On the Numerical Simulation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation

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Boussinesq equation (BE) is the first model for surface waves in shallow fluid layer that accounts for both nonlinearity and dispersion. The balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the waves. In the 60s it was discovered that these permanent waves can behave in many instances as particles and they were called *solitons*. A plethora of deep mathematical results have been obtained for solitons in the 1D case, but it is of crucial importance to investigate also the 2D case, because of the different phenomenology and the practical importance. The accurate derivation of the Boussinesq system combined with an approximation, that reduces the full model to a single equation, leads to the Boussinesq Paradigm Equation (BPE):

$$u_{tt} = \Delta\left[u - F(u) + \beta_1 u_{tt} - \beta_2 \Delta u\right], \quad F(u) := \alpha u^2 \tag{1}$$

where u is the surface elevation, $\beta_1,\ \beta_2 > 0$ are two dispersion coefficients, and α is an amplitude parameter. The main difference of Eq. (1) from BE is that in the former one more term is present for $\beta_1 \neq 0$ called "rotational inertia".

It has been recently shown that the 2D BPE admits stationary soliton solutions as well. Even though no analytical formula for these solutions is available, they can be accurately constructed using either finite differences, perturbation technique, or Galerkin spectral method. Virtually nothing is known about the properties of these solutions when they are allowed to evolve in time and it is of utmost importance to answer the questions about their structural stability.

In order to devise a numerical time-stepping procedure we recast Eq. (1), as the following system:

$$v(x,y,t) := u - \beta_1 \Delta u, \quad v_{tt} = \frac{\beta_2}{\beta_1}\Delta v + \frac{\beta_1 - \beta_2}{\beta_1^2}(u - v) - \alpha \Delta F(u).$$

We design an implicit time stepping scheme for the above coupled system and solve it by the Bi-Conjugate Gradient Stabilized Method with ILU preconditioner. The scheme is second order accurate in space and time and unconditionally stable. We perform all standard tests to validate the algorithm: three different spatial grids and different time increments. The results from our numerical experiments show that for some values of the phase speed and relatively small times the unsteady solutions have a solitonic behaviour, although for large times the solution either transforms into a diverging propagating wave or blows-up. The threshold for the value of phase speed for which blow-up is observed depends mildly on the resolution of the grid, because a rougher grid has additional numerical dispersion that acts to diminish the role of the nonlinear terms.