Finite Difference Schemes for Multidimensional Boussinesq Equation

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In the present work we study the Cauchy problem for the Boussinesq equation

\[
\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \quad x \in \mathbb{R}^n, \quad t > 0,
\]

\[
u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x),
\]

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions

\[
u(x, t) \to 0, \quad \Delta u(x, t) \to 0 \text{ as } |x| \to \infty,
\]

where $\Delta$ is the Laplace operator, $\alpha, \beta_1$ and $\beta_2$ are positive constants.

This is a 4-th order equation in $x$ and $t$ on unbounded region with non-linearity contained in the term $f(u) = u^2$. 
For $\beta_2 > 0$ the problem is *well-posed in the sense of Hadamar* and is usually called “Proper” BE or “good” BE.

- Ch. Christov 1994–2010
- Xu&Liu (2009) – existence of a global weak solution; sufficient conditions for both the existence and the lack of a global solution.
- Polat&Ertas (2009) – local and global solution, blow-up of solutions – under different conditions for the nonlinear function $f(u)$.

We assume that the functions $u_0, u_1$ and $f(u)$ satisfy some regularity conditions so that a unique solution for BE exists and is smooth enough.
By the scaling transformation $\frac{x}{\sqrt{\beta_1}} = y$, $\frac{t\sqrt{\beta_2}}{\beta_1} = \tau$ Boussinesq equation can be rewritten in the form

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \Delta \frac{\partial^2 u}{\partial t^2} - \Delta^2 u + \Delta g(u), \quad x \in \mathbb{R}^n, \quad t > 0, \quad (1)$$

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x), \quad (2)$$

where $g$ is connected to $f$ by

$$g(u) = \frac{\beta_1}{\beta_2} \left( \alpha f(u) + (1 - \frac{\beta_2}{\beta_1})u \right).$$
Properties to the Boussinesq equation

Let $\| \cdot \|$ denote the standard norm in $L_2(\mathbb{R}^n)$. Define the energy functional

$$E(u(t)) = \left\| (-\Delta)^{-1/2} \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 + \| u \|^2 + \| \Delta u \|^2 + \int_{\mathbb{R}^n} G(u) du$$

with

$$G(u) = \int_0^u g(s) ds$$

**Theorem (Conservation law)**

*The solution $u$ to Boussinesq problem satisfies the following energy identity*

$$E(u(t)) = E(u(0)).$$

We obtain similar energy identities for the solutions of the FDS employed in the discretization of problem (1), (2).
Operator form of the Boussinesq equation

\[(I - \Delta) \left( \frac{\partial^2}{\partial t^2} - \Delta \right) u = \Delta g(u), \]

\[(I - \Delta)w = \Delta g(u), \quad \left( \frac{\partial^2}{\partial t^2} - \Delta \right) u = w, \quad \text{(3)}\]

\[\left( \frac{\partial^2}{\partial t^2} - \Delta \right) w = \Delta g(u), \quad (I - \Delta)u = w, \quad \text{(4)}\]

where \(w\) are different auxiliary functions.
The method employed by Ch. Christov can be transformed to splitting (4).
We will exploit splitting (3).
Notations

- Domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, $L_1, L_2$ – sufficiently large;
- a uniform mesh with steps $h_1, h_2$ in $\Omega$: $x_i = ih_1, i = -M_1, M_1; y_j = jh_2, j = -M_2, M_2$;
- $\tau$ - the time step, $t_k = k\tau, k = 0, 1, 2, ...$;
- mesh points $(x_i, y_j, t_k)$;
- $v^k_{(i,j)}$ denotes the discrete approximation $u(x_i, y_j, t_k)$;
- notations for some discrete derivatives of mesh functions:
  - $v^k_{x,(i,j)} = (v^k_{(i+1,j)} - v^k_{(i,j)})/h_1; \quad \bar{v}^k_{x,(i,j)} = (v^k_{(i,j)} - v^k_{(i-1,j)})/h_1$;
  - $v^k_{xx,(i,j)} = \left( v^k_{(i+1,j)} - 2v^k_{(i,j)} + v^k_{(i-1,j)} \right)/h_1^2$;
  - $v^k_{tt,(i,j)} = \left( v^k_{(i,j)} - 2v^k_{(i,j)} + v^k_{(i-1,j)} \right)/\tau^2$;
  - $\Delta_h v = v_{xx} + v_{yy}$ – the 5-point discrete Laplacian.
  - $(\Delta_h)^2 v = v_{x\!x\!x\!x\!x} + v_{y\!y\!y\!y\!y} + 2v_{x\!x\!y\!y}$ – the discrete biLaplacian

Whenever possible the arguments of the mesh functions $v^k_{(i,j)}$ are omitted.
In approximation of $\Delta_h v$ and $(\Delta_h)^2 v$ we use $v^\theta$ – the symmetric $\theta$-weighted approximation to $v_{(i,j)}^k$:

$$v_{(i,j)}^{\theta,k} = \theta v_{(i,j)}^{k+1} + (1 - 2\theta) v_{(i,j)}^k + \theta v_{(i,j)}^{k-1}, \ \theta \in \mathbb{R}.$$  

for approximation of non-linear term $g(u(x_i, y_j, t_k))$ we use

- either $g(v_{(i,j)}^k)$,
- or

$$g_1(v^k) = \frac{G(v^{k+1}) - G(v^{k-1})}{v^{k+1} - v^{k-1}}, \ \ G(u) = \int_0^u g(s)ds. \ \ (5)$$

Note that in the case under consideration function $g(v)$ is a polynomial of $v$, thus the integrals $G(v)$ used in $g_1$ are explicitly evaluated!
Non-iterative Method (NM)

\[ \nu^k_{tt} - \Delta_h \nu^k_{tt} - \Delta_h \nu^\theta,k + (\Delta_h)^2 \nu^\theta,k = \Delta_h g(\nu^k). \]

Iterative Method (IM)

\[ \nu^k_{tt} - \Delta_h \nu^k_{tt} - \Delta_h \nu^\theta,k + (\Delta_h)^2 \nu^\theta,k = \Delta_h g_1(\nu^k). \]

Initial conditions

\[
\begin{align*}
\nu^0_{(i,j)} &= u_0(x_i, y_j), \\
\nu^1_{(i,j)} &= u_0(x_i, y_j) + \tau u_1(x_i, y_j) \\
&\quad + 0.5\tau^2 (I - \Delta_h)^{-1} (\Delta_h u_0 - (\Delta_h)^2 u_0 + \Delta_h g(u_0)) (x_i, y_j).
\end{align*}
\]

The equations, boundary and initial conditions form two families of finite difference schemes.
We factorize the LHS of IM and NM:

\[
(I - \Delta_h - \theta \tau^2 \Delta_h + \theta \tau^2 (\Delta_h)^2) \ v_{tt} - \Delta_h v + (\Delta_h)^2 v
\]

\[
= (I - \Delta_h) \left( (I - \theta \tau^2 \Delta_h) \ v_{tt} - \Delta_h v \right).
\]

and split NM and IM

**Non-iterative Method (NM)**

\[
(I - \Delta_h)w = \Delta_h g(v), \quad (I - \theta \tau^2 \Delta_h) \ v_{tt} - \Delta_h v = w
\]

**Iterative Method (IM)**

\[
(I - \Delta_h)w = \Delta_h g_1(v), \quad (I - \theta \tau^2 \Delta_h) \ v_{tt} - \Delta_h v = w
\]

using an auxiliary function \(w\).
Algorithm for Non-iterative Method (NM)

A. Evaluate $v^{(0)}$, $v^{(1)}$ from the initial conditions

B. For $k = 1, 2, \ldots$ do ($v^{(k-1)}$, $v^{(k)}$ are known)

(a) find $w$ by standard elliptic solver

$$ (I - \Delta_h)w = \Delta_h g(v^{(k)}), \quad w = 0 \text{ (BC)} $$

(b) obtain $v^{(k+1)}$ from

$$ (I - \theta \tau^2 \Delta_h) v^{(k)}_{tt} - \Delta_h v^{(k)} = w, \quad v^{(k+1)} = 0 \text{ (BC)} $$

efficient methods for 2D hyperbolic equation
(ADI, economic factorized schemes

$$ (I - \theta \tau^2 v_{\bar{x}\bar{x}})(I - \theta \tau^2 v_{\bar{y}\bar{y}})v_{tt} - \Delta_h v = w $$
Algorithm for Iterative Method (IM)

A. Evaluate $v^{(0)}$, $v^{(1)}$ from the initial conditions

B. For $k = 1, 2, \ldots$ do ($v^{(k-1)}$, $v^{(k)}$ are known)

1. take $v^{(k+1)[0]} = v^{(k)}$

2. for $s = 1, 2, \ldots$ repeat steps (a), (b) below until $|v^{(k+1)[s+1]} - v^{(k+1)[s]}| < \epsilon |v^{(k+1)[s]}|$

(a) find $w$ by standard elliptic solver

$$(I - \Delta_h)w = \Delta_h \frac{G(v^{(k+1)[s]}) - G(v^{(k-1)})}{v^{(k+1)[s]} - v^{(k-1)}},$$

$w = 0$ (BC),

(b) obtain $v^{(k+1)[s+1]}$ from

$$(I - \theta \tau^2 \Delta_h) v^{(k)[s+1]}_{tt} - \Delta_h v^{(k)[s+1]} = w,$$

$v^{(k+1)[s+1]} = 0$ (BC)

- ADI, economic factorized schemes

$$(I - \theta \tau^2 v_{xx})(I - \theta \tau^2 v_{yy})v_{tt} - \Delta_h v = w$$

3. set $v^{(k+1)} = v^{(k+1)[s+1]}$
Preliminaries:
the space of mesh functions which vanish on $\omega$;
the scalar product at time $t^k$ with respect to the spatial variables
$$\langle v, w \rangle = \sum_{i,j} h_1 h_2 v^{(k)}(i,j) w^{(k)}(i,j);$$
operators $A = -\Delta_h$
$$B = I - \Delta_h + \tau^2 \theta(-\Delta_h + (\Delta_h)^2);$$
$A$ is a self-adjoint positive definite operator.

Operator form of the schemes:

$$Bv_{tt} + Av + A^2 v = -Ag,$$
$$A^{-1} Bv_{tt} + v + Av + g = 0,$$
$$Bv_{tt} + Av + A^2 v = -Ag_1,$$
$$A^{-1} Bv_{tt} + v + Av + g_1 = 0.$$

(derived after applying $A^{-1}$)
The energy functional $E^L_h$ (obtained from the linear part of the equation) at the $k$-th time level is

$$E^L_h(v^{(k)})$$

$$= \left\langle A^{-1}v_t^{(k)}, v_t^{(k)} \right\rangle + \left\langle v_t^{(k)}, v_t^{(k)} \right\rangle + \tau^2(\theta - 1/4) \left\langle (I + A)v_t^{(k)}, v_t^{(k)} \right\rangle$$

$$+ 1/4 \left\langle v^{(k)} + v^{(k+1)} + A(v^{(k)} + v^{(k+1)}), v^{(k)} + v^{(k+1)} \right\rangle$$

The full discrete energy functional is (including the non-linearity)

$$E_h(v^{(k)}) = E^L_h(v^{(k)}) + \left\langle G(v^{(k+1)}), 1 \right\rangle + \left\langle G(v^{(k)}), 1 \right\rangle$$

Theorem (Discrete conservation law)

The solution to the iterative scheme (IM) satisfies the energy equalities

$$E_h(v^{(k)}) = E_h(v^{(0)}), \quad k = 1, 2, \ldots$$

i.e. the discrete energy is conserved in time.
\begin{equation}
\theta > \frac{1}{4} - \frac{1}{\tau^2||A||}
\end{equation}

Note that if parameter $\theta$ satisfies (6), then functional $E_h^L(v^k)$ is nonnegative and can be viewed as a norm. Such combined norms depending on the values of solution on several layers are typical for three-layer schemes.

**Theorem (Discrete identities for NM )**

*The solution to the non-iterative scheme (NM) satisfies the equalities*

\[
E_h^L(v^{(k)}) + (g(v^k), v^{k+1}) = E_h^L(v^{(k-1)}) + (g(v^k), v^{(k-1)}), k = 1, 2, \ldots
\]

**The local truncation error of both NM and IM is $O(|h|^2 + \tau^2).$**
Consider the following linear problem with solution $u$ – Boussinesq equation with nonlinear term $\Delta f(u)$ replaced by $\Delta \psi_1$ ($\psi_1$ is an known function)– and the discrete scheme

$$v_{tt} - \Delta_h v_{tt} - \Delta_h v^\theta + (\Delta_h)^2 v^\theta = \Delta_h \psi_1. \quad (7)$$

**Theorem (stability and convergence)**

Let $\theta$ be such that

$$\theta > \frac{1}{4} - \frac{1}{\tau^2 \|A\|}.$$  

1. Then the finite difference method (7) is stable with respect to the initial data and the right-hand side.

2. If the solution $u$ to the linear problem is smooth enough, then the solution $v$ to (7) converges to the exact solution $u$ and

$$\max_{(x_i, y_j)} |v^k(x_i, y_j) - u(x_i, y_j, t^k)| < C \left( h^2 + \tau^2 \right).$$
An analytical solution of the 1D equation (one solitary wave):

\[ u(x, t; x_0, c) = \frac{3}{2} \frac{c^2 - 1}{\alpha} \text{sech}^2 \left( \frac{x - x_0 - ct}{2} \sqrt{\frac{c^2 - 1}{\beta_1 c^2 - \beta_2}} \right), \]

where \( x_0 \) is the initial position of the peak of the solitary wave,

- Parameters: \( \alpha = 3, \beta_1 = 1.5, \beta_2 = 0.5, c \) is the wave speed.
- Initial conditions for one solitary wave or two solitary waves:

\[
\begin{align*}
  u(x, 0) &= u(x, 0; -40, 2) + u(x, 0; 50, -1.5) \\
  \frac{du}{dt}(x, 0) &= u(x, 0; -40, 2)_t + u(x, 0; 50, -1.5)_t 
\end{align*}
\]

- Two schemes with \( \theta = 0.5 \):
  - non-iterative and
  - iterative (inner iterations until relative error < \( \epsilon, \epsilon = 10^{-13} \)).
### One solitary wave

Rate of convergence and errors for $x \in [-100, 100]$, $t \in [0, 20]$, $c = 2$

<table>
<thead>
<tr>
<th>$h = \tau$</th>
<th>Rate no iter.</th>
<th>Rate with iter.</th>
<th>Error no iter.</th>
<th>Error with iter.</th>
<th>with iter./no iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>–</td>
<td>–</td>
<td>0.02559</td>
<td>0.32271</td>
<td>12.60931</td>
</tr>
<tr>
<td>0.05</td>
<td>2.02762</td>
<td>1.87037</td>
<td>0.00628</td>
<td>0.08826</td>
<td>14.06140</td>
</tr>
<tr>
<td>0.025</td>
<td>2.00675</td>
<td>1.96892</td>
<td>0.00156</td>
<td>0.02255</td>
<td>14.43498</td>
</tr>
<tr>
<td>0.0125</td>
<td>2.00142</td>
<td>1.99221</td>
<td>0.00039</td>
<td>0.00567</td>
<td>14.52742</td>
</tr>
</tbody>
</table>

- The error is the difference between the calculated and the exact solution in uniform norm for $t = 20$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For one solitary wave the non-iterative scheme is about 14 times more precise than the iterative scheme.
Interaction of two solitary waves with different speeds

Rate of convergence and errors
for \( x \in [-150, 150], \ t \in [0, 40], \ c_1 = 2, \ c_2 = -1.5 \)

<table>
<thead>
<tr>
<th>( h = \tau )</th>
<th>Rate no iter.</th>
<th>Rate with iter.</th>
<th>Error no iter.</th>
<th>Error with iter.</th>
<th>with iter./no iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>2.09561</td>
<td>1.97465</td>
<td>0.017375</td>
<td>0.102754</td>
<td>5.913796</td>
</tr>
<tr>
<td>0.02</td>
<td>1.94485</td>
<td>1.99369</td>
<td>0.017375</td>
<td>0.026027</td>
<td>6.187079</td>
</tr>
<tr>
<td>0.01</td>
<td>1.97704</td>
<td>1.99838</td>
<td>0.001084</td>
<td>0.006528</td>
<td>6.021106</td>
</tr>
</tbody>
</table>

- For every \( h \) the error is calculated by Runge method as \( E_1^2 / (E_1 - E_2) \) with \( E_1 = \|u[h] - u[h/2]\|, \ E_2 = \|u[h/2] - u[h/4]\|, \) where \( u[h] \) is the calculated solution with step \( h \) for \( t = 40 \).
- The numerical rate of convergence is \( (\log E_1 - \log E_2) / \log 2 \).
- The calculations confirm the schemes are of order \( O(h^2 + \tau^2) \).
- For two solitary waves the non-iterative scheme is about 6 times more precise than the iterative scheme.
With respect to the error magnitude the non-iterative method performs much better than the iterative method!

**Justification:** Consider the right-hand side of the iterative method. We expand \( g_1(u(x_i, t^k)) \) in Taylor series about the point \((x_i, t^k)\) and get

\[
g_1(u(x_i, t^k)) = g(u(x_i, t^k)) + \tau^2 R + O(\tau^3),
\]

\[
R = \alpha \frac{\beta_1}{\beta_2} \left( \frac{1}{3} \left( \frac{\partial u}{\partial t}(x_i, t^k) \right)^2 + u \frac{\partial^2 u}{\partial t^2}(x_i, t^k) \right) + \frac{1}{2} \left( \frac{\beta_1}{\beta_2} - 1 \right) \frac{\partial^2 u}{\partial t^2}(x_i, t^k).
\]

Thus, the right-hand sides of the two methods

\[
\Delta_h g_1(u(x_i, t^k)) - \Delta_h g(u(x_i, t^k)) = \tau^2 \Delta_h R + O(\tau^3).
\]

differ by terms of order \(O(\tau^2)\). This has essential impact on the error, when the solution has large derivatives!
One solitary wave with $c = 2$, non-iterative scheme
$h = 0.01$, $x \in [-100, 100]$, $t \in [0, 20]$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$L_1$ error</th>
<th>Rate</th>
<th>$L_2$ error</th>
<th>Rate</th>
<th>$L_\infty$ error</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.32</td>
<td>2.781695</td>
<td>-</td>
<td>0.703758</td>
<td>-</td>
<td>0.384748</td>
<td>-</td>
</tr>
<tr>
<td>0.16</td>
<td>0.534777</td>
<td>5.20</td>
<td>0.136372</td>
<td>5.16</td>
<td>0.075703</td>
<td>5.08</td>
</tr>
<tr>
<td>0.08</td>
<td>0.128292</td>
<td>4.17</td>
<td>0.032492</td>
<td>4.19</td>
<td>0.017982</td>
<td>4.21</td>
</tr>
<tr>
<td>0.04</td>
<td>0.031634</td>
<td>4.06</td>
<td>0.007996</td>
<td>4.06</td>
<td>0.004422</td>
<td>4.07</td>
</tr>
<tr>
<td>0.02</td>
<td>0.007748</td>
<td>4.08</td>
<td>0.001956</td>
<td>4.09</td>
<td>0.001082</td>
<td>4.09</td>
</tr>
<tr>
<td>0.01</td>
<td>0.001793</td>
<td>4.32</td>
<td>0.000450</td>
<td>4.34</td>
<td>0.000249</td>
<td>4.33</td>
</tr>
<tr>
<td>0.005</td>
<td>0.000305</td>
<td>5.87</td>
<td>7.439e-5</td>
<td>6.05</td>
<td>4.174e-5</td>
<td>5.98</td>
</tr>
<tr>
<td>0.001</td>
<td>0.000171</td>
<td>4.630e-5</td>
<td></td>
<td></td>
<td>2.478e-5</td>
<td></td>
</tr>
</tbody>
</table>

- For $\tau \geq h/c$ the error behaves as $O(\tau^2)$
- For $\tau < h/(4c)$ the error does not depend on $\tau$.
- The error behavior is similar in every norm.
Error dependence on time step with a fixed space step

One solitary wave with $c = 2$, iterative scheme

$h = 0.01$, $x \in [-100, 100]$, $t \in [0, 20]$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$L_1$ error</th>
<th>Rate</th>
<th>$L_2$ error</th>
<th>Rate</th>
<th>$L_\infty$ error</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>4.961105</td>
<td>-</td>
<td>1.335671</td>
<td>-</td>
<td>0.677222</td>
<td>-</td>
</tr>
<tr>
<td>0.08</td>
<td>1.427490</td>
<td>3.48</td>
<td>0.402828</td>
<td>3.32</td>
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<td>0.04</td>
<td>0.370195</td>
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<td>0.02</td>
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<td>0.01</td>
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<td>0.001727</td>
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<td>0.0025</td>
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<td>3.65</td>
<td>0.000470</td>
<td>3.67</td>
<td>0.000253</td>
<td>3.67</td>
</tr>
<tr>
<td>0.00125</td>
<td>0.000557</td>
<td>2.97</td>
<td>0.000156</td>
<td>3.02</td>
<td>8.3887e-5</td>
<td>3.01</td>
</tr>
</tbody>
</table>

- For $\tau \geq h/c$ the error behaves as $O(\tau^2)$
- For $\tau \leq h/(2c)$ the error does not depend on $\tau$.
- The error behavior is similar in every norm.
Discrete identities errors

The error is maximum for every $t \in [0, 40]$ of the numerical integral for $x \in [-150, 150]$, either for one solitary wave with $c_1 = 2$ or for two solitary waves with $c_1 = 2, c_2 = -1.5$.

<table>
<thead>
<tr>
<th>$\tau = h$</th>
<th>1 soliton no iter.</th>
<th>1 soliton with iter.</th>
<th>2 solitons no iter.</th>
<th>2 soliton with iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.1264e-13</td>
<td>2.3152e-13</td>
<td>9.3245e-11</td>
<td>5.6192e-10</td>
</tr>
<tr>
<td>0.05</td>
<td>3.9790e-13</td>
<td>4.1866e-13</td>
<td>1.3416e-11</td>
<td>7.3909e-11</td>
</tr>
<tr>
<td>0.025</td>
<td>6.2528e-13</td>
<td>5.3321e-13</td>
<td>2.1630e-12</td>
<td>9.3973e-12</td>
</tr>
<tr>
<td>0.0125</td>
<td>1.0232e-13</td>
<td>8.9952e-13</td>
<td>1.2921e-12</td>
<td>1.2091e-12</td>
</tr>
</tbody>
</table>

- The discrete identities are different for the iterative and for the non-iterative schemes (conservation law for IM and discrete identities for NM)
- The table shows the numerical solution satisfies the respective discrete identities.
Interaction of two solitary waves with different speeds
\[ x \in [-120, 120], \ t \in [0, 35], \ c_1 = 2, \ c_2 = -1.5 \]
Interaction of two solitary waves with different speeds
\[ x \in [-80, 120], \ t \in [0, 35], \ c_1 = 2, \ c_2 = -1.5 \]
Interaction of two solitary waves with different speeds

\[ x \in [-80, 120], \ t \in [0, 35], \ c_1 = 2, \ c_2 = -1.5 \]
Interaction of two solitary waves with different speeds
\[ x \in [-80, 120], \ t \in [0, 35], \ c_1 = 2, \ c_2 = -1.5 \]
Interaction of two solitary waves with different speeds
\[ x \in [-80, 120], \ t \in [0, 35], \ c_1 = 2, \ c_2 = -1.5 \]
Thank you for your attention!