Gamma\cap\{i,\dots\max\Gamma\}}(\bigwedge_{j\in\Gamma^{<i}} \neg q_{\psi,j,\Gamma} \Rightarrow \psi)\}$$

for all $i \in \Gamma$. Then, if $i > \min\Gamma$, the inconsistency of y_i implies the inconsistency of $y_{i'}$ where $i' = \max\Gamma^{<i}$, by a single application of the rule *INT* to

$$\bigwedge y_{i'} \Rightarrow \left(\begin{array}{c} \mathsf{P}_{i'}\big(q_{\psi,i',\Gamma} \wedge \bigwedge_{j\in\Gamma^{<i'}} \neg q_{\psi,j,\Gamma} \wedge \neg\psi\big)\vee \\ \mathsf{P}_{\Gamma\cap\{i,\dots\max\Gamma\}}\big(\neg q_{\psi,i',\Gamma} \wedge \bigwedge_{j\in\Gamma^{<i'}} \neg q_{\psi,j,\Gamma} \wedge \neg\psi\big) \end{array} \right),$$

which is a presumedly valid formula expressing the inconsistency of y_i. Hence the inconsistency of $y_{\max\Gamma}$ entails the inconsistency of $y_{\min\Gamma}$ which is $y \cup \{\mathsf{D}_\Gamma\psi\}$. Now choosing y to be $\overline{x}_0 \setminus \{\mathsf{D}_j(p_{\psi,j,\Gamma} \Rightarrow \psi) : j \in \Gamma\}$ entails that the inconsistency of $\overline{x}_0$ can be reduced to the inconsistency of another finite subset of $\overline{x}$, with no occurrences of the variables $q_{\psi,j,\Gamma}$, $j \in \Gamma^{<\max\Gamma}$, nor of any other variables outside AP which do not occur in $\overline{x}_0$. By repeating this reasoning we can eliminate all the formulas with auxiliary variables in them from $\overline{x}_0$ within finitely many steps, because of the finiteness of $\overline{x}_0$, at the cost of adding formulas of the form $\mathsf{D}_\Gamma\psi$, which, by the definition of $\overline{x}$, satisfy $\mathsf{D}_\Gamma\psi \in x$. The resulting inconsistent set is a subset of x, which is a contradiction. $\dashv$ The use of the formulas $\mathsf{D}_{i,\Gamma}\psi$ and the sets of the form $\overline{x}$ becomes clear further below.

4.2 An *IS*-like structure with non-distributed global states

Now we are ready to construct an interpreted system which satisfies the given consistent formula φ. We start by defining a structure which differs from an interpreted system only in the form of its global states, which are not tuples of local states.

In the sequel we denote the set

$$\{\psi, \neg\psi : \psi \in \mathrm{Subf}(\varphi) \cup \bigcup_{i\in\Gamma} \mathrm{Subf}(\mathsf{D}_i\mathsf{I}) \cup \mathrm{Subf}(\langle\langle i\rangle\rangle \circ \top)\}$$

of the subformulas of φ and the above special purpose formulas, and their negations, by C for the sake of brevity. According to $ATL^{\Diamond}_{iR}$ syntax, the well-formedness of $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\varphi$ follows from that of φ immediately. Despite that, in the definition of C above and elsewhere we assume that $\mathsf{D}_\Gamma\varphi$ is a subformula of $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\varphi$ too, like in ATL^{DP}_{iR}.

Below we use the maximal consistent subsets of C as the principal building block of global states, thus assigning to C the part commonly played by an appropriate form of *Fisher-Ladner closure*, which typically includes some more formulas that are related to the target one φ. Whenever some formula ψ is not necessarily in C, we write $\vdash \bigwedge x \Rightarrow \psi$, and, sometimes, $x \cup \{\psi\}$ *is consistent* for appropriate $x \subset C$, instead of $\psi \in x$ for the corresponding subsets x of a closure.

Let W be the set of all the maximal consistent subsets of C. W is a subset of $\mathcal{P}(C)$ and is therefore finite. Given a subset X of W, we denote the formula $\bigvee_{w\in X} \bigwedge w$ by $\widehat{X}$. For every formula of the form $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in C$ we define the sequences $W^{\Gamma,\psi}_n$ and $W^{\Gamma,\psi}_{\leq n}$ of subsets of W, $n < \omega$, by the clauses

$$W^{\Gamma,\psi}_0 = \{w \in W :\vdash \bigwedge w \Rightarrow \mathsf{D}_\Gamma\psi\};$$

$$W^{\Gamma,\psi}_{n+1} = \{w \in W \setminus W^{\Gamma,\psi}_{\leq n} : \langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in w \text{ and } w \text{ is consistent with } \langle\langle\Gamma\rangle\rangle \circ \widehat{W^{\Gamma,\psi}_{\leq n}}\};$$

$$W^{\Gamma,\psi}_{\leq n} = \bigcup_{m\leq n} W^{\Gamma,\psi}_m \text{ for all } n < \omega.$$

The sets $W_n^{\Gamma,\psi}$ ($W_{\leq n}^{\Gamma,\psi}$) consist of those maximal consistent subsets of C which state that Γ can achieve the goal $\Diamond \mathsf{D}_\Gamma\psi$ in (at most) n steps. Since W is finite, there exists an $n_0 \leq |W|-1$ such that $W_{\leq n_0}^{\Gamma,\psi} = W_{\leq n}^{\Gamma,\psi}$ for all $n \geq n_0$:

Lemma 13 $\bigcup\limits_{n<\omega} W_n^{\Gamma,\psi} = \bigcup\limits_{n\leq|W|-1} W_n^{\Gamma,\psi}.$

We write $W_+^{\Gamma,\psi}$ for $\bigcup\limits_{n<\omega} W_n^{\Gamma,\psi}$ and $W_-^{\Gamma,\psi}$ for $W \setminus W_+^{\Gamma,\psi}$, respectively. The following lemma shows that states $w \in W_-^{\Gamma,\psi}$, which, as explained above, rule out Γ achieving $\Diamond \mathsf{D}_\Gamma\psi$ in within $|W|-1$ steps, are not consistent with $\langle\!\langle\Gamma\rangle\!\rangle\Diamond \mathsf{D}_\Gamma\psi$.

Lemma 14 *Let $\langle\!\langle\Gamma\rangle\!\rangle\Diamond \mathsf{D}_\Gamma\psi \in C$ and $w \in W_-^{\Gamma,\psi}$. Then $\langle\!\langle\Gamma\rangle\!\rangle\Diamond \mathsf{D}_\Gamma\psi$ is not consistent with w.*

Proof: If $w \in W_-^{\Gamma,\psi}$, then, by the definition of $W_-^{\Gamma,\psi}$, the duality between $\langle\!\langle\Gamma\rangle\!\rangle\circ$ and D_Γ, and $[\![\Gamma]\!]\circ$ and P_Γ, respectively, and the fact that either $\mathsf{D}_\Gamma\psi \in w$, or (an equivalent of) $\mathsf{P}_\Gamma\neg\psi \in w$ because the former is a subformula of $\langle\!\langle\Gamma\rangle\!\rangle\Diamond \mathsf{D}_\Gamma\psi$, we have

$$\vdash \bigwedge w \Rightarrow \mathsf{P}_\Gamma\neg\psi \text{ and } \vdash \bigwedge w \Rightarrow [\![\Gamma]\!] \circ \mathsf{P}_\Gamma\neg\widehat{W_{\leq m}^{\Gamma,\psi}} \text{ for all } m < \omega,$$

whence

$$\vdash \widehat{W_-^{\Gamma,\psi}} \Rightarrow \mathsf{P}_\Gamma\neg\psi \text{ and } \vdash \widehat{W_-^{\Gamma,\psi}} \Rightarrow [\![\Gamma]\!] \circ \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}},$$

respectively. By applying an extra P_Γ to each side of $\Rightarrow$ in the above formulas, as possible due to $\mathbf{K}_\mathsf{D}$ and N_D, and then using the equivalences

$$[\![\Gamma]\!] \circ \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \Leftrightarrow \mathsf{P}_\Gamma[\![\Gamma]\!] \circ \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \text{ and } \mathsf{P}_\Gamma\mathsf{P}_\Gamma[\![\Gamma]\!] \circ \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \Leftrightarrow \mathsf{P}_\Gamma[\![\Gamma]\!] \circ \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}},$$

which can be derived from the axioms $\mathsf{D}_\circ$, $\mathbf{4}_\mathsf{D}$ and $\mathbf{T}_\mathsf{D}$, and, similarly, the equivalence $\mathsf{P}_\Gamma\mathsf{P}_\Gamma\neg\psi \Leftrightarrow \mathsf{P}_\Gamma\neg\psi$, we obtain

$$\vdash \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \Rightarrow \mathsf{P}_\Gamma\neg\psi \text{ and } \vdash \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \Rightarrow [\![\Gamma]\!] \circ \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}}.$$

By the rule $N_{\langle\!\langle\emptyset\rangle\!\rangle\Box}$, we have $\vdash \langle\!\langle\emptyset\rangle\!\rangle\Box(\mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \Rightarrow [\![\Gamma]\!] \circ \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}})$. Since

$$\mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \wedge \langle\!\langle\emptyset\rangle\!\rangle\Box(\mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \Rightarrow [\![\Gamma]\!] \circ \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}}) \Rightarrow [\![\Gamma]\!]\Box\mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}}$$

is an instance of axiom $LFP_{\Diamond\mathsf{D}}$, we infer $\vdash \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}} \Rightarrow [\![\Gamma]\!]\Box\mathsf{P}_\Gamma\neg\psi$. Together with $\vdash \widehat{W_-^{\Gamma,\psi}} \Rightarrow \mathsf{P}_\Gamma\widehat{W_-^{\Gamma,\psi}}$, this entails the lemma. ⊣ In the rest of the completeness proof we take account of the fact that the possibility of achieving any single goal of the form $\Diamond \mathsf{D}_\Gamma\psi$ within $|W|$ steps is based on the consistency of ws from W with steps which *make progress* on the goal in question, i.e., change a state in $W_{\leq x+1}^{\Gamma,\psi}$ to one in $W_{\leq x}^{\Gamma,\psi}$ for some $x < |W|-1$, and does not imply that progress can be made on two or more $\Diamond \mathsf{D}_\Gamma$-goals in parallel. Next we define the states of the IS-like structure which we are constructing to include a $w \in W$, and two numbers $k < |C||W|$ and $s \in \{0,1\}$. The purpose of k and s is to identify a $\Diamond \mathsf{D}_\Gamma$-goal from C, which, if achievable according to w, is allowed the exclusive opportunity to make progress from the considered state. All the other $\Diamond \mathsf{D}_\Gamma$-goals which are achievable too *preserve their prospects* without necessarily making progress, i.e., the corresponding formulas $\langle\!\langle\Gamma\rangle\!\rangle\Diamond \mathsf{D}_\Gamma\chi \in w$ recur in successor states, until a state with the appropriate value of k is reached.

Definition 15 Let $\langle\!\langle\Gamma_q\rangle\!\rangle\Diamond \mathsf{D}_{\Gamma_q}\psi_q$, $q = 0, \ldots, M-1$, be all the formulas of the form $\langle\!\langle\Gamma\rangle\!\rangle\Diamond \mathsf{D}_\Gamma\psi$ in C. The set of the global states of our IS-like structure is $S = W \times \{0, \ldots, M|W|-1\} \times \{0,1\}$.

As it becomes clear from the definition of the transition relation below, behaviours always start at states $\langle w, k, s\rangle$ with $k = 0$. Transitions increase k modulo $M|W|$. This way k partitions every run into intervals of length $M|W|$, each consisting of M subintervals of length $|W|$, the qth subinterval being a *window of opportunity* for making progress on $\langle\langle\Gamma_q\rangle\rangle\Diamond \mathsf{D}_{\Gamma_q}\psi_q$, $q = 0, \ldots, M-1$. The value of s indicates whether Γ_q has been pursuing $\Diamond \mathsf{D}_{\Gamma_q}\psi_q$ continuously from the beginning of the window of opportunity, that is, whether it is early enough for the goal to be reached within the window of opportunity, as it may take up to $|W|$ steps, which is the full length of the window, for the goal to be achieved. Transition to a state with $s = 1$ is possible in the beginning of the window of opportunity for $\Diamond \mathsf{D}_{\Gamma_q}\psi_q$, and within it, only in case the latest action of Γ_q is aimed at $\Diamond \mathsf{D}_{\Gamma_q}\psi_q$. Throughout the window for $\Diamond \mathsf{D}_{\Gamma_q}\psi_q$, action tuples in which the members of Γ_q are not unanimous about $\Diamond \mathsf{D}_{\Gamma_q}\psi_q$ lead to successor states with $s = 0$, in which $\Diamond \mathsf{D}_{\Gamma_q}\psi_q$ may as well be abandoned.

Definition 16 An *action for agent* $i \in \Sigma$ is a formula of the form $\langle\langle\Gamma\rangle\rangle \circ \gamma$ such that $i \in \Gamma$ and $\gamma \in C$.

Intuitively, action $\langle\langle\Gamma\rangle\rangle \circ \gamma$ is the part of agent i in Γ's effort to achieve the goal γ. Agent i can perform $\langle\langle\Gamma\rangle\rangle \circ \gamma$, regardless of whether the rest of the members of Γ are also doing γ, and regardless of whether Γ can achieve γ in the reference state at all. However, unless these two conditions are met, this is not guaranteed to bring γ.

Definition 17 Let $a = \langle\langle\langle\Gamma_1\rangle\rangle \circ \gamma_1, \ldots, \langle\langle\Gamma_N\rangle\rangle \circ \gamma_N\rangle$ be a vector of actions, one for every agent from Σ, and let $\Gamma \subseteq \Sigma$. Γ is said to be *unanimous* in a if $\Gamma_i = \Gamma$ and γ_i is the same formula for all $i \in \Gamma$.

Unlike agents' actions, environment's actions are aimless; they just ensure that whatever agents do not prevent can actually happen. The description of a successor state always includes the goals achieved by agent actions and some formulas about the past. We assume that the environment works to complete this description by trying to include into it each of the remaining formulas from C in some order of its choosing. Only the formulas which do not destroy consistency become added. The result is a description of the new state as a maximal consistent subset of C and depends on the chosen ordering of C.

Definition 18 An *environment action* is a linear ordering of the formulas from C.

Let Act_i be the set of the actions for agent i, $i \in \Sigma$, and Act_e be the set of the environment actions. We define $t_0 : S \times Act_{\Sigma_e} \to S$ as follows:

Let $\langle w, k, s\rangle \in S$, $a = \langle\langle\langle\Delta_1\rangle\rangle \circ \delta_1, \ldots, \langle\langle\Delta_N\rangle\rangle \circ \delta_N, a_e\rangle \in Act_{\Sigma_e}$, $a_e = \theta_0 < \ldots < \theta_{|C|-1}$. Let the formula $d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i, k, s)$ be defined for $i = 1, \ldots, N$, $k \in \{0, \ldots, M|W|-1\}$ as follows:

$$d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i, k, s) = \begin{cases} \mathsf{D}_{\Gamma_q}\widehat{W^{\Gamma_q,\psi_q}_{\leq |W|-r-2}} & \text{if } \langle\langle\Delta_i\rangle\rangle \circ \delta_i \text{ is } \langle\langle\Gamma_q\rangle\rangle \circ \langle\langle\Gamma_q\rangle\rangle\Diamond \mathsf{D}_{\Gamma_q}\psi_q \\ & \text{and } s = 1; \\ \mathsf{D}_{\Delta_i}\delta_i & \text{otherwise,} \end{cases}$$

where $q = k \text{ div } |W|$, $r = k \bmod |W|$, and $W^{\Gamma_q,\psi_q}_{\leq |W|-r-2}$ is defined with respect to the given w. Then we put[1]

$$t_0(w, k, s, a) = \langle w_{environment} \cup (w_{past} \cup w_{actions}) \cap C, (k+1) \bmod M|W|, s'\rangle$$

where the sets of formulas w_{past} and $w_{actions}$ are meant to encode the properties of $t_0(w, k, s, a)$ which follow from its being a successor of w, and the outcome of the action tuple a, respectively, and the set $w_{environment}$ is meant to add whatever other properties from those expressed by formulas from C can be consistently attriibuted to $t_0(w, k, s, a)$, as a result of the environment action. The three sets of formulas are defined as follows:

$w_{past} = \{\ominus\psi :\vdash \bigwedge w \Rightarrow \psi\}$,

[1]Here and in the sequel we write $t_0(w, k, s, \ldots)$, $f(t, k, s)$, etc., instead of $t_0(\langle w, k, s\rangle, \ldots)$, $f(\langle t, k, s\rangle)$, etc., for better readability.

$w_{actions} = \{d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i, k, s) : \vdash \bigwedge w \Rightarrow \langle\langle\Delta_i\rangle\rangle \circ \delta_i$ and Δ_i is unanimous in $a,\ i \in \Sigma\}$,
$w_{environment}$ is determined by the condition

$$\theta_m \in w_{environment} \text{ iff } w_{past} \cup w_{actions} \cup (w_{environment} \cap \{\theta_0, \ldots, \theta_m\}) \text{ is consistent}$$

for $m = 0, \ldots, |C|-1$. This condition can be also spelled out as including θ_m in $w_{environment}$ at step m iff θ_m is consistent with $w_{past} \cup w_{actions}$ together with those of $\theta_0, \ldots, \theta_{m-1}$ which have been added to $w_{environment}$ at previous steps. To define s', let $q' = (k+1)$ div $|W|$, $r' = (k+1)\text{mod}|W|$; then $s' = 1$ iff either $r' = 0$ and

$$\vdash \bigwedge(w_{environment} \cup (w_{past} \cup w_{actions}) \cap C) \Rightarrow \langle\langle\Gamma_{q'}\rangle\rangle\Diamond\mathsf{D}_{\Gamma_{q'}}\psi_{q'},$$

or $r' \neq 0$, $s = 1$, $\vdash \bigwedge w \Rightarrow \langle\langle\Gamma_{q'}\rangle\rangle \circ \langle\langle\Gamma_{q'}\rangle\rangle\Diamond\mathsf{D}_{\Gamma_{q'}}\psi_{q'}$ and $\langle\langle\Gamma_{q'}\rangle\rangle \circ \langle\langle\Gamma_{q'}\rangle\rangle\Diamond\mathsf{D}_{\Gamma_{q'}}\psi_{q'}$ is $\langle\langle\Delta_i\rangle\rangle \circ \delta_i$ for all $i \in \Gamma_{q'}$.

We need to prove that $w_{environment} \cup (w_{past} \cup w_{actions}) \cap C \in W$.

Lemma 19 *The union $w_{environment} \cup (w_{past} \cup w_{actions}) \cap C$ is a maximal consistent subset of C.*

Proof: Since a_e is an ordering of all the formulas from C, the defining condition for $w_{environment}$ entails the maximality, provided that $w_{past} \cup w_{actions}$ is consistent. Next we prove the consistency of $w_{past} \cup w_{actions}$. Without loss of generality we can assume that $w_{actions}$ consists of $\mathsf{D}_{\Delta_i}\delta_i$, $i = 1, \ldots, m$, for some $m \leq N$, possibly $\top$, and possibly $\mathsf{D}_{\Gamma_q}\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}}$, where $q = k$ div $|W|$ and $r = k\text{mod}|W|$ like previously, in case Γ_q is unanimous in a, $\langle\langle\Gamma_q\rangle\rangle \circ \langle\langle\Gamma_q\rangle\rangle\Diamond\mathsf{D}_{\Gamma_q}\psi_q$ occurs in a, $\vdash \bigwedge w \Rightarrow \langle\langle\Gamma_q\rangle\rangle \circ \langle\langle\Gamma_q\rangle\rangle\Diamond\mathsf{D}_{\Gamma_q}\psi_q$ and $s = 1$. We assume that the latter conditions are met and therefore $\mathsf{D}_{\Gamma_q}\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}} \in w_{actions}$. These conditions entail that $w \in W^{\Gamma_q,\psi_q}_{\leq|W|-r-1}$. The case of no $\mathsf{D}_{\Gamma_q}\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}}$ in $w_{actions}$ is simpler and we skip it. We also ignore the possible presence of $\top$ in $w_{actions}$ as it does not affect the consistency. The condition $w \in W^{\Gamma_q,\psi_q}_{\leq|W|-r-1}$ entails that w is consistent with $\langle\langle\Gamma_q\rangle\rangle \circ \widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}}$. The unanimity of the coalitions $\Delta_1, \ldots, \Delta_m, \Gamma_q$, in $a_1, \ldots, a_N$ entails that these coalitions are disjoint. Hence $\vdash \bigwedge w \Rightarrow \langle\langle\Delta_1\rangle\rangle \circ \delta_1 \wedge \ldots \wedge \langle\langle\Delta_m\rangle\rangle \circ \delta_m$ and the consistency of w with $\langle\langle\Gamma_q\rangle\rangle \circ \widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}}$ imply that w is consistent with $\langle\langle\Sigma\rangle\rangle \circ (\mathsf{D}_{\Delta_1}\delta_1 \wedge \ldots \wedge \mathsf{D}_{\Delta_m}\delta_m \wedge \mathsf{D}_{\Gamma_q}\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}})$, by axioms S and $\mathsf{D}_\circ$. Assume that $\{\mathsf{D}_{\Delta_1}\delta_1, \ldots, \mathsf{D}_{\Delta_m}\delta_m, \mathsf{D}_{\Gamma_q}\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}}\} \cup w_{past}$ is inconsistent for the sake of contradiction. Then there exists a finite $w^0_{past} \subset w_{past}$ such that $\vdash \mathsf{D}_{\Delta_1}\delta_1 \wedge \ldots \wedge \mathsf{D}_{\Delta_m}\delta_m \wedge \mathsf{D}_{\Gamma_q}\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}} \Rightarrow \neg \bigwedge w^0_{past}$. Then, by the rule $Mono_{\langle\langle.\rangle\rangle\circ}$ and the axioms $\langle\langle.\rangle\rangle \circ \ominus$ and $\mathbf{T}_\mathsf{D}$, and the definition of w_{past}, this entails $\vdash \langle\langle\Sigma\rangle\rangle \circ (\mathsf{D}_{\Delta_1}\delta_1 \wedge \ldots \wedge \mathsf{D}_{\Delta_m}\delta_m \wedge \mathsf{D}_{\Gamma_q}\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-r-2}}) \Rightarrow \neg \bigwedge w$, which is a contradiction. ⊣ Importantly, despite that many formulas from w_{past} and $w_{actions}$ may be left out of $t_0(w, k, s, a)$ because of not being in C, all of the logical consequences of these formulas which *are* in C are bound to be included in $t_0(w, k, s, a)$ as members of $w_{environment}$.

We define the remaining components $V_0 \subseteq S \times AP$ and $I_0 \subseteq S$ of our IS-like structure by the clauses

$$V_0(w, k, s, p) \leftrightarrow p \in w \text{ and } I_0 = \{\langle w, 0, s\rangle \in S : \mathsf{I} \in w, s = 1 \text{ iff } \langle\langle\Gamma_0\rangle\rangle\Diamond\mathsf{D}_{\Gamma_0}\psi_0 \in w\}.$$

4.3 An interpreted system satisfying the given consistent formula φ

The structure $IS_0 = \langle S, I_0, \langle Act_i : i \in \Sigma_e\rangle, t_0, V_0\rangle$ differs from an interpreted system by the form of the state space S, which consists of "simple" global states rather than tuples of local states. Consequently the properties of transition functions in interpreted systems which are related to that form cannot be formulated straightforwardly for t_0. Next we build an interpreted system

$$IS = \langle\langle L_i : i \in \Sigma_e\rangle, I, \langle Act_i : i \in \Sigma_e\rangle, t, V\rangle$$

which corresponds to IS_0. We define L_e as $W \times \{0, \ldots, M|W|-1\} \times \{0, 1\}$, i.e., L_e is the state space S of IS_0. L_i is the set of the (not necessarily maximal) consistent subsets of $\overline{C}$ for all $i \in \Sigma$. (Recall the definition of $\overline{x}$ for sets of formulas x from Section 4.1.) We define the mapping $f : S \to L_{\Sigma_e}$ by the clauses

$$(f(w, k, s))_i = \{\psi : \mathsf{D}_i\psi \in \overline{w}\} \text{ for } i \in \Sigma \text{ and } (f(w, k, s))_e = \langle w, k, s\rangle.$$

The consistency of the sets $(f(w, k, s))_i$ follows from the consistency of the corresponding $\overline{w}$ by $\mathbf{T}_{\mathsf{D}}$; the latter follows from the consistency of the respective $w \in W$, by Lemma 12. Since $(f(w, k, s))_e = \langle w, k, s\rangle$, f is injective. Note that $(f(w, k, s))_i$ contains the formulas $\mathsf{D}_{i,\Gamma}\psi$ from Section 4.1 for formulas $\mathsf{D}_\Gamma\psi$ such that $\vdash \bigwedge w \Rightarrow \mathsf{D}_\Gamma\psi$, as they have the form $\mathsf{D}_i\chi$.

We define $I \subseteq L_{\Sigma_e}$ as $f(I_0)$ and V by the equivalence $V(l, p) \leftrightarrow V_0(l_e, p)$.

We define $t : L_{\Sigma_e} \times Act_{\Sigma_e} \to L_{\Sigma_e}$ for arbitrary $a \in Act_{\Sigma_e}$ and $l \in f(S)$ by the equality

$$t(f(w, k, s), a) = f(t_0(w, k, s, a)).$$

This definition entails that $f(S)$ is closed under t. Since $I \subset f(S)$, the states in $L_{\Sigma_e} \setminus f(S)$ are unreachable. This renders definition of t on $L_{\Sigma_e} \setminus f(S)$ irrelevant.

Lemma 20 *If $i \in \Sigma$, $l', l'' \in f(S)$, $l'_i = l''_i$ and $l'_e = l''_e$, then $(t(l', a))_i = (t(l'', a))_i$.*

Proof: The equality $l'_e = l''_e$ alone entails that $l' = l'' = f(w, k, s)$ for $\langle w, k, s\rangle = l'_e = l''_e$. $\dashv$ Below we prove that $IS, l \models \varphi$ for all $l \in I$ as a corollary to a standard *truth lemma*, which, in our setting, is the statement that[2]

$$IS, f(w^0, k^0, s^0)a^0 \ldots a^{n-1} f(w^n, k^n, s^n) \models \psi \text{ iff } \psi \in w^n$$

for all $\psi \in C$ and all $f(w^0, k^0, s^0)a^0 \ldots a^{n-1} f(w^n, k^n, s^n) \in R^{fin}(IS)$. We prove this statement by induction on the construction of formulas ψ. The proof is partitioned into lemmata, two for each of the possible main connectives D_Γ, $\langle\langle\Gamma\rangle\rangle\circ$ and $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma$ in ψ. The proofs of the lemmata below can be found in Appendix A.

Lemma 21 *Let $n < \omega$, $\Gamma \subseteq \Sigma$, $\mathrm{Var}(\psi) \subseteq AP$ and $\vdash \bigwedge w^n \Rightarrow \mathsf{D}_\Gamma\psi$. Then $\vdash \bigwedge v^n \Rightarrow \mathsf{D}_\Gamma\psi$ and $\vdash \bigwedge v^n \Rightarrow \psi$ for all $f(v^0, k^0, t^0)b^0 \ldots b^{n-1} f(v^n, k^n, t^n) \in [r]_\Gamma$.*

Definition 22 Given $w \in W$, we write $D_\Gamma(w)$ for the set of formulas

$$\{\psi : \vdash \bigwedge w \Rightarrow \psi \text{ and } \psi \text{ is of one of the forms } \mathsf{D}_\Gamma\chi \text{ and } \neg\mathsf{D}_\Gamma\chi\}.$$

Lemma 21 entails that if

$$f(v^0, k^0, t^0)b^0 \ldots b^{n-1} f(v^n, k^n, t^n) \sim_\Gamma f(w^0, k^0, s^0)a^1 \ldots a^n f(w^n, k^n, s^n),$$

then $D_\Gamma(v^n) = D_\Gamma(w^n)$.

Lemma 23 *Let $n < \omega$, $r \in R^n(IS)$, $\Gamma \subseteq \Sigma$. Let $\vdash \bigwedge v^n \Rightarrow \psi$ for all $f(v^0, k^0, t^0)b^0 \ldots b^{n-1} f(v^n, k^n, t^n) \in [r]_\Gamma$. Let $f(w^n, k^n, s^n)$ be the last state of r. Then $\vdash \bigwedge w^n \Rightarrow \mathsf{D}_\Gamma\psi$.*

Lemma 24 *Let $n < \omega$, $r \in R^n(IS)$,*

$$r = f(w^0, k^0, s^0)a^0 \ldots a^{n-1} f(w^n, k^n, s^n),$$

$\Gamma \subseteq \Sigma$ and $\langle\langle\Gamma\rangle\rangle\circ\psi \in w^n$. Then there exists a $g \in Act_\Gamma$ such that, if $f(v^0, k^0, t^0)b^0 \ldots b^{n-1} f(v^n, k^n, t^n) \in [r]_\Gamma$, $b^n \in Act_{\Sigma_e}$, $(b^n)_\Gamma = g$ and $t_0(v^n, k^n, t^n, b^n) = \langle v^{n+1}, k^{n+1}, t^{n+1}\rangle$, then $\psi \in v^{n+1}$.

[2] Here and below the symbols in the superscript position in expressions such as k^n and s^n denote that these expressions are different names for numbers and not exponentiation.

Lemma 25 *Given* n, r *and* Γ *as in Lemma 24,* $\psi \in C$ *and* $g \in Act_\Gamma$*, if* $t_0(v^n, k^n, t^n, b^n) = \langle v^{n+1}, k^{n+1}, t^{n+1}\rangle$ *entails* $\psi \in v^{n+1}$ *for all* $f(v^0, k^0, t^0)b^0 \dots b^{n-1} f(v^n, k^n, t^n) \in [r]_\Gamma$ *and all* $b^n \in Act_{\Sigma_e}$ *such that* $(b^n)_\Gamma = g$*, then* w^n *is consistent with* $\langle\langle\Gamma\rangle\rangle \circ \psi$*.*

Lemma 26 *Let* $n < \omega$*,* $r \in R^n(IS)$*,* $\Gamma \subseteq \Sigma$ *and let* $\langle\langle\Gamma\rangle\rangle\Diamond \mathsf{D}_\Gamma\psi$ *appear in the last state of* r*. Then there exists a strategy* s *for* Γ *such that for every* $f(w^0, k^0, s^0)a^0 \dots a^{m-1} f(w^m, k^m, s^m) \dots \in \mathrm{out}([r]_\Gamma, s)$ *there exists a* $k < \omega$ *such that* $\mathsf{D}_\Gamma\psi \in w^{n+k}$*.*

Lemma 27 *Let* $n < \omega$*,* $r \in R^n(IS)$*,* $\Gamma \subseteq \Sigma$ *and* $\langle\langle\Gamma\rangle\rangle\Diamond \mathsf{D}_\Gamma\psi \in C$*. Let there exist a strategy* s *for* Γ *such that for every*

$$r' = f(w^0, k^0, s^0)a^0 \dots a^{m-1} f(w^m, k^m, s^m) \dots \in \mathrm{out}([r]_\Gamma, s)$$

there exists a $k < \omega$ *such that* $\mathsf{D}_\Gamma\psi \in w^{n+k}$*. Then* $\langle\langle\Gamma\rangle\rangle\Diamond \mathsf{D}_\Gamma\psi \in w^n$*.*

Now we are ready to formulate and prove the truth lemma itself. This essentially concludes the completeness proof for $ATL_{iR}^{\Diamond}$ because the equivalence between $\varphi \in w_0$ and $IS, w_0 \models \varphi$, which the truth lemma entails, shows that the given consistent formula φ is also satisfiable.

Lemma 28 (Truth Lemma) *Let* $\psi \in C$ $m < \omega$ *and* $r = f(w^0, k^0, s^0)a^0 \dots a^{m-1} f(w^m, k^m, s^m) \in R^m(IS)$*. Then* $IS, r \models \psi$ *is equivalent to* $\psi \in w^m$*.*

Proof: Induction on the construction of ψ. We skip the trivial cases of ψ being $\bot$, a propositional variable, or of the form $\chi_1 \Rightarrow \chi_2$.

Let ψ be $\ominus\chi$. In case $m = 0$, we have $\mathsf{I} \in w^m$, whence by the validity of $\psi \Rightarrow \top$ and the rule $Mono_\ominus$ and axiom $\ominus\bot$ we obtain $\ominus\psi \notin w^m$, which concurs with the fact that $IS, r \not\models \ominus\psi$ for $r \in R^0(IS)$.

Let $m > 0$. Then $IS, r \models \ominus\psi$ is equivalent to

$$IS, f(w^0, k^0, s^0)a^0 \dots a^{m-1} f(w^{m-1}, k^{m-1}, s^{m-1}) \models \psi.$$

By the inductive hypothesis this is equivalent to $\psi \in w^{m-1}$, whence $\ominus\psi \in w^{m-1}_{past}$ as defined with respect to a^{m-1}. Now the definition of t implies that $\ominus\psi \in w^m$.

Let ψ be $(\chi_1 \mathsf{S} \chi_2)$. Then $IS, r \models \psi$ is equivalent to the existence of an $i \leq m$ such that

$$IS, f(w^0, k^0, s^0)a^0 \dots a^{m-1} f(w^i, k^i, s^i) \models \chi_2$$

and

$$IS, f(w^0, k^0, s^0)a^0 \dots a^{m-1} f(w^j, k^j, s^j) \models \chi_1 \text{ for } j = i+1, \dots, m.$$

By the induction hypothesis this is equivalent to $\chi_2 \in w^i$ and $\chi_1 \in w^{i+1}, \dots, w^m$. By a repeated use of the axiom $FP_{(.\mathsf{S}.)}$ the latter implies $\psi \in w^i, \dots, w^m$. It remains to be shown that $\psi \in w^m$ implies the existence of an $i \leq m$ such that $\chi_2 \in w^i$ and $\chi_1 \in w^{i+1}, \dots, w^m$. We do this by induction on m. If $m = 0$, then $\vdash \bigwedge w^m \Rightarrow \neg\ominus\psi$ and therefore $\chi_2 \in w^m$ by axiom $FP_{(.\mathsf{S}.)}$. Hence we can put $i = m = 0$. Let $m > 0$. Then, if $\chi_2 \in w^m$, i can be chosen to be m again. If $\chi_2 \notin w^m$, then axiom $FP_{(.\mathsf{S}.)}$ entails that $\vdash \bigwedge w^m \Rightarrow \ominus\psi \wedge \chi_1$, whence $\chi_1 \in w^m$ and $\psi \in w^{m-1}$ can be established by reasoning as in the case about ψ being a $\ominus$-formula. Now, by our induction hypothesis for $m-1$, there exists an $i \leq m-1$ such that $\chi_2 \in w^i$ and $\chi_1 \in w^{i+1}, \dots, w^{m-1}$. Together with $\chi_2 \in w^m$, which we have established already, this completes the proof that an i satisfying $\chi_2 \in w^i$ and $\chi_1 \in w^{i+1}, \dots, w^m$ exists.

Let ψ be $\mathsf{D}_\Gamma\chi$ where $\Gamma \subseteq \Sigma$. Now $IS, r \models \psi$ is equivalent to $IS, r' \models \chi$ for all $r' \in [r]_\Gamma$. By the induction hypothesis this is equivalent to $\chi \in v^m$ for all v^m such that there is an $r' \in [r]_\Gamma$ of the form $f(v^0, k^0, t^0)a^0 \dots a^{m-1} f(v^m, k^m, t^m)$. Now, by Lemmata 21 and 23, the latter is equivalent to $\psi \in v^m$ for all such v^m, including $\psi \in w^m$.

The cases of ψ being of one of the forms $\langle\langle\Gamma\rangle\rangle \circ \chi$ and $\langle\langle\Gamma\rangle\rangle\Diamond \mathsf{D}_\Gamma\chi$ are handled similarly by means of Lemmata 24 and 25, and Lemmata 26 and 27, respectively. The case of ψ being of the form $\langle\langle\Gamma\rangle\rangle\Box\chi$ is handled by repeated use of Lemmata 24 and 25 in combination with the corresponding instance of axiom $FP_\Box$. $\dashv$

5 Completeness of flat ATL_{iR}^{DP}

Despite that we do not know whether adding (1) and (2) to our system for $ATL_{iR}^{\Diamond}$ is sufficient for the complete axiomatisation of ATL_{iR}^{DP} on the class of interpreted systems with finite branching, completeness and finite model property can be established for a subset of the logic which is substantially greater than $ATL_{iR}^{\Diamond}$ using a validity preserving translation into $ATL_{iR}^{\Diamond}$ based on (1) and (2). We call this subset *flat* ATL_{iR}^{DP} because of the restriction of $\langle\langle.\rangle\rangle(.\mathsf{U}.)$- and $[\![.]\!](.\mathsf{U}.)$-subformulas not to occur in the scope of some of the temporal or cooperation modalities, namely $\langle\langle.\rangle\rangle(.\mathsf{U}.)$, $[\![.]\!](.\mathsf{U}.)$ and $(.\mathsf{S}.)$. Ruling out the occurrences of $\langle\langle.\rangle\rangle(.\mathsf{U}.)$- and $[\![.]\!](.\mathsf{U}.)$-subformulas in the scope of $\langle\langle.\rangle\rangle(.\mathsf{U}.)$, $[\![.]\!](.\mathsf{U}.)$ and $(.\mathsf{S}.)$ makes it straightforward to determine the lengths of the runs at which, according to the definition of $\models$, the satisfaction of these subformulas can affect the satisfaction of the given formula, provided that we are interested in the satisfaction of the given formula at an initial state. The role of the parameter d in the translation of flat ATL_{iR}^{DP} into $ATL_{iR}^{\Diamond}$ below is to keep track of reference runs' lengths.

To prove the following proposition, it is sufficient to notice that the satisfaction of all the occurrences of $\langle\langle.\rangle\rangle(.\mathsf{U}.)$, $[\![.]\!](.\mathsf{U}.)$ which are substantially affected by $\mathsf{t}(.,d)$ is relevant only at runs of length d, which satisfy $\mathrm{level}_\Gamma\ominus^d\mathsf{I}$.

Proposition 29 (completeness of flat ATL_{iR}^{DP}) *Let φ be a formula in ATL_{iR}^{DP} in which no $\langle\langle.\rangle\rangle(.\mathsf{U}.)$- or $[\![.]\!](.\mathsf{U}.)$-subformulas occur in the scope of the operators $\langle\langle.\rangle\rangle(.\mathsf{U}.)$, $[\![.]\!](.\mathsf{U}.)$ or $(.\mathsf{S}.)$, except if these subformulas abbreviate to one of the forms $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi$ and $[\![\Gamma]\!]\Diamond\psi$. Let $\mathsf{t}(\psi,d)$, for formulas ψ and (possibly negative) integers d, be defined by the clauses*

$$\begin{array}{lcl}
\mathsf{t}(\bot,d) & \rightleftharpoons & \bot;\\
\mathsf{t}(p,d) & \rightleftharpoons & p;\\
\mathsf{t}(\psi_1\Rightarrow\psi_2,d) & \rightleftharpoons & \mathsf{t}(\psi_1,d)\Rightarrow\mathsf{t}(\psi_2,d);\\
\mathsf{t}(\mathsf{D}_\Gamma\psi,d) & \rightleftharpoons & \mathsf{D}_\Gamma\mathsf{t}(\psi,d);\\
\mathsf{t}(\langle\langle\Gamma\rangle\rangle\circ\psi,d) & \rightleftharpoons & \langle\langle\Gamma\rangle\rangle\circ\mathsf{t}(\psi,d+1);\\
\mathsf{t}(\langle\langle\Gamma\rangle\rangle(\psi_1\mathsf{U}\psi_2),d) & \rightleftharpoons & \begin{cases}\langle\langle\Gamma\rangle\rangle(\psi_1\mathsf{U}\psi_2) & \textit{if } \langle\langle\Gamma\rangle\rangle(\psi_1\mathsf{U}\psi_2) \textit{ is of the form } \langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\chi;\\ \langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\diamondminus(\psi_2\wedge(\ominus\psi_1\mathsf{S}\ominus^d\mathsf{I})), & \textit{if } \langle\langle\Gamma\rangle\rangle(\psi_1\mathsf{U}\psi_2) \textit{ is not of the above form and } d\geq 0;\\ \bot, & \textit{otherwise;}\end{cases}\\
\mathsf{t}([\![\Gamma]\!](\psi_1\mathsf{U}\psi_2),d) & \rightleftharpoons & \begin{cases}[\![\Gamma]\!](\psi_1\mathsf{U}\psi_2) & \textit{if } [\![\Gamma]\!](\psi_1\mathsf{U}\psi_2)i \textit{ is of the form } [\![\Gamma]\!]\Diamond\chi;\\ [\![\Gamma]\!]\Diamond\mathsf{P}_\Gamma\diamondminus(\psi_2\wedge(\ominus\psi_1\mathsf{S}\ominus^d\mathsf{I})), & \textit{if } [\![\Gamma]\!](\psi_1\mathsf{U}\psi_2) \textit{ is not of the above form and } d\geq 0;\\ \bot, & \textit{otherwise;}\end{cases}\\
\mathsf{t}(\ominus\psi,d) & \rightleftharpoons & \ominus\mathsf{t}(\psi,d-1);\\
\mathsf{t}((\psi_1\mathsf{S}\psi_2),d) & \rightleftharpoons & (\psi_1\mathsf{S}\psi_2).
\end{array}$$

Then φ is valid at the runs of length 0 of all IS with finitely many initial states and finitely many successors to every state iff $\models_{ATL_{iR}^{\Diamond}}\mathsf{t}(\varphi,0)$ or, by our completeness result, $\vdash_{ATL_{iR}^{\Diamond}}\mathsf{t}(\varphi,0)$.

6 Model-checking ATL_{iR}^{DP}

The *model-checking problem* for ATL_{iR}^{DP} is to decide whether $IS\models\varphi$ for a given formula φ and interpreted system IS. Our model-checking procedure for ATL_{iR}^{DP} builds on model checking techniques for CTL with knowledge modalities and ATL with complete information. It works by recursion on the construction of formulas. Given a formula φ with the main connective being either $\langle\langle\Gamma\rangle\rangle$ or D_Γ, the procedure involves refining the given interpreted system IS to an interpreted system $\widehat{IS}_\Gamma$ in which for every state l, either all the finite runs which end at l satisfy φ, or all the finite runs which end at l satisfy $\neg\varphi$. This essentially means that in the refined interpreted systems all the information about the satisfaction of formulas in which the main connective is related to the distributed knowledge for coalition Γ can be recovered from the *states* of the system $\widehat{IS}_\Gamma$ (instead of the *runs* of the original system IS). To this end the refinement step includes as a variant of the *subset construction* as known from [Rei84]. The construction of refined state spaces is inspired by [CDHR06] where two-player games in which one of the players has incomplete information are transformed into equivalent complete information games. Unlike that setting, we handle objectives which may be unobservable to the respective coalitions. The main differences in our technique arise from the need to handle goals of the form $(\varphi \mathsf{U} \psi)$ despite that the considered coalition is not in a position to detect a ψ-state when it encounters one immediately.

Before going to the construction, we give some preliminaries on automata on infinite trees which we use in some key steps in the construction of refined arenas. Given a set Δ, a Δ-*labeled tree* is a partial function $T : \mathbb{N}^* \rightharpoonup \Delta$ such that the set $\mathrm{supp}(T)$ of the tree *nodes* contains ε and is prefix-closed; trees are "full", that is, if $xi \in \mathrm{supp}(T)$, then $xj \in \mathrm{supp}(T)$ for all $j \leq i$ too, and all tree branches are infinite. Infinite branches are also called *paths*. A path is *initialized* if its first node is ε. We denote the *set* $\{T(x_k) : k \geq 0\}$ of the labels on path $\pi = x_0 \ldots x_1 \ldots x_k \ldots$ by $T(\pi)$.

Below we use tree automata of the form $\langle Q, \Sigma, \delta, q_0, \mathcal{F}\rangle$ with set of *states* Q, *alphabet* Σ, *initial* state $q_0 \subseteq Q$, *transition relation* $\delta \subseteq Q \times \Sigma \times (2^Q \setminus \emptyset)$ and *acceptance condition* $\mathcal{F} \subseteq 2^Q$.

A tree automaton accepts $Q \times \Sigma$-labelled trees. Given a tree $T : \mathbb{N}^* \rightharpoonup Q \times \Sigma$, let $T_Q(x)$ and $T_\Sigma(x)$ denote q and σ, respectively, for $x \in \mathrm{supp}(T)$ and $\langle q, \sigma\rangle = T(x)$. Then T is *accepted* iff:

$T_Q(\varepsilon) = q_0$;
if $xi, xj \in \mathrm{supp}(t)$ and $T_Q(xi) = T_Q(xj)$, then $i = j$;
$\langle T_Q(x), T_\Sigma(x), \{T_Q(xi) : xi \in \mathrm{supp}(t)\}\rangle \in \delta$ for all $x \in \mathrm{supp}(T)$;
$\{T_Q(x) : x \in \pi\} \in \mathcal{F}$ for all initialized paths $\pi \subseteq \mathrm{supp}(T)$.

The set of the trees accepted by automaton $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$.

The automata we consider have "occurrence" accepting conditions: an initialized path is accepted if the set of states *occurring* in it is one of those in $\mathcal{F}$, even if some of these states occur only finitely many times.

Theorem 30 (cf e.g. [Tho97]) *Checking whether $\mathcal{L}(\mathcal{A}) = \emptyset$ for a tree automaton $\mathcal{A}$ with "occurrence" accepting condition is decidable.*

6.1 The state-splitting construction

Let IS be the interpreted system $\langle\langle L_i : i \in \Sigma_e\rangle, I, \langle Act_i : i \in \Sigma_e\rangle, t, V\rangle$ for the set of atomic propositions AP and the set of agents $\Sigma = \{1, \ldots, N\}$.

Definition 31 (labelled outcomes) Given a coalition $\Gamma \subseteq \Sigma$, a set of global states $M \subseteq L$, $a \in Act_\Gamma$, and $v \in L_\Gamma$, we denote the set

$$\{t(m, b) : m \in M, b \in Act_\Sigma, b_\Gamma = a, (t(m, b))_\Gamma = v)\}$$

by $\mathrm{out}(M, a, v)$.

The set $\mathrm{out}(M, a, v)$ consists of the global states which can be reached from a state from M by an action tuple in which the actions of the members of Γ are as in a, and have v as their local states for the members of Γ.

Given a coalition Γ, we construct a new interpreted system $\widehat{IS}_\Gamma = \langle\langle \widehat{L}_i : i \in \Sigma_e\rangle, \widehat{I}, \langle Act_i : i \in \Sigma_e\rangle, \widehat{t}, \widehat{V}\rangle$ as follows:

$\widehat{L}_i = L_i$ for $i \in \Sigma$, $\qquad \widehat{L}_e = \{\langle l, M\rangle : M \subseteq L, l \in M, l_\Gamma = m_\Gamma \text{ for all } m \in M\}$
$\widehat{I} = \{\langle l_\Sigma, \langle l, \{m \in I : m_\Gamma = l_\Gamma\}\rangle\rangle : l \in I\}$
$\widehat{t}(\langle l_\Sigma, \langle l, M\rangle\rangle, a) = \langle (t(l, a))_\Sigma, \langle t(l, a), \mathrm{out}(M, a_\Gamma, t(l, a)_\Gamma)\rangle\rangle$
$\widehat{V}(\langle l_\Sigma, \langle l, M\rangle\rangle, p)$ iff $V(l, p)$

$\widehat{IS}_\Gamma$ is a refinement of IS, as its states are obtained by state-splitting of the states of IS. The construction of $\widehat{IS}$ is remniscent of the subset construction: the last component of each state in $\widehat{IS}_\Gamma$ corresponds to the *set of states* in which coalition Γ considers that the system can be after a certain sequence of observations. Hence, in $\widehat{IS}_\Gamma$, every global state of IS is augmented with a set of global states which are indistinguishable from it to the considered coalition Γ, at the end of some finite run. For initial states, the augmenting set consists of *all* the indistinguishable initial IS states. In order to preserve the observational abilities of agents, the augmenting sets do not affect the local states of agents. Instead they are made part of the local state of the environment.

Note that the definition of $\widehat{t}$ above applies only to $\widehat{IS}_\Gamma$ global states of the form $\langle l_\Sigma, \langle l, M\rangle\rangle$ where l is an arbitrary global state in IS. Since all the states in $\widehat{I}$ have this form, and the values of $\widehat{t}$ have this form too, no other states can appear in $\widehat{IS}_\Gamma$ runs. That is why the definition of $\widehat{t}$ on other states is irrelevant. The same holds about $\widehat{V}$.

Note also that every run $r = l^0a^1l^1a^2\dots$ in IS corresponds to a unique run $\widehat{r} = \langle l^0_\Sigma, \langle l^0, M^0\rangle\rangle a^1\langle l^1_\Sigma, \langle l^1, M^1\rangle\rangle a^2\dots$ in $\widehat{IS}_\Gamma$ because the set M^0 is determined by l^0, and M^{i+1} is determined by l^i, M^i and a^i for every $i < \omega$. The converse holds too: for every $r' \in R^\omega(\widehat{IS}_\Gamma)$ there exists a unique $r \in R^\omega(IS)$ such that $r' = \widehat{r}$. Furthemore, since local runs in IS and $\widehat{IS}_\Gamma$ have the same form, strategies for A in IS are strategies for A in $\widehat{IS}_\Gamma$ too.

Proposition 32 *Let $\Gamma, \Delta \subseteq \Sigma$, $r, r' \in R(IS)$ and let $\widehat{r}$ and $\widehat{r'}$ denote their corresponding runs in $\widehat{IS}_\Gamma$.*

(i) *$r \sim_\Delta r'$ is equivalent to $\widehat{r} \sim_\Delta \widehat{r'}$.*

(ii) *If $s \in S(\Delta, IS)$, and r is finite, then $r' \in \mathrm{out}(r, s)$ in IS iff $\widehat{r'} \in \mathrm{out}(\widehat{r}, s)$ in $\widehat{IS}_\Gamma$.*

(iii) *If $p \in AP$, r is finite and $\langle l_\Sigma, \langle l, M\rangle\rangle$ is the last state of $\widehat{r}$, then $IS, r \models \mathsf{D}_\Gamma p$ and $\widehat{IS}_\Gamma, \widehat{r} \models \mathsf{D}_\Gamma p$ are both equivalent to $V(m, p)$ for all $m \in M$.*

(iv) *If φ is an arbitrary ATL^{DP}_{iR} formula and r is finite, then $IS, r \models \varphi$ is equivalent to $\widehat{IS}_\Gamma, \widehat{r} \models \varphi$.*

Proof: Items (i)-(iii) follow directly from definition; (iv) is proved by induction on the construction of φ. The cases given in detail below are representative:

φ is $\mathsf{D}_\Delta\psi$ for some $\Delta \subseteq \Sigma$: $\widehat{IS}_\Gamma, \widehat{r} \models \mathsf{D}_\Delta\psi$ iff $\widehat{IS}_\Gamma, \widehat{r'} \models \psi$ for all $\widehat{r'} \in [\widehat{r}]_\Delta$. The latter is equivalent to $IS, r' \models \psi$ for all $r' \in [r]_\Delta$ by the induction hypothesis. This is equivalent to $IS, r \models \mathsf{D}_\Delta\psi$.

φ is $\langle\langle\Delta\rangle\rangle(\psi_1 \mathsf{U} \psi_2)$ for some $\Delta \subseteq \Sigma$: $\widehat{IS}_\Gamma, \widehat{r} \models \varphi$ iff there exists an $s \in S(\Delta, \widehat{IS}_\Gamma)$ such that for every $\widehat{r'} \in \mathrm{out}([\widehat{r}]_\Delta, s)$ there exists a $k < \omega$ such that $\widehat{IS}_\Gamma, \widehat{r'}[0..|r| + k] \models \psi_2$ and $\widehat{IS}_\Gamma, \widehat{r'}[0..|r| + i] \models \psi_1$ for $i = 0, \dots, k - 1$. It follows from (i) and (ii) that $\widehat{r'} \in \mathrm{out}([\widehat{r}]_\Delta, s)$ in $\widehat{IS}_\Gamma$ is equivalent to $r' \in \mathrm{out}([r]_\Delta, s)$ in IS. Now the induction hypothesis entails that satisfaction condition for φ at $\widehat{IS}_\Gamma, \widehat{r}$ is equivalent to the satisfaction condition for φ at IS, r. $\dashv$

Proposition 32, (iii) explains the purpose of the construction of $\widehat{IS}_\Gamma$. Given a $p \in AP$, and $r \in R^{fin}(IS)$, if $\langle l_\Sigma, \langle l, M\rangle\rangle$ is the last state of $\widehat{r}$, then $\widehat{IS}_\Gamma, \widehat{r} \models \mathsf{D}_\Gamma p$ iff $V(m, p)$ for all $m \in M$. Hence, in $\widehat{IS}_\Gamma$ the information about the satisfaction of a formula of the form $\mathsf{D}_\Gamma p$ is hardcoded in the last state of the reference run.

6.2 The state labeling constructions

In this section, given an interpreted system $IS = \langle\langle L_i : i \in \Sigma_e\rangle, I, \langle Act_i : i \in \Sigma_e\rangle, t, V\rangle$ and a formula φ, we show how to construct an interpreted system IS' whose runs are equivalent to those of IS and which has the property that for every subformula ψ of φ and any finite run r in IS the condition $IS', r \models \psi$ is determined by the last state of r. This means that we can define the sets

$$\begin{aligned}[\![\psi]\!]_{IS'} &= \{l : IS', r \models \psi \text{ for some } r \in R(IS') \text{ which ends at } l\}\\ &= \{l : IS', r \models \psi \text{ for all } r \in R(IS') \text{ which end at } l\}\end{aligned}$$

for the subformulas ψ of φ. In other words, every state of IS' can be labelled with the subformulas of φ which hold at the finite runs that end at this state.

We obtain IS' by a sequence of refinements of the given interpreted system IS. If $[\![\psi]\!]_{IS}$ can be defined for a formula ψ, then the vocabulary of IS can be extended by an atomic proposition p_ψ so that $V(l, p_\psi)$

iff $l \in [\![\psi]\!]_{IS}$. Then substituting p_ψ for ψ in ATL_{iR}^{DP} formulas preserves their meaning in IS. Therefore IS' can obtained by a sequence of refinements each of which enables the definition of $[\![.]\!]$ to be extended to some formula ψ built using just one ATL_{iR}^{DP} connective and propositional variables. No transformations on the given IS are needed in order to define the $[\![\bot]\!]_{IS}$, $[\![p]\!]_{IS}$ for $p \in AP$, and $[\![p \Rightarrow q]\!]_{IS}$, which is equal to $[\![q]\!]_{IS} \cup L \setminus [\![p]\!]_{IS}$. Here follow the constructions for the other possible forms of ψ.

$\ominus p$: We construct $\overline{IS} = \langle\langle \overline{L}_i : i \in \Sigma_e\rangle, \overline{I}, \langle Act_i : i \in \Sigma_e\rangle, \overline{t}, \overline{V}\rangle$ where

$$\begin{aligned}
&\overline{L}_i = L_i \text{ for } i \in \Sigma\\
&\overline{L}_e = L_e \times \{0,1\}\\
&\overline{I} = \{\langle l_\Sigma, \langle l_e, 0\rangle : l \in I\}\\
&\overline{t}(\langle l_\Sigma, \langle l, x\rangle\rangle, a) = \langle (t(l,a))_\Sigma, \langle (t(l,a)), V(l,p)\rangle\rangle\\
&\overline{V}(\langle l_\Sigma, \langle l_e, x\rangle\rangle, p) \text{ iff } V(l,p)
\end{aligned}$$

Again, states which do not have the form $\langle l_\Sigma, \langle l_e, x\rangle\rangle$ for some $l \in L$ are not reachable and therefore the definition t on such states is irrelevant.

A direct check shows that every run $r = l^0 a^1 l^1 \ldots \in R(IS)$ corresponds to a unique run $\overline{r} = \langle l_\Sigma^0, \langle l_e^0, x^0\rangle\rangle a^1 \langle l_\Sigma^1, \langle l_e^1, x^1\rangle\rangle \ldots \in R(\overline{IS})$. Moreover, if $\overline{r}$ is finite, then $\overline{IS}, \overline{r} \models \ominus p$ iff $x^{|r|} = 1$. Hence

$$[\![\ominus p]\!]_{\overline{IS}} = \{\langle l_\Sigma, \langle l_e, 1\rangle\rangle : l \in L\}.$$

$(p\mathsf{S}q)$: We construct $\overline{IS} = \langle\langle \overline{L}_i : i \in \Sigma_e\rangle, \overline{I}, \langle Act_i : i \in \Sigma_e\rangle, \overline{t}, \overline{V}\rangle$ where $\overline{L}_i$, $i \in \Sigma_e$, and $\overline{V}$ are as in the case of $\ominus p$, and

$$\begin{aligned}
&\overline{I} = \{\langle l_\Sigma, \langle l_e, V(l,q)\rangle : l \in I\}\\
&\overline{t}(\langle l_\Sigma, \langle l, x\rangle\rangle, a) = \langle (t(l,a))_\Sigma, \langle t(l,a)_e, \max\{V(t(l,a),q)), \min\{V(t(l,a),p), x\}\}\rangle\rangle
\end{aligned}$$

Here $\max\{V(t(l,a),q)), \min\{V(t(l,a),p), x\}\}$ is an expression for the fixpoint expansion $q \vee (p \wedge \ominus(p\mathsf{S}q))$ of $(p\mathsf{S}q)$ in which x stands for the truth value of $(p\mathsf{S}q)$ at the predecessor state $\langle l_\Sigma, \langle l_e, x\rangle\rangle$. Again, states which do not have the form $\langle l_\Sigma, \langle l_e, x\rangle\rangle$ for some $l \in L$ can be ignored.

An induction on n shows that $\overline{r} = \langle l_\Sigma^0, \langle l_e^0, x^0\rangle\rangle a^1 \ldots a^n \langle l_\Sigma^n, \langle l_e^n, x^n\rangle\rangle \in R^{fin}(\overline{IS})$ implies $\overline{IS}, \overline{r} \models (p\mathsf{S}q)$ iff $x^n = 1$. Hence

$$[\![(p\mathsf{S}q)]\!]_{\overline{IS}} = \{\langle l_\Sigma, \langle l_e, 1\rangle\rangle : l \in L\}.$$

$\mathsf{D}_\Gamma p$: For this case we use the interpreted system $\widehat{IS}_\Gamma$ constructed in Section 6.1. Proposition 32, (iii) entails that

$$[\![\mathsf{D}_\Gamma p]\!]_{\widehat{IS}_\Gamma} = \{\langle l_\Sigma, \langle l, M\rangle\rangle \in \widehat{L} : V(m,p) \text{ for all } m \in M\}.$$

$\langle\!\langle\Gamma\rangle\!\rangle \circ p$: We use $\widehat{IS}_\Gamma$ from Section 6.1 again. A direct check shows that

$$[\![\langle\!\langle\Gamma\rangle\!\rangle \circ p]\!]_{\widehat{IS}_\Gamma} = \{\langle l_\Sigma, \langle l, M\rangle\rangle \in \widehat{L} : (\exists a \in Act_\Gamma)(\forall b \in Act_{\Sigma_e})\Big(b_\Gamma = a \rightarrow (\forall m \in M) V(\widehat{t}(\langle m_\Sigma, \langle m, M\rangle\rangle, b), p)\Big)\}.$$

$\langle\!\langle\Gamma\rangle\!\rangle(p\mathsf{U}q)$: Let φ be $\langle\!\langle\Gamma\rangle\!\rangle(p\mathsf{U}q)$. We use the interpreted system $\widehat{IS}_\Gamma$ from Section 6.1. For every state $\langle l_\Sigma, \langle l, M\rangle\rangle \in \widehat{L}$ we build a tree automaton $\mathcal{A} = \mathcal{A}^{IS,\varphi}_{\langle l_\Sigma, \langle l, M\rangle\rangle}$ such that $\mathcal{L}(\mathcal{A}) \neq \emptyset$ iff $\widehat{IS}_\Gamma, \widehat{r} \models \langle\!\langle\Gamma\rangle\!\rangle(p\mathsf{U}q)$ for some and, equivalently, all $\widehat{r} \in R^{fin}(\widehat{IS}_\Gamma)$ which have $\langle l_\Sigma, \langle l, M\rangle\rangle$ as their last state. The states of $\mathcal{A}$ represent classes of Γ-indistinguishable runs that end in a given state of IS, extended with a special mechanism is needed for checking whether the objective $(p\mathsf{U}q)$ is satisfied on all paths of an accepted tree. This mechanism is explained in the following.

Clearly, there are runs along which the coalition Γ may be unable to determine the truth values of $(p\mathsf{U}q)$ at some steps. That is, it is possible to have $r, r' \in R^n(IS)$ such that $r \sim_\Gamma r'$ and $r[0..n]$ satisfies $(p\mathsf{U}q)$ whereas $r'[0..n]$ does not. On the other hand, if Γ has the objective to enforce $(p\mathsf{U}q)$, any winning strategy s for Γ has the property that in the tree T representing the runs which are the outcome of s, there exists a level x such that on all the runs in T, the objective $(p\mathsf{U}q)$ has been reached at x or before x, and, for some runs which end on a level $y < x$, the objective $(p\mathsf{U}q)$ has not yet been accomplished. For this reason, for checking that Γ has a winning strategy in some state, for each finite run r starting in that state we need to record the set R_2 of states which represent endpoints of runs that are Γ-indistinguishable from r, and, together with this, the subset of states $R_1 \subseteq R_2$ which are endpoints of such runs on which the objective $(p\mathsf{U}q)$ has not been accomplished before their endpoint. Moreover, in our search of winning strategies, we will only be interested in the finite runs whose associated set R_2 contains either endpoints of runs that have accomplished, in their past, the objective of $(p\mathsf{U}q)$, or endpoints of runs along which p has always been true.

Formally, given $\langle l_\Sigma, \langle l, M\rangle\rangle \in \widehat{L}$, consider the tree automaton $\mathcal{A}_{\langle l_\Sigma, \langle l, M\rangle\rangle} = \langle Q, Act_\Sigma, \delta, \{q_0\}, \mathcal{F}\rangle$ where

$$Q = \{\bot\} \cup \{\langle R_1, R_2\rangle : R_1 \subseteq R_2 \subseteq L, R_1 \subseteq [\![p]\!]_{IS} \setminus [\![q]\!]_{IS}, \mathsf{card}\big((R_2)_\Gamma\big) = 1\}$$

$$q_0 = \begin{cases} \langle M \setminus [\![q]\!]_{IS}, M\rangle & \text{if } M \subseteq [\![p]\!]_{IS} \cup [\![q]\!]_{IS}; \\ \bot & \text{otherwise.} \end{cases}$$

$$\delta(\bot, a) = \{\bot\} \text{ for all } a \in Act_\Gamma$$

$$\delta(\langle R_1, R_2\rangle, a) = \begin{cases} \{\bot\} & \text{if there exist } l \in R_1, \langle l'_\Sigma, \langle l', M'\rangle\rangle \in \widehat{L} \text{ and } b \in Act_{\Sigma_e} \text{ such that} \\ & b_\Gamma = a, \widehat{t}(\langle l_\Sigma, \langle l, R_2\rangle\rangle, b) = \langle l'_\Sigma, \langle l', M'\rangle\rangle \text{ and } l' \notin [\![p]\!]_{IS} \cup [\![q]\!]_{IS}; \\ \{\langle \mathrm{out}(R_1, a, m) \setminus [\![q]\!]_{IS}, \mathrm{out}(R_2, a, m)\rangle : m \in L_\Gamma, \mathrm{out}(R_2, a, m) \neq \emptyset\} & \\ & \text{otherwise.} \end{cases}$$

$$\mathcal{F} = \{\mathcal{R} \subseteq Q : \langle \emptyset, R\rangle \in \mathcal{R} \text{ for some } R \subseteq L\}$$

As explained above, the first component R_1 of a pair $\langle R_1, R_2\rangle \in Q$ consists of those states in R_2 which correspond to runs where the satisfaction of $(p\mathsf{U}q)$ has not been accomplished yet. Hence tree node labels of the form $\langle \emptyset, R\rangle$ indicate that the satisfaction of $(p\mathsf{U}q)$ is accomplished along all runs which end at a state from R. Furthermore, whenever an a-successor of $\langle R_1, R_2\rangle$ does not contain a state labelled with $\bot$, $l \in [\![p]\!]_{IS} \cup [\![q]\!]_{IS}$ holds for all $l \in out(R_2, a, m)$, $m \in L_\Gamma$. Next we prove that $[\![\langle\!\langle \Gamma\rangle\!\rangle(p\mathsf{U}q)]\!]_{\widehat{IS}_\Gamma}$ can be defined as

$$\{\langle l_\Sigma, \langle l, M\rangle\rangle \in \widehat{L} : \mathcal{L}(\mathcal{A}_{\langle l_\Sigma, \langle l, M\rangle\rangle}) \neq \emptyset\}.$$

Proposition 33 *Let $\langle l_\Sigma, \langle l, M\rangle\rangle \in \widehat{L}$ be the last state of $\widehat{r} \in R^{fin}(\widehat{IS}_\Gamma)$. Then*

$$\widehat{IS}_\Gamma, \widehat{r} \models \langle\!\langle \Gamma\rangle\!\rangle(p\mathsf{U}q) \text{ iff } \mathcal{L}(\mathcal{A}_{\langle l_\Sigma, \langle l, M\rangle\rangle}) \neq \emptyset.$$

Proof: ($\rightarrow$) Let $\widehat{IS}_\Gamma, \widehat{r} \models \langle\!\langle \Gamma\rangle\!\rangle(p\mathsf{U}q)$. Then there exists an $s \in S(\Gamma, \widehat{IS}_\Gamma)$ such that for every $\widehat{r'} \in \mathrm{out}([\widehat{r}]_\Gamma, s)$ there exists a $k < \omega$ such that $\widehat{IS}_\Gamma, \widehat{r'}[0..|r| + k] \models q$ and $\widehat{IS}_\Gamma, \widehat{r'}[0..|r| + i] \models p$, $i = 0, \ldots, k - 1$.

We construct the tree $T : \mathbb{N}^* \rightharpoonup Q \times Act_\Gamma$. For the root of T we put

$$T(\varepsilon) = \langle M \setminus [\![q]\!]_{IS}, M, s(\widehat{r}_\Gamma)\rangle.$$

Given a finite path $\varepsilon = x_0 \prec \ldots \prec x_j$ of T, assume that $T(x_i) = \langle\langle X_1^i, X_2^i\rangle, a^i\rangle$, $i = 1, \ldots, j$, and let $\delta(\langle X_1^i, X_2^i\rangle, a_i) = \{\langle R_1^1, R_2^1\rangle, \ldots, \langle R_1^k, R_2^k\rangle\}$. According to the definition of the set of states of $\mathcal{A}$, for every node x_m $(1 \leq i \leq j)$ the projection $(X_2^m)_\Gamma$ consists of a singleton set, denote it $\{l_\Gamma^i\}$. The same holds for each of the sets R_2^p, $1 \leq p \leq k$, hence denote $(R_2^p)_\Gamma$ as $\bar{l}_\Gamma^p$. We set the degree of branching of node x_j to k and put

$$t(x_j p) = \langle\langle R_1^p, R_2^p\rangle, s(l_\Gamma^1 a^1 \Gamma \ldots l_\Gamma^j a_\Gamma^j \bar{l}_\Gamma^p)\rangle,\ p = 1, \ldots, k.$$

A lengthy but otherwise trivial direct check shows that $t \in \mathcal{L}(\mathcal{A}_{\langle l_\Sigma, \langle l, M\rangle\rangle})$

($\leftarrow$) Assume that $T \in \mathcal{L}(\mathcal{A}_{\langle l_\Sigma, \langle l, M\rangle\rangle})$. We construct an $s \in S(\Gamma, \widehat{IS}_\Gamma)$ which is a witness for $\widehat{IS}_\Gamma, \widehat{r} \models \langle\!\langle A\rangle\!\rangle(p\mathsf{U}q)$. We only need to define s on finite Γ-local runs which are extensions to $\widehat{r}_\Gamma$. Let $\widehat{r}_\Gamma = v^0 b^1 \ldots b^n v^n \in$

$\widehat{L}_\Gamma(Act_\Gamma \widehat{L}_\Gamma)^n$. Let $x \in \mathrm{supp}(T)$ and $x_0 = \varepsilon \prec x_1 \prec \ldots \prec x_k = x$ be all the nodes leading to x in T. Let $t(x_i) = \langle\langle R_1^i, R_2^i\rangle, b^{n+i}\rangle$, and $v^{n+i} = v(R_2^i)$, $i = 1, \ldots, k$. Then we put

$$s(v^0 b^1 \ldots b^n v^n v^{n+1} b^{n+1} \ldots b^{n+k-1} v^{n+k-1}) = b^{n+k}.$$

The values of s for other sequences from $\in \widehat{L}_\Gamma(Act_\Gamma \widehat{L}_\Gamma)^*$ are irrelevant because, as it becomes clear below, s enables Γ to avoid the corresponding runs.

A direct check using the fact that every path in T contains a node x such that $T(x)$ has the form $\langle\langle\emptyset, R\rangle, b\rangle$ shows that if

$$\langle l_\Sigma^0, \langle l^0, M^0\rangle\rangle a^1 \ldots a^n \langle l_\Sigma^n, \langle l^n, M^n\rangle\rangle a^{n+1} \langle l_\Sigma^{n+1}, \langle l^{n+1}, M^{n+1}\rangle\rangle \ldots \in \mathrm{out}([\widehat{r}]_\Gamma, s)$$

then there exists a $k < \omega$ such that $V(m,p)$ for all $m \in M^n \cup \ldots \cup M^{n+k-1}$ and $V(m,q)$ for all $m \in M^{n+k}$. This follows from the fact that every path π in T contains a node x such that $T(x)$ has the form $\langle\langle\emptyset, R\rangle, b\rangle$, and all the nodes along π which precede x have the form $\langle\langle R_1, R_2\rangle, b\rangle$ with R_2 consisting of states m such that $V(m,p)$. $\dashv$

$\langle\langle\Gamma\rangle\rangle(p\mathsf{W}q)$: Just like in the case of $\langle\langle\Gamma\rangle\rangle(p\mathsf{U}q)$, for every $\langle l_\Sigma, \langle l, M\rangle\rangle \in \widehat{L}$ we build an automaton $\mathcal{A}_{\langle l_\Sigma, \langle l, M\rangle\rangle}$ such that $\mathcal{L}(\mathcal{A}_{\langle l_\Sigma, \langle l, M\rangle\rangle}) \neq \emptyset$ is equivalent to $\widehat{IS}_\Gamma, \widehat{r} \models \langle\langle\Gamma\rangle\rangle(p\mathsf{W}q)$ for finite runs $\widehat{r}$ having $\langle l_\Sigma, \langle l, M\rangle\rangle$ as the last state. The definition of $\mathcal{A}_{\langle l_\Sigma, \langle l, M\rangle\rangle}$ is the same as in the case of $\langle\langle\Gamma\rangle\rangle(p\mathsf{U}q)$ except for the acceptance condition, which is changed to take account of the possibility to satisfy $(p\mathsf{W}q)$ at infinite runs consistsing only of p-states. The acceptance condition in this case is

$$\mathcal{F} = \{\mathcal{R} \subseteq Q : \langle\emptyset, R\rangle \in \mathcal{R} \text{ for some } R \subseteq L\} \cup \{R \subseteq Q : \bot \notin R\},$$

that is, only paths which refute $(p\mathsf{W}q)$ within finitely many steps are rejected.

6.3 The model-checking algorithm

As explained in the beginning of this section, given that $\psi_1, \ldots \psi_n$ is an enumeration of the subformulas of φ such that $\psi_i \in \mathrm{Subf}(\psi_j)$ entails $i \leq j$, our model-checking algorithm works by constructing a sequence of interpreted systems $IS_0 = IS, IS_1, \ldots, IS_n = IS'$ such that $[\![\psi_j]\!]_{IS_k}$ is defined for $j, k = 1, \ldots, n$, $j \leq k$. The construction of IS_{j+1} from IS_j is according to the main connective of ψ_{j+1}, as described above. Finally, φ is satisfiable at IS iff $[\![\varphi]\!]_{IS'}$ contains at least one initial state of IS'.

Complexity The state-splitting construction in our algorithm involves a procedure that produces an exponential blowup of the state-space of the model. Since this procedure is applied for every subformula involving an epistemic or a coalition operator, the state-space constructed at the end of the algorithm is proportional to a tower of exponentials whose height equals the *epistemic depth* of the given formula, that is, the maximal number of nested epistemic or coalition operators. Hence, the complexity of our algorithm is nonelementary in the size of the given formula.

Concluding remarks

Interestingly, *constant* strategies turn sufficient for the satisfaction of the target formula in the model involved in our completeness argument for $ATL_{iR}^{\Diamond}$. This implies that giving up the restriction on strategies to be coordinated, which is essential for our model-checking algorithm, does not affect the *logic* $ATL_{iR}^{\Diamond}$, i. e., the set of valid formulas. This is a consequence of the design of interpreted systems, in which transitions are defined through a transition function, whereas agents are required to just *name* the actions they choose. The presence of an environment with its local state directly influencing the successor local states of every agent makes it possible to supply the transition function with informative state properties without having to make these properties known to any of the proper agents.

The axioms (1) and (2) extend our complete proof system for $ATL_{iR}^{\Diamond}$ to a system for ATL_{iR}^{DP}, and represent the "unfinished" part of our work. We do not know whether their deductive power is sufficient for the completeness of the extended system. It can be shown how, by adding some appropriate ω-rules, one can use these axioms in order to obtain a system that is ω-complete for ATL_{iR}^{DP}, or equivalently, these axioms can be claimed to be complete for ATL_{iR}^{DP} *relative* to $ATL_{iR}^{\diamond}$. This is so because (1) and (2) actually define $\langle\langle\Gamma\rangle\rangle(\varphi \mathsf{U} \psi)$ in terms of $ATL_{iR}^{\diamond}$. The totality of this definition follows from the fact that the set of the formulas $\Xi = \{\neg\ominus^k \mathsf{I} : k < \omega\}$ is unsatisfiable. The shortcoming of this definition is that the defining $ATL_{iR}^{\diamond}$ formula depends on the (length of) the reference finite run: by considering formulas of the form $\ominus^k \mathsf{I}$ as ξ in (1), we obtain equivalents of the form $\langle\langle\Gamma\rangle\rangle\diamond \mathsf{D}_\Gamma \diamondminus(\psi \wedge (\ominus\varphi \mathsf{S} \xi))$ to $\langle\langle\Gamma\rangle\rangle(\varphi \mathsf{U} \psi)$ at runs of length k. Unfortunately, the unsatisfiability of Ξ takes an ω-rule to encode in a proof system. Another infinitary rule is needed to exclude infinite degrees of branching, which would render (1) and (2) unsound. It is an open question whether ATL_{iR}^{DP} admits a finitary axiomatisation, and whether validity in it is recursively enumerable at all.

Another direction of future research is to embark on the study of a fully-fledged ATL_{iR}^{DP*}. Developing a model-checking algorithm of the same general form for arbitrary *LTL* path objectives appears to be technically challenging because of subtleties which arise in connection with the construction that leads from system states to coalition mental states, which were given in Section 6.1 for the case of ATL_{iR}^{DP}.

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A Proofs of lemmata from Section 4.3

Proof:[of Lemma 21] Let $r \in R^n(IS)$, $r = f(w^0,k^0,s^0)a^1\dots a^n f(w^n,k^n,s^n)$ and $f(v^0,k^0,t^0)b^0\dots b^{n-1}f(v^n,k^n,t^n) \in [r]_\Gamma$. This implies that

$$(f(v^n,k^n,t^n))_i = \{\chi : \mathsf{D}_i\chi \in \overline{v^n}\} = (f(w^n,k^n,s^n))_i$$

for all $i \in \Gamma$. Next we use this in order to prove that $\vdash \bigwedge w^n \Rightarrow \mathsf{D}_\Gamma\psi$ implies $\vdash \bigwedge v^n \Rightarrow \mathsf{D}_\Gamma\psi$. There is nothing to prove for $\Gamma = \emptyset$ and singleton Γs. Let $|\Gamma| \geq 2$ and $\vdash \bigwedge w^n \Rightarrow \mathsf{D}_\Gamma\psi$. Then, by the definition of $\overline{w^n}$, $\mathsf{D}_i(p_{\psi,i,\Gamma} \Rightarrow \psi) \in \overline{w^n}$ for all $i \in \Gamma$. Now $(f(v^n,k^n,t^n))_i = (f(w^n,k^n,s^n))_i$ implies that $\mathsf{D}_i(p_{\psi,i,\Gamma} \Rightarrow \psi) \in \overline{v^n}$. By the construction of v^n, this entails $\vdash \bigwedge v^n \Rightarrow \mathsf{D}_\Gamma\psi$, which implies that $\vdash \bigwedge v^n \Rightarrow \psi$ as well, by axiom $\mathbf{T}_\mathsf{D}$. $\dashv$ **Proof:**[of Lemma 23] Induction on n. Let $n = 0$ and $r = f(w^0,k^0,s^0)$. Assume that $\neg\mathsf{D}_\Gamma\psi \in D_\Gamma(w^0)$ for the sake of contradiction. Then $D_\Gamma(w^0)$ is consistent with $\neg\psi$ as well. If not, then $\vdash \bigwedge D'_\Gamma(w^0) \Rightarrow \psi$ holds for some finite $D'_\Gamma(w^0) \subset D_\Gamma(w^0)$, whence $\vdash \bigwedge D'_\Gamma(w^0) \Rightarrow \mathsf{D}_\Gamma\psi$ by the rule N_D and the axioms $\mathbf{K}_\mathsf{D}$, $\mathbf{4}_\mathsf{D}$ and $\mathbf{5}_\mathsf{D}$, and this contradicts the consistency of $\neg\psi$ with $D_\Gamma(w^0)$. Hence there exists a $w' \in W$ such that $D_\Gamma(w^0) \subseteq D_\Gamma(w')$ and $\neg\psi$ is consistent with w'. Since $f(w^0,k^0,s^0) \in R^0(IS)$, which is equivalent to $f(w^0,k^0,s^0) \in I$, we have $\mathsf{D}_\Gamma \mathsf{I} \in D_\Gamma(w^0)$. By axioms $\mathbf{T}_\mathsf{D}$ and D_I this implies $\mathsf{D}_\Gamma \mathsf{I} \in w'$, whence $f(w',k^0,s^0) \in R^0(IS)$ too. $D_\Gamma(w^0) \subseteq D_\Gamma(w')$ entails $D_i(w^0) \subseteq D_i(w')$, by $\mathbf{4}_\mathsf{D}$, $\mathbf{5}_\mathsf{D}$ and $Mono_\mathsf{D}$, and $D_i(w^0) \cap C = D_i(w') \cap C$, by the maximality of C, whence we conclude $(\underline{f(w',k^0,s^0)})_i = (f(w^0,k^0,s^0))_i$, for all $i \in \Gamma$. Hence $f(w',k^0,s^0) \in [f(w^0,k^0,s^0)]_\Gamma$, by the definition of $\overline{w^0}$, $\overline{w'}$. This contradicts the assumption that $\vdash \bigwedge w' \Rightarrow \psi$ for all w' such that $f(w',k^0,s^0) \in [f(w^0,k^0,s^0)]_\Gamma$.

Next we prove that the lemma holds for any

$$r = f(w^0,k^0,s^0)a^0 \dots a^{n-1} f(w^n,k^n,s^n)a^n f(w^{n+1},k^{n+1},s^{n+1}) \in R^{n+1}(IS),$$

provided that it holds for all runs of length n. Assume that $\neg\mathsf{D}_\Gamma\psi \in D_\Gamma(w^{n+1})$ for the sake of contradiction. $D_\Gamma(w^{n+1})$ is consistent with $w^n_{past,\mathsf{D}} = \{\ominus\chi : \chi \in D_\Gamma(w^n)\}$ because, by the definition of $t(f(w^n,k^n,s^n),a^n)$, the union $w^n_{past} \cup w^n_{actions} \cup w^n_{environment}$ is consistent (the latter three sets being defined with respect to a^n), $D_\Gamma(w^{n+1})$ consists of formulas which are derivable from that union by MP, and $w^n_{past,\mathsf{D}} \subseteq w^n_{past}$. Assume that $w^n_{past,\mathsf{D}} \cup D_\Gamma(w^{n+1})$ is not consistent with $\neg\psi$ for the sake of contradiction. Then there exist finite subsets ${w'}^n_{past,\mathsf{D}} \subset w^n_{past,\mathsf{D}}$ and $D'_\Gamma(w^{n+1}) \subset D_\Gamma(w^{n+1})$ such that $\vdash \bigwedge({w'}^n_{past,\mathsf{D}} \cup D'_\Gamma(w^{n+1})) \Rightarrow \psi$. Then, by PR, $\mathbf{4}_\mathsf{D}$, $\mathbf{5}_\mathsf{D}$ and $Mono_\ominus$, which provide that $\vdash \ominus\mathsf{D}_\Gamma\chi \Leftrightarrow \mathsf{D}_\ominus\chi$ and $\vdash \ominus\mathsf{D}_\Gamma\chi \Leftrightarrow \ominus\chi$ for $\ominus\chi \in {w'}^n_{past}$, as these formulas have the forms $\ominus\mathsf{D}_\Gamma\eta$ and $\ominus\neg\mathsf{D}_\Gamma\eta$, this entails $\vdash \bigwedge({w'}^n_{past,\mathsf{D}} \cup D'_\Gamma(w^{n+1})) \Rightarrow \mathsf{D}_\Gamma\psi$, which contradicts $\neg\mathsf{D}_\Gamma\psi \in D_\Gamma(w^{n+1})$.

Let θ be $\bigwedge\{\chi \in C : D_\Gamma(w^{n+1}), \neg\psi \vdash_{MP} \ominus\chi\}$, where $\vdash_{MP}$ indicates derivability from the indicated premises and $ATL^\diamond_{iR}$ theorems with MP as the only proof rule. Then the consistency of $w^n_{past,\mathsf{D}} \cup D_\Gamma(w^{n+1}) \cup \{\neg\psi\}$ entails that $\{\theta\} \cup D_\Gamma(w^n)$ is consistent and therefore if $\mathsf{D}_\Gamma\neg\theta \not\in D_\Gamma(w^n)$. By the inductive hypothesis there exists an $f(v^0,k^0,t^0)b^0 \dots b^{n-1} f(v^n,k^n,t^n) \in [f(w^0,k^0,s^0)a^0 \dots a^{n-1} f(w^n,k^n,s^n)]_\Gamma$ such that $v^n \cup \{\theta\}$ is consistent.

We need to define a $b^n \in Act_{\Sigma_e}$ such that $b^n_\Gamma = a^n_\Gamma$ and the v^{n+1} determined by $f(v^{n+1},k^{n+1},t^{n+1}) = t(f(v^n,k^n,t^n),b^n)$ satisfies $D_\Gamma(v^{n+1}) = D_\Gamma(w^{n+1})$ and is consistent with $\neg\psi$. We put $b^n_i = \langle\langle i\rangle\rangle \circ \top$ for all $i \in \Sigma \setminus \Gamma$. We put $b^n_e = \xi_1 < \dots < \xi_{|C|}$ where $\{\xi_1,\dots,\xi_j\}$ is a maximal consistent subset of C which is consistent with $D_\Gamma(w^{n+1}) \cup \{\neg\psi\}$ for some appropriate $j \in \{1,\dots,|C|\}$. For $i \in \Gamma$ we put $b^n_i = a^n_i$ in fulfillment of the requirement $b^n_\Gamma = a^n_\Gamma$. We conclude the proof by showing that the corresponding v^{n+1} has the desired properties, that is, $D_\Gamma(w^{n+1}) \subseteq v^{n+1}$ and $\neg\psi$ is consistent with v^{n+1}.

Without loss of generality we assume that the actions from $\{b^n_i : i \in \Gamma\}$ for which Δ_i is unanimous in b^n are $\langle\langle\Delta_i\rangle\rangle \circ \delta_i$, $i = 1,\dots,m$. By the definition of b^n_i for $i \in \Sigma \setminus \Gamma$, the only coalitions $\Delta \not\subseteq \Gamma$ which are unanimous in b have $\langle\langle\Delta\rangle\rangle \circ \top$ as their actions. Since $\vdash \langle\langle\Delta\rangle\rangle \circ \top$ regardless of Δ, we ignore such coalitions in the sequel. The definition of $t(f(v^n,k^n,t^n),b^n)$ entails that $v^n_{actions}$ consists of possibly $\mathsf{D}_\Delta\top$ for some Δ, which we ignore, and $d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,t^n)$ for those $i \in \{1,\dots,m\}$ for which $\langle\langle\Delta_i\rangle\rangle \circ \delta_i \in v^n$. The latter entails $\mathsf{D}_\Gamma\langle\langle\Delta_i\rangle\rangle \circ \delta_i \in v^n$ by axioms $\mathsf{D}_\circ$ and $Mono_\mathsf{D}$, because $\Delta_i \subseteq \Gamma$. Therefore, since $D_\Gamma(v^n) = D_\Gamma(w^n)$, we conclude that $d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,t^n) \in v^n_{actions}$ implies $\langle\langle\Delta_i\rangle\rangle \circ \delta_i \in w^n$. Since $b^n_\Gamma = a^n_\Gamma$, $\Delta_1,\dots,\Delta_m$ are unanimous in a^n too. Hence, by the definition of $t(f(w^n,k^n,s^n),a^n)$, $d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,t^n) \in v^n_{actions}$ implies $\mathsf{D}_{\Delta_i} d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,s^n) \in w^n_{actions}$, $i = 1,\dots,k$. Consequently, due to $\Delta_i \subseteq \Gamma$ again, $D_\Gamma(w^{n+1}) \vdash_{MP} \mathsf{D}_{\Delta_i} d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,s^n)$ for all $d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,t^n) \in v^n_{actions}$. We prove $D_\Gamma(w^{n+1}) \vdash_{MP} \mathsf{D}_{\Delta_i} d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,t^n)$ by establishing $d(\Delta_i,\delta_i,k^n,t^n) = d(\Delta_i,\delta_i,k^n,s^n)$. To conclude that, note that $s^n = t^n$ for $\langle\langle\Delta_i\rangle\rangle \circ \delta_i$ of the form $\langle\langle\Gamma_q\rangle\rangle \circ \langle\langle\Gamma_q\rangle\rangle \diamond_{\Gamma_q} \psi_q$ and n in intervals of the form

$$\{(zM+q)|W|,\dots,(zM+q+1)|W|-1\} \tag{4}$$

because

$$f(v^0,k^0,t^0)b^0 \dots b^{n-1} f(v^n,k^n,t^n) \sim_\Gamma f(w^0,k^0,s^0)a^0 \dots a^{n-1} f(w^n,k^n,s^n),$$

and $d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,t^n) = d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i,k^n,s^n)$ regardless of the values of s^n and t^n for n outside such intervals.

We chose the members of a maximal consistent subset of C, which is consistent with $D_\Gamma(w^{n+1}) \cup \{\neg\psi\}$ to come first in the ordering b^n_e of C. Therefore proving that the consistency of v^{n+1} with $\neg\psi$ and $D_\Gamma(w^{n+1}) = D_\Gamma(v^{n+1})$, which would entail that $v^{n+1}_\Gamma = w^{n+1}_\Gamma$, amounts to proving that $v^n_{actions} \cup v^n_{past} \cup D_\Gamma(w^{n+1}) \cup \{\neg\psi\}$ is consistent. Above we proved that the formulas from $v^n_{actions}$ can be derived from those from $D_\Gamma(w^{n+1})$. Hence we need to prove the consistency of just $v^n_{past} \cup D_\Gamma(w^{n+1}) \cup \{\neg\psi\}$. Assume the contrary for the sake of contradiction. Then, since v^n_{past} is logically equivalent to $\{\ominus\chi : \chi \in v^n\}$, $v^n_{past} \cup \{\ominus\theta\}$ is inconsistent too, whereas $v^n \cup \{\theta\}$ is consistent, which is a contradiction, by the definition of θ and v^n_{past}. ⊣ **Proof:**[of Lemma 24] An immediate check shows that putting $g_i = \langle\langle\Gamma\rangle\rangle \circ \psi$ for all $i \in \Gamma$ brings the required properties. In case Γ is the empty coalition, $\psi \in v^{n+1}$ follows from $\psi \in v^n_{past}$ by the second axiom $\langle\langle.\rangle\rangle \circ \ominus$. ⊣ **Proof:**[of Lemma 25] Let $f(v^0, k^0, t^0)b^0 \dots b^{n-1} f(v^n, k^n, t^n) \in [r]_\Gamma$. Let $b^n_\Gamma = g$ and $b^n_i = \langle\langle i\rangle\rangle \circ \top$ for all $i \in \Sigma \setminus \Gamma$. Then $v^n_{actions}$ consists of the formulas $d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i, k^n, t^n)$ which correspond to the actions $\langle\langle\Delta_i\rangle\rangle \circ \delta_i$ from g such that $\Delta_i \subseteq \Gamma$ is unanimous in b^n and $\langle\langle\Delta_i\rangle\rangle \circ \delta_i \in v^n$, and possibly some formulas of the form $\mathsf{D}_i\top$, which we ignore. Note that, since $\Delta_i \subseteq \Gamma$, $\langle\langle\Delta_i\rangle\rangle \circ \delta_i \in v^n$ for the same $i \in \Gamma$ for all the $v^n \in W$ which appear in the last states of runs from $[r]_\Gamma$. Furthermore, $d(\langle\langle\Delta_i\rangle\rangle \circ \delta_i, k^n, t^n)$ does not depend on t^n for k^n outside intervals of the form (4) where $\langle\langle\Delta_i\rangle\rangle \circ \delta_i$ is $\langle\langle\Gamma_q\rangle\rangle \circ \langle\langle\Gamma_q\rangle\rangle\Diamond\mathsf{D}_{\Gamma_q}\psi_q$, and $t^n = s^n$ for k^n inside intervals of this form. Hence $v^n_{actions}$ does not depend on the choice of v^n as long as b^n is as chosen above. Obviously $\vdash \ominus(\bigwedge v^n) \Rightarrow \alpha$ is equivalent to $v^n_{past} \vdash_{MP} \alpha$ for any formula α. As the environment action b^n_e ranges over all the orderings of C, $v^n_{environment}$ ranges over all the completions of $v^n_{actions} \cup \{\ominus \bigwedge v^n\}$ to a maximal consistent subset of C. The premiss $\psi \in v^{n+1}$ for v^{n+1} such that $\langle v^{n+1}, \dots\rangle = t_0(v^n, k^n, t^n, b^n)$ of the lemma entails $\vdash \bigwedge v^n_{actions} \wedge \ominus(\bigwedge v^n) \wedge \bigwedge v^n_{environment} \Rightarrow \psi$. Since the conjunction on the left of $\Rightarrow$ in this formula is consistent, choosing an environment action b^n_e with $\neg\psi$ as the least element shows that $\vdash \bigwedge v^n_{actions} \wedge \ominus(\bigwedge v^n) \Rightarrow \psi$. Now let

$$\theta_n \rightleftharpoons \bigvee_{f(v^0,k^0,t^0)b^0\dots b^{n-1}f(v^n,k^n,t^n)\in[r]_\Gamma} \bigwedge v^n.$$

Then $\vdash \bigwedge v^n_{actions} \wedge \ominus\theta_n \Rightarrow \psi$ again. Hence $\vdash \langle\langle\Gamma\rangle\rangle \circ (\bigwedge v^n_{actions} \wedge \ominus\theta_n) \Rightarrow \langle\langle\Gamma\rangle\rangle \circ \psi$ by $Mono_{\langle\langle.\rangle\rangle\circ}$. Since $\vdash \bigwedge v^n \Rightarrow \theta_n$ for every possible v^n, we have $\vdash \bigwedge w^n \Rightarrow \mathsf{D}_\Gamma\theta_n$ by Lemma 23. The fact that the unanimous Δ_is are disjoint sub-coalitions of Γ implies that $\langle\langle\Gamma\rangle\rangle \circ \bigwedge v^n_{actions}$ is consistent with every possible v^n, including w^n itself. By $\langle\langle.\rangle\rangle \circ \ominus$, this entails that $\langle\langle\Gamma\rangle\rangle \circ (\bigwedge v^n_{actions} \wedge \ominus\theta_n)$ is consistent with w^n too. As shown above, this entails $\langle\langle\Gamma\rangle\rangle \circ \psi$ is consistent with w^n. ⊣ **Proof:**[of Lemma 26] We are going to prove that an s with the required property can be defined by putting $(s(r'_\Gamma))_i = \langle\langle\Gamma\rangle\rangle \circ \langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi$ for all $i \in \Gamma$ and r'_Γ such that

$$|r'| \in \{n, \dots, n + (M+1)|W| - 1\}. \qquad (5)$$

The value of s for runs of other lengths is irrelevant.

Let $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi$ be $\langle\langle\Gamma_q\rangle\rangle\Diamond\mathsf{D}_{\Gamma_q}\psi_q$ for some $q \in \{0, \dots, M-1\}$. Let

$$f(w^0, k^0, s^0)a^0 \dots a^{m-1} f(w^m, k^m, s^m) \dots \in \mathrm{out}([r]_\Gamma, s).$$

Then the interval from (5) has a subinterval of the form (4). Let $\{x, \dots, x + |W| - 1\}$ be the leftmost such subinterval. (There may be at most two.) Then either there exists a $y_0 \in \{n, \dots, x\}$ such that $\mathsf{D}_\Gamma\psi \in w^{y_0}$, or $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in w^y$ for all $y \in \{n, \dots, x\}$. In both cases this is established by induction on y using the axiom $FP_{\Diamond\mathsf{D}}$ and the fact that $d(\langle\langle\Gamma\rangle\rangle \circ \langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi, k^y, s^y) = \mathsf{D}_\Gamma\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in w^y_{actions}$ for all the relevant ys.

In the first case the k which appears in the lemma can be chosen to be y_0. In the second case $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in w^x$. Then an induction on $y \in \{0, \dots, |W| - 2\}$ shows that $s^{x+y} = 1$, $\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-y-1}}$ is consistent with w^{x+y} and

$$d(\langle\langle\Gamma\rangle\rangle \circ \langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi, k^{x+y}, s^{x+y}) = \mathsf{D}_\Gamma\widehat{W^{\Gamma_q,\psi_q}_{\leq|W|-y-2}} \in w^{x+y}_{actions}.$$

This entails that $w^{x+y_0} \in W^{\Gamma_q,\psi_q}_0$ for some $y_0 < |W|$. Then k from the lemma can be chosen to be $x + y_0$ where y_0 is the least one with the above property. ⊣ **Proof:**[of Lemma 27] Given an r' as above, there exists a least $k < \omega$ such that $\psi \in w^{n+k}$, which we denote by $k(r')$. Consider the set H_r of the runs $r'[0..m]$ where $r' \in \mathrm{out}([r]_\Gamma, s)$ and $n \leq m \leq n + k(r')$. Let $h' \prec h''$ if $h' = h''[0..|h''| - 1]$. $\langle H_r, \prec\rangle$ is the union of finitely many trees, each with its root in $[r]_\Gamma$. Since S is finite, these trees have finite degree of branching. Therefore, by König's Lemma, H_r is finite. Let $k^{\max}_r = \max\{|h| - n : h \in H_r\}$. Note that the conditions of the lemma hold for every $h \in H_r$, that is, if $h \in H_r$, then for every

$$f(w^0, k^0, s^0)a^0 \dots a^{m-1} f(w^m, k^m, s^m) \dots \in \mathrm{out}([h]_\Gamma, s)$$

there exists a $k < \omega$ such that $\mathsf{D}_\Gamma \psi \in w^{|h|+k}$. Furthermore, $H_h \subseteq H_r$ and $k_h^{\max} \leq k_r^{\max} + n - |h|$.

We shall prove the lemma for $h \in H_r$ by induction on $k_h^{\max}$. Let $k_h^{\max} = 0$ and $f(w^m, k^m, s^m)$ be the last state of h. Then $\mathsf{D}_\Gamma \psi \in w^m$, whence $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in w^m$ by axiom $FP_{\Diamond\mathsf{D}}$. For the induction step, assume that $k_h^{\max} > 0$. If $\mathsf{D}_\Gamma \psi \in w^m$, then we reason like in the base case. If not, then for all h' of the form $ha^m f(w^{m+1}, k^{m+1}, s^{m+1})$ such that $a_\Gamma^m = s(h_\Gamma)$ we have $k_{h'}^{\max} < k_h^{\max}$. By the induction hypothesis this entails $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in w^{m+1}$, whence, by Lemma 25, $\langle\langle\Gamma\rangle\rangle \circ \langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in w^m$, and therefore, by axiom $FP_{\Diamond\mathsf{D}}$ again, we conclude $\langle\langle\Gamma\rangle\rangle\Diamond\mathsf{D}_\Gamma\psi \in w^m$. ⊣