Numerical Computation of High Order Derivatives of Erlang B and C Functions

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This paper focuses on the high transcendental Erlang B and C functions

$$B(a,x) = \frac{a^x/x!}{\sum_{j=0}^{x} a^j/j!} = \left(a\int_0^{+\infty} e^{-az}(1+z)^x\,dz\right)^{-1}, \text{ for } a,x>0, \quad (1)$$

$$C(a,x) = \frac{\frac{a^x}{x!}\frac{x}{x-a}}{\sum_{j=0}^{x-1}\frac{a^j}{j!}+\frac{x}{x-a}} = \left(a\int_0^{+\infty} e^{-az}(1+z)^{x-1}z\,dz\right)^{-1}, x>a>0. \quad (2)$$

Those two functions are very well known in the context of queueing theory and teletraffic engineering, where the variable a is the offered traffic and x (positive integer) is the number of servers of the queue.

A method for calculating the derivatives of order n of the Erlang's B and C function in the number of servers, which can be considered the natural generalization of the classical recursive algorithm for the calculation of the B function itself, was developed by the author and presented in some previous papers. Extensive computation has shown that the proposed method is very accurate for a wide range of values of a and x and compares favorably, in terms of efficiency, with the method proposed by D.L. Jagerman, excepting for very high values of the arguments where the situation is the inverse.

In the present work it is shown that for high values of a and x (say $a, x \geq 100$) it is possible to obtain a significant improvement in the method efficiency, without jeopardizing the required precision, by defining a *reduced recursion* starting from a point closer to the desired value of x. The problem of estimating the initial values was by-passed, accepting the value zero for all initial values. It will be shown that this option does not jeopardize the accuracy of the method, since the absolute value of the relative error decreases rapidly during the recursive calculation.

The essential problem dealt with in this paper is focused on how to estimate the value of the initial point which allows obtaining the required precision. The proposed process is very efficient and inherently based on closed formulæ avoiding iterative procedures.