Finite Difference Schemes for Multidimensional Boussinesq Equation

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Introduction

In the present work we study the Cauchy problem for the Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \quad x \in \mathbb{R}^n, \ t > 0,$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions

$u(x,t) \to 0$, $\Delta u(x,t) \to 0$ as $|x| \to \infty$,

where Δ is the Laplace operator, α, β_1 and β_2 are positive constants.

This is a 4-th order equation in x and t on unbounded region with non-linearity contained in the term $f(u) = u^2$.

Referencies

For $\beta_2 > 0$ the problem is *well-posed in the sense of Hadamar* and is usually called "Proper" BE or "good" BE.

- Ch. Christov 1994–2010
- Xu&Liu (2009) – existence of a global weak solution; sufficient conditions for both the existence and the lack of a global solution.
- Polat&Ertas (2009) – local and global solution, blow-up of solutions – under different conditions for the nonlinear function $f(u)$.

We assume that the functions u_0, u_1 and $f(u)$ satisfy some regularity conditions so that a unique solution for BE exists and is smooth enough.

Simplification

By the scaling transformation $\frac{x}{\sqrt{\beta_1}} = y$, $\frac{t\sqrt{\beta_2}}{\beta_1} = \tau$ Boussinesq equation can be rewritten in the form

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \Delta \frac{\partial^2 u}{\partial t^2} - \Delta^2 u + \Delta g(u), \quad x \in \mathbb{R}^n, \ t > 0, \qquad (1)$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \qquad (2)$$

where g is connected to f by

$$g(u) = \frac{\beta_1}{\beta_2}\left(\alpha f(u) + (1 - \frac{\beta_2}{\beta_1})u\right).$$

Properties to the Boussinesq equation

Let $\|\cdot\|$ denote the standard norm in $L_2(R^n)$.
Define the energy functional

$$E\left(u(t)\right) = \left\|(-\Delta)^{-1/2}\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|u\|^2 + \|\Delta u\|^2 + \int_{R^n} G(u)du$$

with

$$G(u) = \int_0^u g(s)ds$$

Theorem (Conservation law)

The solution u to Boussinesq problem satisfies the following energy identity

$$E\left(u(t)\right) = E\left(u(0)\right).$$

We obtain similar energy identities for the solutions of the FDS employed in the discretization of problem (1), (2).

Operator form of the Boussinesq equation

$$(I - \Delta)\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = \Delta g(u),$$

(I - the identity operator).

Two splittings are possible:

$$(I - \Delta)w = \Delta g(u), \quad \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = w, \tag{3}$$

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) w = \Delta g(u), \quad (I - \Delta)u = w, \tag{4}$$

where w are different auxiliary functions.
The method employed by Ch. Christov can be transformed to splitting (4).
We will exploit splitting (3).

Notations

- Domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, L_1, L_2 – sufficiently large;
- a uniform mesh with steps h_1, h_2 in Ω:
  $x_i = ih_1, i = -M_1, M_1;\ y_j = jh_2, j = -M_2, M_2;$
- τ - the time step, $t_k = k\tau, k = 0, 1, 2, ...;$
- mesh points (x_i, y_j, t_k);
- $v^k_{(i,j)}$ denotes the discrete approximation $u(x_i, y_j, t_k)$;
- notations for some discrete derivatives of mesh functions:
  - $v^k_{x,(i,j)} = (v^k_{(i+1,j)} - v^k_{(i,j)})/h_1;\quad v^k_{\bar{x},(i,j)} = (v^k_{(i,j)} - v^k_{(i-1,j)})/h_1;$
  - $v^k_{\bar{x}x,(i,j)} = \left(v^k_{(i+1,j)} - 2v^k_{(i,j)} + v^k_{(i-1,j)}\right)/h_1^2;$
  - $v^k_{\bar{t}t,(i,j)} = \left(v^{k+1}_{(i,j)} - 2v^k_{(i,j)} + v^{k-1}_{(i,j)}\right)/\tau^2;$
  - $\Delta_h v = v_{\bar{x}x} + v_{\bar{y}y}$ – the 5-point discrete Laplacian.
  - $(\Delta_h)^2 v = v_{\bar{x}x\bar{x}x} + v_{\bar{y}y\bar{y}y} + 2v_{\bar{x}x\bar{y}y}$ – the discrete biLaplacian

Whenever possible the arguments of the mesh functions $^k_{(i,j)}$ are omitted.

Finite Difference Schemes

In approximation of $\Delta_h v$ and $(\Delta_h)^2 v$ we use v^θ – the symmetric θ-weighted approximation to $v^k_{(i,j)}$:

$v^{\theta,k}_{(i,j)} = \theta v^{k+1}_{(i,j)} + (1-2\theta) v^k_{(i,j)} + \theta v^{k-1}_{(i,j)}, \ \theta \in R.$

for approximation of non-linear term $g(u(x_i, y_j, t_k))$ we use

- either $g(v^k_{(i,j)})$,
- or

$$g_1(v^k) = \frac{G(v^{k+1}) - G(v^{k-1})}{v^{k+1} - v^{k-1}}, \quad G(u) = \int_0^u g(s)ds. \qquad (5)$$

Note that in the case under consideration function $g(v)$ is a polynomial of v, thus the integrals $G(v)$ used in g_1 are explicitly evaluated!

Non-iterative Method (NM)

$$v^k_{\bar{t}t} - \Delta_h v^k_{\bar{t}t} - \Delta_h v^{\theta,k} + (\Delta_h)^2 v^{\theta,k} = \Delta_h g(v^k).$$

Iterative Method (IM)

$$v^k_{\bar{t}t} - \Delta_h v^k_{\bar{t}t} - \Delta_h v^{\theta,k} + (\Delta_h)^2 v^{\theta,k} = \Delta_h g_1(v^k).$$

Initial conditions

$$v^0_{(i,j)} = u_0(x_i, y_j),$$
$$v^1_{(i,j)} = u_0(x_i, y_j) + \tau u_1(x_i, y_j) + 0.5\tau^2 (I - \Delta_h)^{-1} \left(\Delta_h u_0 - (\Delta_h)^2 u_0 + \Delta_h g(u_0)\right)(x_i, y_j).$$

The equations, boundary and initial conditions form two families of finite difference schemes.

We factorize the LHS of IM and NM:

$$\left(I - \Delta_h - \theta\tau^2\Delta_h + \theta\tau^2(\Delta_h)^2\right) v_{\bar{t}t} - \Delta_h v + (\Delta_h)^2 v$$
$$= (I - \Delta_h)\left(\left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t} - \Delta_h v\right).$$

and split NM and IM

Non-iterative Method (NM)

$$(I - \Delta_h)w = \Delta_h g(v), \quad \left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t} - \Delta_h v = w$$

Iterative Method (IM)

$$(I - \Delta_h)w = \Delta_h g_1(v), \quad \left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t} - \Delta_h v = w$$

using an auxiliary function w.

Algorithm for Non-iterative Method (NM)

A. Evaluate $v^{(0)}$, $v^{(1)}$ from the initial conditions

B. For $k = 1, 2, \ldots$ do ($v^{(k-1)}$, $v^{(k)}$ are known)

(a) find w by standard elliptic solver
$(I - \Delta_h)w = \Delta_h g(v^{(k)})$, $w = 0$ (BC),

(b) obtain $v^{(k+1)}$ from
$$\left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t}^{(k)} - \Delta_h v^{(k)} = w, \; v^{(k+1)} = 0 \text{ (BC)}$$

efficient methods for 2D hyperbolic equation
(ADI, economic factorized schemes
$(I - \theta\tau^2 v_{\bar{x}x})(I - \theta\tau^2 v_{\bar{y}y})v_{\bar{t}t} - \Delta_h v = w)$

Algorithm for Iterative Method (IM)

A. Evaluate $v^{(0)}$, $v^{(1)}$ from the initial conditions

B. For $k = 1, 2, \ldots$ do ($v^{(k-1)}$, $v^{(k)}$ are known)

1. take $v^{(k+1)[0]} = v^{(k)}$
2. for $s = 1, 2, \ldots$ repeat steps (a), (b) below until $\left|v^{(k+1)[s+1]} - v^{(k+1)[s]}\right| < \epsilon \left|v^{(k+1)[s]}\right|$
   - (a) find w by standard elliptic solver
     $$(I - \Delta_h)w = \Delta_h \frac{G(v^{(k+1)[s]}) - G(v^{(k-1)})}{v^{(k+1)[s]} - v^{(k-1)}},$$
     $w = 0$ (BC),
   - (b) obtain $v^{(k+1)[s+1]}$ from
     $$\left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t}^{(k)[s+1]} - \Delta_h v^{(k)[s+1]} = w,$$
     $v^{(k+1)[s+1]} = 0$ (BC)
     - ADI, economic factorized schemes
     $$(I - \theta\tau^2 v_{\bar{x}x})(I - \theta\tau^2 v_{\bar{y}y})v_{\bar{t}t} - \Delta_h v = w$$
3. set $v^{(k+1)} = v^{(k+1)[s+1]}$

Analysis of the nonlinear schemes

Preliminaries:

the space of mesh functions which vanish on ω;

the scalar product at time t^k with respect to the spatial variables $\langle v, w\rangle = \sum_{i,j} h_1 h_2 v^{(k)}_{(i,j)} w^{(k)}_{(i,j)}$;

operators $A = -\Delta_h$

$$B = I - \Delta_h + \tau^2\theta(-\Delta_h + (\Delta_h)^2);$$

A is a self-adjoint positive definite operator.

Operator form of the schemes:

$$Bv_{\bar{t}t} + Av + A^2 v = -Ag, \quad Bv_{\bar{t}t} + Av + A^2 v = -Ag_1,$$

$$A^{-1}Bv_{\bar{t}t} + v + Av + g = 0, \quad A^{-1}Bv_{\bar{t}t} + v + Av + g_1 = 0.$$

(derived after applying A^{-1})

The energy functional E_h^L (obtained from the linear part of the equation) at the k-th time level is

$$E_h^L(v^{(k)})$$
$$= \left\langle A^{-1} v_t^{(k)}, v_t^{(k)} \right\rangle + \left\langle v_t^{(k)}, v_t^{(k)} \right\rangle + \tau^2(\theta - 1/4) \left\langle (I + A) v_t^{(k)}, v_t^{(k)} \right\rangle$$
$$+ 1/4 \left\langle v^{(k)} + v^{(k+1)} + A(v^{(k)} + v^{(k+1)}), v^{(k)} + v^{(k+1)} \right\rangle$$

The *full discrete energy functional* is (including the non-linearity)

$$E_h(v^{(k)}) = E_h^L(v^{(k)}) + \left\langle G(v^{(k+1)}), 1 \right\rangle + \left\langle G(v^{(k)}), 1 \right\rangle$$

Theorem (Discrete conservation law)

The solution to the iterative scheme (IM) satisfies the energy equalities

$$E_h(v^{(k)}) = E_h(v^{(0)}), \qquad k = 1, 2, \ldots .$$

i.e. the discrete energy is conserved in time.

$$\theta > \frac{1}{4} - \frac{1}{\tau^2||A||} \qquad (6)$$

Note that if parameter θ satisfies (6), then functional $E_h^L(v^k)$ is nonnegative and can be viewed as a norm. Such combined norms depending on the values of solution on several layers are typical for three-layer schemes.

Theorem (Discrete identities for NM)

The solution to the non-iterative scheme (NM) satisfies the equalities

$$E_h^L(v^{(k)}) + (g(v^k), v^{k+1}) = E_h^L(v^{(k-1)}) + (g(v^{(k)}), v^{(k-1)}), k = 1, 2, \dots$$

The local truncation error of both NM and IM is $O(|h|^2 + \tau^2)$.

Consider the following linear problem with solution u – Boussinesq equation with nonlinear term $\Delta f(u)$ replaced by $\Delta\psi_1$ (ψ_1 is an known function)– and the discrete scheme

$$v_{\bar{t}t} - \Delta_h v_{\bar{t}t} - \Delta_h v^\theta + (\Delta_h)^2 v^\theta = \Delta_h \psi_1. \tag{7}$$

Theorem (stability and convergence)

Let θ be such that

$$\theta > \frac{1}{4} - \frac{1}{\tau^2||A||}.$$

1. *Then the finite difference method (7) is stable with respect to the initial data and the right-hand side.*
2. *If the solution u to the linear problem is smooth enough, then the solution v to (7) converges to the exact solution u and*

$$\max_{(x_i,y_j)} |(v^k(x_i,y_j) - u(x_i,y_j,t^k))| < C\left(h^2+\tau^2\right).$$

Preliminaries

- An analytical solution of the 1D equation (one solitary wave):

$$u(x,t;x_0,c) = \frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-x_0-ct}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}\right),$$

where x_0 is the initial position of the peak of the solitary wave,
- Parameters: $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, c is the wave speed.
- Initial conditions for one solitary wave or two solitary waves:

$$u(x,0) = u(x,0;-40,2) + u(x,0;50,-1.5)$$

$$\frac{du}{dt}(x,0) = u(x,0;-40,2)_t + u(x,0;50,-1.5)_t$$

- Two schemes with $\theta = 0.5$:
  - non-iterative and
  - iterative (inner iterations until relative error $< \epsilon$, $\epsilon = 10^{-13}$).

One solitary wave

Rate of convergence and errors
for $x \in [-100, 100]$, $t \in [0, 20]$, $c = 2$

$h = \tau$	Rate no iter.	Rate with iter.	Error no iter.	Error with iter.	with iter./ no iter.
0.1	–	–	0.02559	0.32271	12.60931
0.05	2.02762	1.87037	0.00628	0.08826	14.06140
0.025	2.00675	1.96892	0.00156	0.02255	14.43498
0.0125	2.00142	1.99221	0.00039	0.00567	14.52742

- The error is the difference between the calculated and the exact solution in uniform norm for $t = 20$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For one solitary wave the non-iterative scheme is about 14 times more precise than the iterative scheme.

Interaction of two solitary waves with different speeds

Rate of convergence and errors
for $x \in [-150, 150]$, $t \in [0, 40]$, $c_1 = 2$, $c_2 = -1.5$

$h = \tau$	Rate no iter.	Rate with iter.	Error no iter.	Error with iter.	with iter./ no iter.
0.04	2.09561	1.97465	0.017375	0.102754	5.913796
0.02	1.94485	1.99369	0.017375	0.026027	6.187079
0.01	1.97704	1.99838	0.001084	0.006528	6.021106

- For every h the error is calculated by Runge method as $E_1^2/(E_1 - E_2)$ with $E_1 = \|u_{[h]} - u_{[h/2]}\|$, $E_2 = \|u_{[h/2]} - u_{[h/4]}\|$, where $u_{[h]}$ is the calculated solution with step h for $t = 40$.
- The numerical rate of convergence is $(\log E_1 - \log E_2)/\log 2$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For two solitary waves the non-iterative scheme is about 6 times more precise than the iterative scheme.

With respect to the error magnitude the non-iterative method performs much better than the iterative method!

Justification: Consider the right-hand side of the iterative method. We expand $g_1(u(x_i, t^k))$ in Taylor series about the point (x_i, t^k) and get

$$g_1(u(x_i, t^k)) = g(u(x_i, t^k)) + \tau^2 R + O(\tau^3),$$

$$R = \alpha \frac{\beta_1}{\beta_2} \left(\frac{1}{3} \left(\frac{\partial u}{\partial t}(x_i, t^k) \right)^2 + u \frac{\partial^2 u}{\partial t^2}(x_i, t^k) \right) + \frac{1}{2} \left(\frac{\beta_1}{\beta_2} - 1 \right) \frac{\partial^2 u}{\partial t^2}(x_i, t^k).$$

Thus, the right-hand sides of the two methods

$$\Delta_h g_1(u(x_i, t^k)) - \Delta_h g(u(x_i, t^k)) = \tau^2 \Delta_h R + O(\tau^3).$$

differ by terms of order $O(\tau^2)$. This has essential impact on the error, when the solution has large derivatives!

Error dependence on time step with a fixed space step

One solitary wave with $c = 2$, non-iterative scheme

$h = 0.01$, $x \in [-100, 100]$, $t \in [0, 20]$

τ	L_1 error	Rate	L_2 error	Rate	L_∞ error	Rate
0.32	2.781695	-	0.703758	-	0.384748	-
0.16	0.534777	5.20	0.136372	5.16	0.075703	5.08
0.08	0.128292	4.17	0.032492	4.19	0.017982	4.21
0.04	0.031634	4.06	0.007996	4.06	0.004422	4.07
0.02	0.007748	4.08	0.001956	4.09	0.001082	4.09
0.01	0.001793	4.32	0.000450	4.34	0.000249	4.33
0.005	0.000305	5.87	7.439e-5	6.05	4.174e-5	5.98
0.001	0.000171		4.630e-5		2.478e-5	

- For $\tau \geq h/c$ the error behaves as $O(\tau^2)$
- For $\tau < h/(4c)$ the error does not depend on τ.
- The error behavior is similar in every norm.

Error dependence on time step with a fixed space step

One solitary wave with $c = 2$, iterative scheme

$h = 0.01$, $x \in [-100, 100]$, $t \in [0, 20]$

τ	L_1 error	Rate	L_2 error	Rate	L_∞ error	Rate
0.16	4.961105	-	1.335671	-	0.677222	-
0.08	1.427490	3.48	0.402828	3.32	0.214408	3.16
0.04	0.370195	3.86	0.105644	3.81	0.056685	3.78
0.02	0.093553	3.96	0.026768	3.95	0.014385	3.94
0.01	0.023588	3.97	0.006751	3.97	0.003630	3.96
0.005	0.006044	3.90	0.001727	3.91	0.000929	3.91
0.0025	0.001654	3.65	0.000470	3.67	0.000253	3.67
0.00125	0.000557	2.97	0.000156	3.02	8.3887e-5	3.01

- For $\tau \geq h/c$ the error behaves as $O(\tau^2)$
- For $\tau \leq h/(2c)$ the error does not depend on τ.
- The error behavior is similar in every norm.

Discrete identities errors

The error is maximum for every $t \in [0, 40]$
of the numerical integral for $x \in [-150, 150]$,
either for one solitary wave with $c_1 = 2$
or for two solitary waves with $c_1 = 2$, $c_2 = -1.5$.

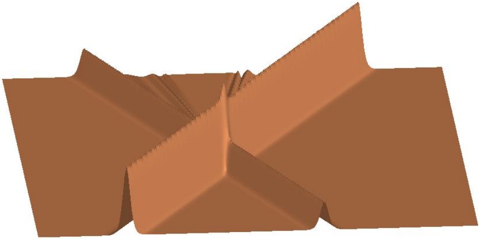
$\tau = h$	1 soliton no iter.	1 soliton with iter.	2 solitons no iter.	2 soliton with iter.
0.1	3.1264e-13	2.3152e-13	9.3245e-11	5.6192e-10
0.05	3.9790e-13	4.1866e-13	1.3416e-11	7.3909e-11
0.025	6.2528e-13	5.3321e-13	2.1630e-12	9.3973e-12
0.0125	1.0232e-13	8.9952e-13	1.2921e-12	1.2091e-12

- The discrete identities are different for the iterative and for the non-iterative schemes (conservation law for IM and discrete identities for NM)
- The table shows the numerical solution satisfies the respective discrete identities.

Movie

Interaction of two solitary waves with different speeds
$x \in [-120, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

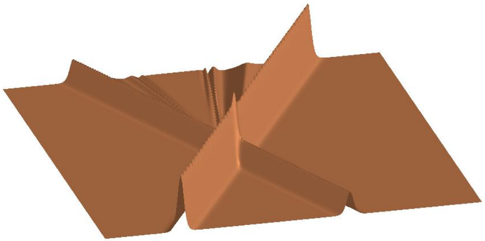
Graphics



Interaction of two solitary waves with different speeds

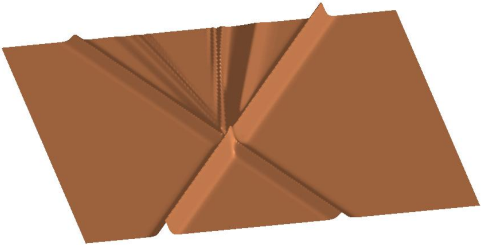
$x \in [-80, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Graphics



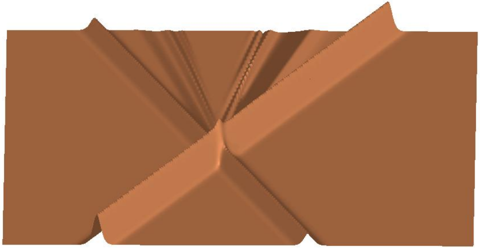
Interaction of two solitary waves with different speeds
$x \in [-80, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Graphics



Interaction of two solitary waves with different speeds
$x \in [-80, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Graphics



Interaction of two solitary waves with different speeds

$x \in [-80, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Thank you
for your attention!