Convergence of Finite Difference Schemes for a Multidimensional Boussinesq Equation

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Introduction

In the present work we study the Cauchy problem for the Boussinesq type equation (called Boussinesq Paradigm Equation):

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \quad x \in \mathbb{R}^n, \ t > 0,$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions

$u(x,t) \to 0$, $\Delta u(x,t) \to 0$ as $|x| \to \infty$,

where Δ is the Laplace operator, α, β_1 and β_2 are positive constants.

This is a 4-th order equation in x and t on unbounded region with non-linearity contained in the term $f(u) = u^2$.

Referencies

BPE and similar: "good BE", "damped BE", "improved BE", ...:

- 1D: existence (local and global in time), uniqueness of weak and strong solutions: Pani& Saranga (1997); Wang& Chen(2002, 2006);
- 1D: blow-up : Liu& Xu(2008); Wang&Chen (2002)
- 1D: numerical solutions – FDS, FEM, spectral and pseudo-spectral methods: Christov & Velarde (1994); Ortega & Sanz-Serna (1990); El-Zoheiry:(2002)
- multidimensional BE: existence, smoothness and blow-up: Varlamov (2007); Xu& Liu (2009); Polat&Ertas (2009)
- 2D BE, numerical investigation: Chertock, Christov& Kurganov (submitted)

A Simplified Form

By the scaling transformation $\frac{x}{\sqrt{\beta_1}} = y$, $\frac{t\sqrt{\beta_2}}{\beta_1} = \tau$ Boussinesq equation can be rewritten in the form

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \Delta \frac{\partial^2 u}{\partial t^2} - \Delta^2 u + \Delta g(u), \quad x \in \mathbb{R}^n, \; t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \tag{2}$$

where g is connected to f by

$$g(u) = \frac{\beta_1}{\beta_2}\left(\alpha f(u) + (1 - \frac{\beta_2}{\beta_1})u\right).$$

We assume that the functions u_0, u_1 and f satisfy some regularity conditions so that a unique solution for BE exists and is smooth enough, say $u \in C^{6,4}\left(\mathbb{R}^d \times (0,T)\right)$.

Properties to the Boussinesq equation

Let $\|\cdot\|$ denote the standard norm in $L_2(\mathbb{R}^d)$.
Define the energy functional

$$E\left(u(\cdot,t)\right) = \left\|(-\Delta)^{-1/2}\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|u\|^2 + \|\Delta u\|^2 + \int_{\mathbb{R}^n} G(u)dx$$

with

$$G(u) = \int_0^u g(s)ds$$

Theorem (Conservation law)

The solution u to Boussinesq problem satisfies the following energy identity

$$E\left(u(\cdot,t)\right) = E\left(u(\cdot,0)\right).$$

We obtain similar energy identities for the solutions of the FDS employed in the discretization of problem (1), (2).

The Simplified form allow us to write BE as

Operator form of the Boussinesq equation

$$(I-\Delta)\left(\frac{\partial^2}{\partial t^2}-\Delta\right)u=\Delta g(u)$$

(I - the identity operator).

Two splittings of BE are possible:

$$(I-\Delta)w=\Delta g(u),\quad \left(\frac{\partial^2}{\partial t^2}-\Delta\right)u=w, \tag{3}$$

$$\left(\frac{\partial^2}{\partial t^2}-\Delta\right)\tilde{w}=\Delta g(u),\quad (I-\Delta)u=\tilde{w}, \tag{4}$$

where w, $\tilde{w}$ are auxiliary functions.
In the presentation we exploit splitting (3).

Notations

- Domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, L_1, L_2 – sufficiently large;
- a uniform mesh with steps h_1, h_2 in Ω: $x_i = ih_1, i = -N_1, N_1$; $y_j = jh_2, j = -N_2, N_2$;
- τ - the time step, $t_k = k\tau, k = 0, 1, 2, \ldots$;
- mesh points (x_i, y_j, t_k);
- $v^k_{(i,j)}$ denotes the discrete approximation $u(x_i, y_j, t_k)$;
- notations for some discrete derivatives of mesh functions:
    - $v^k_{x,(i,j)} = (v^k_{(i+1,j)} - v^k_{(i,j)})/h_1$; $\quad v^k_{\bar{x},(i,j)} = (v^k_{(i,j)} - v^k_{(i-1,j)})/h_1$;
    - $v^k_{\bar{x}x,(i,j)} = \left(v^k_{(i+1,j)} - 2v^k_{(i,j)} + v^k_{(i-1,j)}\right)/h_1^2$;
    - $v^k_{\bar{t}t,(i,j)} = \left(v^{k+1}_{(i,j)} - 2v^k_{(i,j)} + v^{k-1}_{(i,j)}\right)/\tau^2$;
    - $\Delta_h v = v_{\bar{x}x} + v_{\bar{y}y}$ – the 5-point discrete Laplacian.
    - $(\Delta_h)^2 v = v_{\bar{x}x\bar{x}x} + v_{\bar{y}y\bar{y}y} + 2v_{\bar{x}x\bar{y}y}$ – the discrete biLaplacian

Whenever possible the arguments of the mesh functions ${}^k_{(i,j)}$ are omitted.

Finite Difference Schemes

In the approximations $\Delta_h v$ and $(\Delta_h)^2 v$ we use v^θ – the symmetric θ-weighted approximation to $v^k_{(i,j)}$:

$$v^{\theta,k}_{(i,j)} = \theta v^{k+1}_{(i,j)} + (1-2\theta) v^k_{(i,j)} + \theta v^{k-1}_{(i,j)}, \quad \theta \in \mathbb{R}.$$

We approximate the non-linear term $g(u(x_i, y_j, t_k))$ by:

- either $g(v^k_{(i,j)})$;
- or $g_1(v^k_{(i,j)})$,

$$g_1(v^k) = \frac{G(v^{k+1}) - G(v^{k-1})}{v^{k+1} - v^{k-1}}, \quad G(u) = \int_0^u g(s)ds. \qquad (5)$$

Note that in the classical case of polynomial f the function $g(v)$ is a polynomial of v, thus the integrals $G(v)$ used in g_1 are explicitly evaluated!

$$v_{\bar{t}t}^{k} - \Delta_h v_{\bar{t}t}^{k} - \Delta_h v^{\theta,k} + (\Delta_h)^2 v^{\theta,k} = \Delta_h g(v^k). \quad (6)$$

$$v_{\bar{t}t}^{k} - \Delta_h v_{\bar{t}t}^{k} - \Delta_h v^{\theta,k} + (\Delta_h)^2 v^{\theta,k} = \Delta_h g_1(v^k). \quad (7)$$

Initial conditions

$$v_{(i,j)}^{0} = u_0(x_i, y_j), \quad (8)$$

$$v_{(i,j)}^{1} = u_0(x_i, y_j) + \tau u_1(x_i, y_j) \quad (9)$$
$$+ 0.5\tau^2 (I - \Delta_h)^{-1} \left(\Delta_h u_0 - (\Delta_h)^2 u_0 + \Delta_h g(u_0)\right)(x_i, y_j).$$

The equations, boundary and initial conditions form two families of finite difference schemes.

We factorize the LHS of FDS:

$$\left(I - \Delta_h - \theta\tau^2\Delta_h + \theta\tau^2(\Delta_h)^2\right) v_{\bar{t}t} - \Delta_h v + (\Delta_h)^2 v$$
$$= (I - \Delta_h)\left(\left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t} - \Delta_h v\right).$$

and split FDS

Non-iterative Method (NM)

$$(I - \Delta_h)w = \Delta_h g(v), \quad \left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t} - \Delta_h v = w$$

Iterative Method (IM)

$$(I - \Delta_h)\tilde{w} = \Delta_h g_1(v), \quad \left(I - \theta\tau^2\Delta_h\right) v_{\bar{t}t} - \Delta_h v = \tilde{w}$$

using auxiliary functions w, $\tilde{w}$.

Analysis of the nonlinear schemes

Preliminaries:

the space of mesh functions which vanish on $\partial\Omega$;
the scalar product at time t^k with respect to the spatial variables

$$\langle v, w \rangle = \sum_{i,j} h_1 h_2 v_{(i,j)}^{(k)} w_{(i,j)}^{(k)};$$

operators $A = -\Delta_h$

$$B = I - \Delta_h + \tau^2 \theta(-\Delta_h + (\Delta_h)^2);$$

A and B are self-adjoint positive definite operator.

Operator form of the schemes:

$$Bv_{\bar{t}t} + Av + A^2 v = -Ag, \tag{10}$$

$$Bv_{\bar{t}t} + Av + A^2 v = -Ag_1. \tag{11}$$

The energy functional E_h^L (obtained from the linear part of the equation) at the k-th time level is

$$E_h^L(v^{(k)}) =$$
$$\left\langle A^{-1}v_t^{(k)}, v_t^{(k)}\right\rangle + \left\langle v_t^{(k)}, v_t^{(k)}\right\rangle + \tau^2(\theta - 1/4)\left\langle (I+A)v_t^{(k)}, v_t^{(k)}\right\rangle$$
$$+ 1/4\left\langle v^{(k)} + v^{(k+1)} + A(v^{(k)} + v^{(k+1)}), v^{(k)} + v^{(k+1)}\right\rangle$$

Note that if parameter θ satisfies

$$\theta > \frac{1}{4} - \frac{1}{\tau^2||A||}, \tag{12}$$

then functional $E_h^L(v^k)$ is nonnegative and can be viewed as a norm. Such combined norms depending on the values of solution on several layers are typical for three-layer schemes.

Theorem (Discrete identities for NM)

The solution to the non-iterative scheme (NM) satisfies the equalities (k=1,2,...)

$$E_h^L(v^{(k)}) + (g(v^k), v^{k+1}) = E_h^L(v^{(k-1)}) + (g(v^{(k)}), v^{(k-1)}).$$

The *full discrete energy functional* is (including the non-linearity)

$$E_h(v^{(k)}) = E_h^L(v^{(k)}) + \left\langle G(v^{(k+1)}), 1 \right\rangle + \left\langle G(v^{(k)}), 1 \right\rangle$$

Theorem (Discrete conservation law)

The solution to the iterative scheme (IM) satisfies the energy equalities

$$E_h(v^{(k)}) = E_h(v^{(0)}), \qquad k = 1, 2, \ldots .$$

i.e. the discrete energy is conserved in time.

Consider the following linear FDS with $\Delta_h \psi_1$ as RHS (where ψ_1 is a known function)

$$v_{\bar{t}t} - \Delta_h v_{\bar{t}t} - \Delta_h v^\theta + (\Delta_h)^2 v^\theta = \Delta_h \psi_1. \qquad (13)$$

Theorem (Stability of the linear FDS)

Let γ be a positive real number and θ be such that

$$\theta > \frac{1+\gamma}{4} - \frac{1}{\tau^2 ||A||}.$$

Then the finite difference method (13) is stable with respect to the initial data and the right-hand side:

$$\left(v^{(k)}, v^{(k)}\right) + \left(Av^{(k)}, v^{(k)}\right) \le C\frac{1+\gamma}{\gamma}\left[\left(Bv^{(0)}, v^{(0)}\right) + \left(A^{-1}Bv_t^{(0)}, A^{-1}Bv_t^{(0)}\right) + \sum_{k=1}^{k-1} \tau \left(\psi_1^{(k)}, \psi_1^{(k)}\right)\right].$$

Theorem (Convergence of the NM)

Assume $g \in W^1_\infty(\mathbb{R})$, *the parameter* θ *satisfies*

$$\theta > \frac{1+\gamma}{4} - \frac{1}{\tau^2||A||}$$

for some $\gamma > 0$ *and the solution* u *to the problem (1) – (2) obey* $u \in C^{6,4}\left(\mathbb{R}^2 \times (0,T)\right)$. *Then the solution* v *to the finite difference scheme (10), (8), (9) converges to* u *as* $|h|, \tau \to 0$ *and the following estimate holds for the error* $z = y - u$ *of the scheme:*

$$\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \leq \frac{1+\gamma}{\gamma} Ce^{Mt_k} \left(|h|^2 + \tau^2\right)^2 \qquad (14)$$

with a constant M *chosen so that* $\max_{i,j,s \leq k} \left(|u(x_i, y_j, t_s)|, |v^{(s)}_{i,j}|\right) \leq M$.

Theorem (Convergence of the IM)

Assume $g \in W^2_\infty(\mathbb{R})$ and the parameter θ satisfies (12) with some $\gamma > 0$. Assume that the solution u to (1) – (2) obeys $u \in C^{6,4}\left(\mathbb{R}^2 \times (0,T)\right)$ and the solution v to the finite difference scheme (11), (8), (9) is bounded in the maximal norm. Let M be a constant such that

$$M \geq \max_{i,j,s \leq k} \left(|u(x_i, y_j, t_s)|, \left| \frac{\partial^2 u}{\partial t^2}(x_i, y_j, t_s) \right|, |v^{(s)}_{i,j}| \right)$$

and τ be sufficiently small, $\tau < \gamma \left(C_2(1+\gamma)M\right)^{-1}$. Then v converges to the exact solution u as $|h|, \tau \to 0$ and the following estimate holds for the error $z = y - u$:

$$\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \leq \frac{1+\gamma}{\gamma} Ce^{Mt_k} \left(|h|^2 + \tau^2\right)^2. \quad (15)$$

The main feature of Theorems 5 and 6 is the established second order of convergence in discrete W_2^1 norm, which is compatible with the rate of convergence of the similar linear problem.

Corollary

(i) The convergence of the solution to FDS's with $\theta > 0.25$ to the exact solution is of second order when $|h|$ and τ go independently to zero.

(ii) The convergence of the solution to the explicit FDS's with $\theta = 0$ to the exact solution is of second order when $|h|$ and τ go to 0 provided: $\tau < \frac{|h|}{\sqrt{1+\gamma}}$ for the 1D problem or $\tau < \frac{|h|}{\sqrt{2(1+\gamma)}}$ for the 2D case.

Corollary

Under the assumptions of Theorems 5 or 6 the FDS's admit the following error estimate in the uniform norm $(z = y - u)$:

$$\max_i |z_i^{(k)}| < Ce^{Mt_k}\sqrt{\frac{1+\gamma}{\gamma}}\left(|h|^2 + \tau^2\right), \qquad 1D;$$

$$\max_{i,j} |z_{i,j}^{(k)}| < Ce^{Mt_k}\sqrt{\ln N}\sqrt{\frac{1+\gamma}{\gamma}}\left(|h|^2 + \tau^2\right), \qquad d = 2.$$

The above estimates are optimal for the 1D case and *almost* optimal (up to a logarithmic factor) for the 2D case.

- The boundedness of the exact solution u to the BE on the time interval $[0, T]$ is a main assumption in the convergence theorems.
- BE may have both bounded on the time interval $[0, \infty)$ solutions or blowing up solutions
- the L_∞ norm of the solution is included in the exponent in the right-hand sides of the error estimates
- if u blows up at a moment T_0, $T_0 > T$, then: $\|u\|_{L_\infty[0,T]}$ will be big ; the term e^{MT} will be big ; the convergence will slow up!
- additional restriction on the time step is

$$\tau < \gamma \left(C_2(1+\gamma)M\right)^{-1}, \quad M \geq \|u\|_{L_\infty[0,T]},$$

  in the convergence theorem for the IM.

In any case the FDS should be applied with very small τ's if one would like to evaluate the solution in a neighborhood of the blow up moment.

Movie

Interaction of two solitary waves with different speeds

$x \in [-120, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Preliminaries

- An analytical solution of the 1D equation (one solitary wave):

$$u(x,t;x_0,c) = \frac{3}{2}\frac{c^2-1}{\alpha}\text{sech}^2\left(\frac{x-x_0-ct}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}\right),$$

  where x_0 is the initial position of the peak of the solitary wave,
- Parameters: $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, c is the wave speed.
- Initial conditions for one solitary wave or two solitary waves:

$$u(x,0) = u(x,0;-40,2) + u(x,0;50,-1.5)$$

$$\frac{du}{dt}(x,0) = u(x,0;-40,2)_t + u(x,0;50,-1.5)_t$$

- Two schemes with $\theta = 0.5$:
  - non-iterative and
  - iterative (inner iterations until relative error $< \epsilon$, $\epsilon = 10^{-13}$).

One solitary wave

Rate of convergence and errors
for $x \in [-100, 100]$, $t \in [0, 20]$, $c = 2$

$h = \tau$	Rate no iter.	Rate with iter.	Error no iter.	Error with iter.	with iter./ no iter.
0.1	–	–	0.02559	0.32271	12.60931
0.05	2.02762	1.87037	0.00628	0.08826	14.06140
0.025	2.00675	1.96892	0.00156	0.02255	14.43498
0.0125	2.00142	1.99221	0.00039	0.00567	14.52742

- The error is the difference between the calculated and the exact solution in uniform norm for $t = 20$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For one solitary wave the non-iterative scheme is about 14 times more precise than the iterative scheme.

Interaction of two solitary waves with different speeds

Rate of convergence and errors
for $x \in [-150, 150]$, $t \in [0, 40]$, $c_1 = 2$, $c_2 = -1.5$

$h = \tau$	Rate no iter.	Rate with iter.	Error no iter.	Error with iter.	with iter./ no iter.
0.04	2.09561	1.97465	0.017375	0.102754	5.913796
0.02	1.94485	1.99369	0.017375	0.026027	6.187079
0.01	1.97704	1.99838	0.001084	0.006528	6.021106

- For every h the error is calculated by Runge method as $E_1^2/(E_1 - E_2)$ with $E_1 = \|u_{[h]} - u_{[h/2]}\|$, $E_2 = \|u_{[h/2]} - u_{[h/4]}\|$, where $u_{[h]}$ is the calculated solution with step h for $t = 40$.
- The numerical rate of convergence is $(\log E_1 - \log E_2)/\log 2$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For two solitary waves the non-iterative scheme is about 6 times more precise than the iterative scheme.

With respect to the error magnitude the non-iterative method performs much better than the iterative method!

Justification: Consider the right-hand side of the iterative method. We expand $g_1(u(x_i,t^k))$ in Taylor series about the point (x_i,t^k) and get

$$g_1(u(x_i,t^k)) = g(u(x_i,t^k)) + \tau^2 R + O(\tau^3),$$

$$R = \alpha\frac{\beta_1}{\beta_2}\left(\frac{1}{3}\left(\frac{\partial u}{\partial t}(x_i,t^k)\right)^2 + u\frac{\partial^2 u}{\partial t^2}(x_i,t^k)\right) + \frac{1}{2}\left(\frac{\beta_1}{\beta_2}-1\right)\frac{\partial^2 u}{\partial t^2}(x_i,t^k).$$

Thus, the right-hand sides of the two methods

$$\Delta_h g_1(u(x_i,t^k)) - \Delta_h g(u(x_i,t^k)) = \tau^2\Delta_h R + O(\tau^3).$$

differ by terms of order $O(\tau^2)$. This has essential impact on the error, when the solution has large derivatives!

Error dependence on time step with a fixed space step

One solitary wave with $c = 2$, non-iterative scheme

$h = 0.01$, $x \in [-100, 100]$, $t \in [0, 20]$

τ	L_1 error	Rate	L_2 error	Rate	L_∞ error	Rate
0.32	2.781695	-	0.703758	-	0.384748	-
0.16	0.534777	5.20	0.136372	5.16	0.075703	5.08
0.08	0.128292	4.17	0.032492	4.19	0.017982	4.21
0.04	0.031634	4.06	0.007996	4.06	0.004422	4.07
0.02	0.007748	4.08	0.001956	4.09	0.001082	4.09
0.01	0.001793	4.32	0.000450	4.34	0.000249	4.33
0.005	0.000305	5.87	7.439e-5	6.05	4.174e-5	5.98
0.001	0.000171		4.630e-5		2.478e-5	

- For $\tau \geq h/c$ the error behaves as $O(\tau^2)$
- For $\tau < h/(4c)$ the error does not depend on τ.
- The error behavior is similar in every norm.

Error dependence on time step with a fixed space step

One solitary wave with $c = 2$, iterative scheme
$h = 0.01$, $x \in [-100, 100]$, $t \in [0, 20]$

τ	L_1 error	Rate	L_2 error	Rate	L_∞ error	Rate
0.16	4.961105	-	1.335671	-	0.677222	-
0.08	1.427490	3.48	0.402828	3.32	0.214408	3.16
0.04	0.370195	3.86	0.105644	3.81	0.056685	3.78
0.02	0.093553	3.96	0.026768	3.95	0.014385	3.94
0.01	0.023588	3.97	0.006751	3.97	0.003630	3.96
0.005	0.006044	3.90	0.001727	3.91	0.000929	3.91
0.0025	0.001654	3.65	0.000470	3.67	0.000253	3.67
0.00125	0.000557	2.97	0.000156	3.02	8.3887e-5	3.01

- For $\tau \geq h/c$ the error behaves as $O(\tau^2)$
- For $\tau \leq h/(2c)$ the error does not depend on τ.
- The error behavior is similar in every norm.

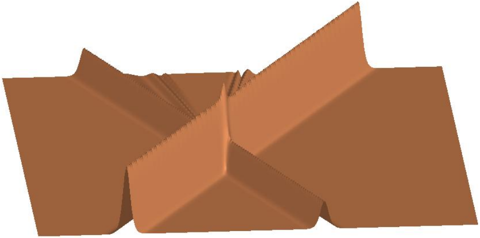
Discrete identities errors

The error is maximum for every $t \in [0, 40]$
of the numerical integral for $x \in [-150, 150]$,
either for one solitary wave with $c_1 = 2$
or for two solitary waves with $c_1 = 2$, $c_2 = -1.5$.

$\tau = h$	1 soliton no iter.	1 soliton with iter.	2 solitons no iter.	2 soliton with iter.
0.1	3.1264e-13	2.3152e-13	9.3245e-11	5.6192e-10
0.05	3.9790e-13	4.1866e-13	1.3416e-11	7.3909e-11
0.025	6.2528e-13	5.3321e-13	2.1630e-12	9.3973e-12
0.0125	1.0232e-13	8.9952e-13	1.2921e-12	1.2091e-12

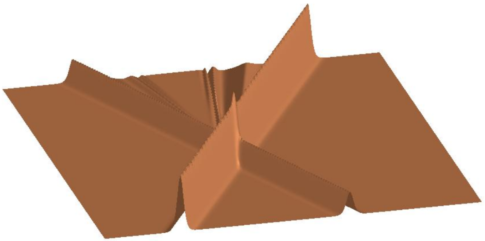
- The discrete identities are different for the iterative and for the non-iterative schemes (conservation law for IM and discrete identities for NM)
- The table shows the numerical solution satisfies the respective discrete identities.

Graphics



Interaction of two solitary waves with different speeds
$x \in [-80, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Graphics



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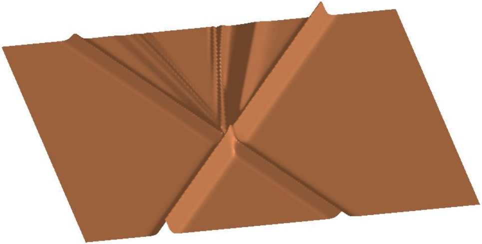

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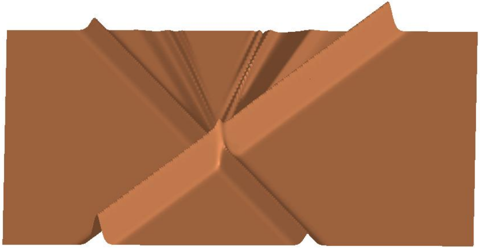

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$x \in [-80, 120],\ t \in [0, 35],\ c_1 = 2,\ c_2 = -1.5$

Thank you
for your attention!