Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SOME BOUNDARY PROPERTIES OF A SERIES IN LAGUERRE'S POLYNOMIALS

PETÂR K. RUSEV

In this paper our aim is to show that some classical results about power series as Hadamard's gap-theorem, Fatou's theorem and Jentzsch's theorem are also valid for series in Laguerre's polynomials. The proofs are based on the asymptotic formula for Laguerre's polynomials in the region $\mathbb{C}-[0, +\infty)$.

1. Definition and asymptotic properties of Laguerre's polynomials. Let $\alpha \neq -1, -2, \ldots$ be an arbitrary complex number. The polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ defined by the equalities

$$L_n^{(\alpha)}(z) = \frac{1}{n!} z^{-\alpha} e^z \frac{d^n}{dz^n} \{z^{n+\alpha} e^{-z}\} \quad (n = 0, 1, 2, \ldots) \tag{1}$$

are called Laguerre's polynomials with a parameter α [1, (5.1.5)]. It is well known that Laguerre's polynomials are special confluent hypergeometric functions [2, p. 268, (36) i. e.

$$L_n^{(\alpha)}(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \Phi(-n, \alpha+1; z). \tag{2}$$

The asymptotic behaviour of $\Phi(a, c; z)$ as a function of a, c, z and especially the case "$a \to \infty$ and z bounded" is investigated by O. Perron [3]. In particular, for Laguerre's polynomials one gets the following asymptotic formula in the region $\mathbb{C}-[0, +\infty)$ [1, (8.22.3)] (α real)

$$L_n^{(\alpha)}(z) = \frac{1}{2\sqrt{\pi}} e^{\frac{z}{2}} (-z)^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} e^{2-(z)^{1/2}\sqrt{n}} \{1+\lambda_n^{(\alpha)}(z)\}, \tag{3}$$

where $\{\lambda_n^{(\alpha)}(z)\}_{n=1}^{\infty}$ are complex functions analytic in the region $\mathbb{C}-[0, +\infty)$ and such that $\lim_{n\to\infty} \lambda_n^{(\alpha)}(z) = 0$ uniformly on every compact subset of this region.

The asymptotic behaviour of Laguerre's polynomials on the ray $[0, +\infty)$ can be described very simply if we are interested only in the "growth" of $L_n^{(\alpha)}(x)$ as a function of n. Namely, the following formula holds [1, (7.6.11)] (α real)

SERDICA. Bulgaricae mathematicae publicationes. Vol. 1, 1975, p. 64—76.

(4) $$L_n^{(\alpha)}(x)=O(n^a),\ a=\max\left(\frac{\alpha}{2}-\frac{1}{4},\ \alpha\right)$$

uniformly on every interval $[0,\omega]$ $(0<\omega<+\infty)$.

2. Convergence of series in Laguerre's polynomials. Using the asymptotic formulas (3) and (4) it is not difficult to describe the region of convergence of a series in Laguerre's polynomials

(5) $$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z).$$

In fact, there is a formula of Cauchy — Hadamard's type for series of the kind (5) [1, 9.2]. More precisely, let

(6) $$\lambda_0=-\overline{\lim_{n\to\infty}}\frac{\ln|a_n|}{2\sqrt{n}}>0$$

and $\Delta(\lambda_0):=[z\in\mathbb{C}:\mathrm{Re}\{(-z)^{1/2}\}<\lambda_0]$. If $\lambda_0=+\infty$, $\Delta(\lambda_0)$ is the whole complex plane. If $\lambda_0<+\infty$, $\Delta(\lambda_0)$ is the interior of the parabola $p(\lambda_0)$ defined by the equality $\mathrm{Re}\{(-z)^{1/2}\}=\lambda_0$. Then:

(a) if $\lambda_0<+\infty$, the series (5) is absolutely uniformly convergent (we say that a series $\sum_{n=1}^{\infty} f_n(z)$ of complex functions is absolutely uniformly convergent on a set $M\subset\mathbb{C}$, if the series $\sum_{n=0}^{\infty}|f_n(z)|$ is uniformly convergent on M) on every compact subset K of the region $\Delta(\lambda_0)-[0,+\infty)$ and on every compact subset of the ray $[0,+\infty)$ and is divergent outside $p(\lambda_0)$;

(b) if $\lambda_0=+\infty$, the series (5) is absolutely uniformly convergent on every compact subset of the region $\mathbb{C}-[0,+\infty)$ and on every compact subset of the ray $[0,+\infty)$.

Let us note that by applying only the asymptotic formulas (3) and (4) it is not possible to describe the mode of convergence of a series of the kind (5) on an arbitrary compact subset of the region $\Delta(\lambda_0)$. A solution of the last problem is given in our paper [4]. In fact, the statement (a) is valid if K is any compact subset of the region $\Delta(\lambda_0)$. Therefore, if $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers satisfying the condition (6), the function

(7) $$f(z)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)$$

is analytic in the region $\Delta(\lambda_0)$.

3. Overconvergence of series in Laguerre's polynomials. The first example of a power series which is analytically noncontinuable outside the unit disk $[z\in\mathbb{C}:|z|<1]$ was given by Weierstrass. Hadamard [5] showed that every power series of the kind $\sum_{k=0}^{\infty} a_{n_k} z^{n_k}$, $n_{k+1}\geq(1+\theta)n_k$, $\theta>0$, has the property discovered by Weierstrass. A. Ostrowski [6] proved a more general gap — theorem namely

Theorem 1 (A. Ostrowski). *Let*

$$\sum_{n=0}^{\infty} a_n z^n \tag{8}$$

be a power series and $\overline{\lim}\sqrt[n]{|a_n|} \neq 0, +\infty$. *If there exist two increasing sequences* $\{p_k\}_{k=1}^{\infty}$ *and* $\{q_k\}_{k=1}^{\infty}$ *of positive integers such that* $q_k \geq (1+\theta)p_k$ $(\theta>0)$ *and* $a_n=0$ *for* $p_k<n<q_k$ $(k=1, 2, 3, \ldots)$, *the sequence of partial sums*

$$\left\{s_{p_k}(z)=\sum_{n=0}^{p_k} a_n z^n\right\}_{k=1}^{\infty} \tag{9}$$

is convergent in the neighbourhood of every point which is regular for the analytic function defined by the series (8).

Hadamard's result can be obtained as a corollary of Ostrowski's theorem. Indeed, for a series of the kind $\sum_{k=0}^{\infty} a_{n_k} z^{n_k}$ the corresponding Ostrowski's sequence (9) coincides with the sequence of its partial sums ($p_k=n_k$, $q_k=n_{k+1}$).

Our aim in this paragraph is to show that for series in Laguerre's polynomials holds a theorem of Ostrowski type. Instead of series of the kind (7) we shall consider series of the type

$$F(\zeta)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(-\zeta^2) \tag{10}$$

with arbitrary complex coefficients. It is not difficult to show that if the sequence $\{a_n\}_{n=0}^{\infty}$ satisfies the condition (6) the region of convergence of the series (10) is the stripe $[\zeta \in \mathbb{C}: |\mathrm{Re}\{\zeta\}| < \lambda_0]$. It is clear also that if a point $z_0 \in p(\lambda_0)$ is regular resp. singular point for the function (7), the points ζ_0 and $-\zeta_0$, where $-\zeta_0^2=z_0$, are regular resp. singular points for the function (10). Conversly, if a point ζ_0, such that $|\mathrm{Re}\{\zeta_0\}|=\lambda_0$, is a regular resp. singular point for $F(\zeta)$, the point $z_0=-\zeta_0^2$ is a regular resp. singular point for $f(z)$.

The corresponding result for overconvergence we shall formulate for series in Laguerre's polynomials.

Theorem 2. *Let the sequence* $\{a_n\}_{n=0}^{\infty}$ *satisfy the condition* (6) *and there exist two increasing sequences of positive integers* $\{p_k\}_{k=1}^{\infty}$, $\{q_k\}_{k=1}^{\infty}$ *such that* $q_k \geq (1+\theta)p_k$ $(k=1, 2, 3, \ldots)$ *for some* $\theta>0$ *and* $a_n=0$ *for* $p_k<n<q_k$ $(k=1, 2, 3, \ldots)$. *Then, the sequence of partial sums*

$$\left\{\sum_{n=0}^{p_k} a_n L_n^{(\alpha)}(z)\right\}_{k=1}^{\infty}$$

is convergent in a neighbourhood of every point $z_0 \in p(\lambda_0)$, *which is regular for the function* (7).

Before giving the proof of theorem 2, we shall prove a very useful lemma namely

Lemma 1. *For every positive* λ:

(a) $$\sum_{n=1}^{k} n^{-1/2} e^{\lambda\sqrt{n}} = O(e^{\lambda\sqrt{k+1}});$$

(b) $$\sum_{n=k+1}^{\infty} n^{-1/2} e^{-\lambda\sqrt{n}} = O(e^{-\lambda\sqrt{k}}).$$

Proof. (a) The function $t^{-1/2}e^{\lambda\sqrt{t}}$ is increasing for $t \geq \lambda^{-2}$. Therefore, if $\nu = \max\{1, [\lambda^{-2}]\}$, we obtain

$$\sum_{n=1}^{k} n^{-1/2} e^{\lambda\sqrt{n}} = \sum_{n=1}^{\nu} n^{-1/2} e^{\lambda\sqrt{n}} + \sum_{n=\nu+1}^{k} n^{-1/2} e^{\lambda\sqrt{n}}$$

$$< \sum_{n=1}^{\nu} n^{-1/2} e^{\lambda\sqrt{n}} + \int_{\nu+1}^{k+1} t^{-1/2} e^{\lambda\sqrt{t}}\, dt$$

$$= \sum_{n=1}^{\nu} n^{-1/2} e^{\lambda\sqrt{n}} + 2\int_{\sqrt{\nu+1}}^{\sqrt{k+1}} e^{\lambda t}\, dt$$

$$= \sum_{n=1}^{\nu} n^{-1/2} e^{\lambda\sqrt{n}} + \frac{2}{\lambda}\left(e^{\lambda\sqrt{k+1}} - e^{\lambda\sqrt{\nu+1}}\right) = O(e^{\lambda\sqrt{k+1}}).$$

(b) The function $t^{-1/2}e^{-\lambda\sqrt{t}}$ is decreasing for $t \geq k$ if k is large enough. Therefore

$$\sum_{n=k+1}^{\infty} n^{-1/2} e^{-\lambda\sqrt{n}} < \int_{k}^{\infty} t^{-1/2} e^{-\lambda\sqrt{t}}\, dt = 2\int_{\sqrt{k}}^{\infty} e^{-\lambda t}\, dt = \frac{2}{\lambda} e^{-\lambda\sqrt{k}}.$$

Proof of Theorem 2. Let $-\zeta_0^2 = z_0$ and $\operatorname{Re}\{\zeta_0\} > 0$. If z_0 is a regular point of $f(z)$, ζ_0 is a regular point of $F(\zeta)$. To prove Theorem 2 it is sufficient to show that the sequence of partial sums

(11) $$\sigma_{p_k}(\zeta) = \sum_{n=0}^{p_k} a_n L_n^{(\alpha)}(-\zeta^2) \quad (k = 0, 1, 2, \ldots)$$

is convergent in a neighbourhood of the point ζ_0. The idea of the proof is the same as the idea of Ostrowski, namely to use Hadamard's three — circles theorem. First of all, there exists a region G, which contains the stripe $[\zeta \in \mathbb{C} : 0 < \operatorname{Re}\{\zeta\} < \lambda_0]$ and also the point ζ_0 and such that $F(\zeta)$ is regular and single-valued in G. Let $\omega_0 = r_0 + i\eta_0$, where $\eta_0 = \operatorname{Im}\{\zeta_0\}$, $\frac{\lambda_0}{2} < r_0 < \lambda_0$ and γ_1, γ_2, γ_3 be circles in G with center ω_0 and radii resp. equal to $(1 - u\delta)(\lambda_0 - r_0)$, $(1+\varepsilon)(\lambda_0 - r_0)$, $(1+\delta)(\lambda_0 - r_0)$, where $1 < u < \sqrt{1+\theta}$ and $0 < \varepsilon < \delta$. Let us note, that the condition $\gamma_j \subset G$ $(j=1, 2, 3)$ can be ensured while for every sufficiently small δ holds the inequality $r_0-(1+\delta)(\lambda_0-r_0)>0$. From the equality

$$\lim_{\delta\to 0}\frac{1}{\delta}\ln\{(1-u\delta)(1+\delta)^{\sqrt{1+\theta}}\}=\sqrt{1+\theta}-u$$

it follows that for every sufficiently small δ holds the inequality

$$(1-u\delta)(1+\delta)^{\sqrt{1+\theta}}>1. \tag{12}$$

Let δ be so chosen that $\gamma_j\subset G$ $(j=1, 2, 3)$ and (12) holds. Further, if we define

$$\sigma=2(\lambda_0-\eta)-2\{r_0+(1-u\delta)(\lambda_0-r_0)\}, \tag{13}$$

$$\tau=2\{r_0+(1+\delta)(\lambda_0-r_0)\}-2(\lambda_0-\eta), \tag{14}$$

where

$$0<\eta<(\lambda_0-r_0)\,\frac{u-1}{2}\,\delta,$$

holds the inequality $\sigma>\tau$.

From the condition (6) it follows that

$$-\overline{\lim_{n\to\infty}}\frac{\ln|n^{\frac{\alpha}{2}+\frac{1}{4}}a_n|}{2\sqrt{n}}=\lambda_0.$$

That is why, there exists a positive constant A such that

$$|a_n|\leq An^{-\frac{\alpha}{2}-\frac{1}{4}}e^{-2(\lambda_0-\eta)\sqrt{n}} \tag{15}$$

for $n=1, 2, 3, \ldots$

We define further for $k=1, 2, 3, \ldots$ and $j=1, 2, 3$

$$\varphi_k(\zeta):=F(\zeta)-\sigma_{p_k}(\zeta),$$
$$M_{k,j}:=\max_{\zeta\in\gamma_j}|\varphi_k(\zeta)|.$$

Using Hadamard's three — circles theorem we obtain the inequality

$$M_{k,2}^{\ln\frac{1+\delta}{1-u\delta}}\leq M_{k,1}^{\ln\frac{1+\delta}{1+\varepsilon}}M_{k,3}^{\ln\frac{1+\varepsilon}{1-u\delta}}. \tag{16}$$

From the asymptotic formula (3) it follows that

$$\max_{\zeta\in\gamma_1}|L_n^{(\alpha)}(-\zeta^2)|=O\left(n^{\frac{\alpha}{2}-\frac{1}{4}}e^{a\sqrt{n}}\right),$$

where $a=2\{r_0+(1-u\delta)(\lambda_0-r_0)\}$. Then, using (15) we get that

$$M_{k,1}=O\left(\sum_{n=q_k}^{\infty} n^{-1/2}e^{-\sigma\sqrt{n}}\right),$$

where σ is defined by (13) and Lemma 1 gives

(17) $$M_{k,1}=O(e^{-\sigma\sqrt{q_k}})=O(e^{-\sigma\sqrt{(1+\theta)p_k}}).$$

On the circle γ_3 we have

$$\max_{\zeta\in\gamma_3} |L_n^{(\alpha)}(-\zeta^2)|=O(n^{\frac{\alpha}{2}-\frac{1}{4}}e^{b\sqrt{n}}),$$

where $b=2\{r_0+(1+\delta)(\lambda_0-r_0)\}$ and using (15) we obtain that

$$M_{k,3}\leq M+|a_0|+L\sum_{n=1}^{p_k} n^{-1/2}e^{\tau\sqrt{n}},$$

where $M=\max_{\zeta\in\gamma_3}|F(\zeta)|$, L is a constant which does not depend on k and τ is defined by (14). Then, Lemma 1 gives

(18) $$M_{k,3}=O(e^{\tau\sqrt{p_k+1}})=O(e^{\tau\sqrt{p_k}}).$$

Further, using (16), (17) and (18) we get that

$$M_{k,2}^{\ln\frac{1+\delta}{1-u\delta}}\leq K(\varepsilon,\delta)\exp\left\{-\left[\sigma\sqrt{1+\theta}\ln\frac{1+\delta}{1+\varepsilon}-\tau\ln\frac{1+\varepsilon}{1-u\delta}\right]p_k^{1/2}\right\},$$

where $K(\varepsilon,\delta)$ depends only on ε and δ.

For every sufficiently small $\varepsilon>0$ holds the inequality

(19) $$\sigma\sqrt{1+\theta}\ln\frac{1+\delta}{1+\varepsilon}-\tau\ln\frac{1+\varepsilon}{1-u\delta}>0.$$

Indeed, according to the choice of δ, for every sufficiently small $\varepsilon>0$ holds the inequality

$$(1+\varepsilon)^{\sqrt{1+\theta}+1}<(1-u\delta)(1+\delta)^{\sqrt{1+\theta}}$$

i. e.

$$\frac{1+\varepsilon}{1-u\delta}<\left(\frac{1+\delta}{1+\varepsilon}\right)^{\sqrt{1+\theta}}.$$

While $0<\tau<\sigma$ and $\frac{1+\varepsilon}{1-u\delta}>1$, we obtain the inequality

$$\left(\frac{1+\varepsilon}{1-u\delta}\right)^{\tau/\sigma}<\left(\frac{1+\delta}{1+\varepsilon}\right)^{\sqrt{1+\theta}}$$

which is equivalent to (19). Then, from (16) it follows that $\lim_{k\to\infty} M_{k,2}=0$ and thus Theorem 2 is proved.

As a corollary we can formulate a theorem Hadamard's type for series in Laguerre's polynomials.

Theorem 3. *Let* $\{n_k\}_{k=0}^{\infty}$ *be a sequence of positive integers such that* $n_{k+1} \geq (1+\theta)n_k$ *for some* $\theta > 0$ *and* $k=0, 1, 2, \ldots$ *Then, if the sequence* $\{a_{n_k}\}_{k=0}^{\infty}$ *satisfies the condition*

$$\lambda_0 = -\overline{\lim_{k\to\infty}} \frac{\ln |a_{n_k}|}{2\sqrt{n_k}} > 0,$$

the *series*

$$\sum_{k=0}^{\infty} a_{n_k} L_{n_k}^{(\alpha)}(z)$$

defines a function which is analytically noncontinuable outside the region $\Delta(\lambda_0)$.

4. Fatou's theorem for series in Laguerre's polynomials. Let $\psi(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with finite and different from zero radius r of convergence. It is well known that if the coefficients $\{a_n\}_{n=0}^{\infty}$ satisfy some additional conditions, there is a relation between the regular points of the function $\psi(z)$ and the behaviour of the power series under consideration on the circumference C_r of convergence. More precisely, the following theorem is valid [7, p. 389; 8, p. 15, 73].

Theorem 4 (P. Fatou). *If* $\lim_{n\to\infty} r^n a_n = 0$, *the series* $\sum_{n=0}^{\infty} a_n z^n$ *is uniformly convergent on every arc* $\gamma \subset C_r$ *all points of which are regular for the function* $\psi(z)$.

We shall see that similar results can be obtained for series in Laguerre's polynomials. The idea of the proof is the same as the idea of the proof of the classical Fatou's theorem. Of course, one needs some modifications. First of all, instead of series in Laguerre's polynomials we consider series of the kind (10). We shall formulate only the corresponding result for series in Laguerre's polynomials.

Theorem 5. *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers satisfying the following conditions:*

(a) $$0 < -\overline{\lim_{n\to\infty}} \frac{\ln |a_n|}{2\sqrt{n}} = \lambda_0 < +\infty;$$

(b) $$\lim_{n\to\infty} n^{\frac{\alpha}{2}+\frac{1}{4}} e^{2\lambda_0\sqrt{n}} a_n = 0.$$

Then, the series

(7) $$f(z) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)$$

is uniformly convergent on every arc $\gamma\subset p(\lambda_0)$ every point of which is regular for the function $f(z)$.

Proof. Let $[a_1, a_2]$, $\mathrm{Im}\{a_1\}<\mathrm{Im}\{a_2\}$, be a segment of the straight line $[\zeta\in\mathbb{C}:\mathrm{Re}\{\zeta\}=\lambda_0]$ and every point of this segment be a regular point for the function

$$F(\zeta)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(-\zeta^2). \tag{10}$$

Let δ be a positive number and define $\zeta_1=a_1-i\delta$, $\zeta_2=a_2+i\delta$. With R we denote the rectangle with vertisses at the points $\zeta_1-\delta$, $\zeta_1+\delta$, $\zeta_2+\delta$, $\zeta_2-\delta$. If δ is sufficiently small, there exists a region G, containing either the stripe $[\zeta\in\mathbb{C}: |\mathrm{Re}\{\zeta\}|<\lambda_0]$ or the closed set $R\cup \mathrm{int}\,R$ and such that the function $F(\zeta)$ is regular and single-valued in G.

For $\zeta\in G$ and $k=1, 2, 3, \ldots$ we define

$$F_k(\zeta)=F(\zeta)-\sum_{n=0}^{k} a_n L_n^{(\alpha)}(-\zeta^2),$$

$$\omega_k(\zeta)=e^{-2(\zeta-\lambda_0)\sqrt{k+1}}(\zeta-\zeta_1)(\zeta-\zeta_2)F_k(\zeta)$$

and

$$\alpha_k=k^{\frac{\alpha}{2}+\frac{1}{4}}e^{2\lambda_0\sqrt{k}}\,a_k. \tag{20}$$

While $\lim_{k\to\infty}\alpha_k=0$, for every $\varepsilon>0$ there exists $\nu_0=\nu_0(\varepsilon)$ such that $|\alpha_k|<\varepsilon$ for $k>\nu_0$.

To prove that the sequence

$$\left\{\sum_{n=0}^{k} a_n L_n^{(\alpha)}(-\zeta^2)\right\}_{k=1}^{\infty}$$

is uniformly convergent on the segment $[a_1, a_2]$, it is sufficient to show that the sequence $\{\omega_k(\zeta)\}_{k=1}^{\infty}$ tends uniformly to zero on the rectangle R. Indeed, if $\lim_{k\to\infty}\omega_k(\zeta)=0$ uniformly on R, from the maximum modulus principle if follows that $\lim_{k\to\infty}\omega_k(\zeta)=0$ uniformly on $[a_1, a_2]$. But $|e^{-2(\zeta-\lambda_0)\sqrt{k+1})}|=1$ if $\zeta\in[a_1, a_2]$, the function $(\zeta-\zeta_1)^{-1}(\zeta-\zeta_2)^{-1}$ is continuous on $[a_1, a_2]$ and we obtain that $\lim_{k\to\infty}F_k(\zeta)=0$ uniformly on $[a_1, a_2]$.

Now we must investigate the behaviour of the sequence $\{\omega_k(\zeta)\}_{k=1}^{\infty}$ on the rectangle R.

(a) $\zeta\in[\zeta_1-\delta, \zeta_2-\delta]$: in this case the series on the right side of (10) is convergent and $F(\zeta)$ is its sum. Having in view the asymptotic formula (3) for Laguerre's polynomials, the equality (20) and Lemma 1 (b), we get that uniformly on $\zeta\in[\zeta_1-\delta, \zeta_2-\delta]$ and $k>\nu_0$

$$\omega_k(\zeta)=O\left(e^{2\delta\sqrt{k+1}}\sum_{n=k+1}^{\infty}|\alpha_n|\,n^{-1/2}e^{-2\delta\sqrt{n}}\right)$$

$$= O\left(\varepsilon e^{2\delta\sqrt{k+1}} \sum_{n=k+1}^{\infty} n^{-1/2} e^{-2\delta\sqrt{n}} \right)$$

$$= O(\varepsilon e^{2\delta(\sqrt{k+1}-\sqrt{k})}) = O(\varepsilon).$$

(b) $\zeta \in [\zeta_1 - \delta, \zeta_1)$ i. e. $\zeta = \xi + i\eta_1$, where $\lambda_0 - \delta \leqq \xi < \lambda_0$ and $\eta_1 = \mathrm{Im}\{\zeta\}$. In this case we obtain that uniformly on ζ and $k > \nu_0$

$$\omega_k(\zeta) = O\left(\varepsilon e^{2(\lambda_0 - \xi)\sqrt{k+1}} (\lambda_0 - \xi) \sum_{n=k+1}^{\infty} n^{-1/2} e^{-2(\lambda_0 - \xi)\sqrt{n}} \right)$$

$$= O\left(\varepsilon e^{2(\lambda_0 - \xi)\sqrt{k+1}} (\lambda_0 - \xi) \int_{\sqrt{k}}^{\infty} e^{-2(\lambda_0 - \xi)t} dt \right)$$

$$= O(\varepsilon e^{2\delta(\sqrt{k+1}-\sqrt{k})}) = O(\varepsilon).$$

(c) $\zeta \in (\zeta_1, \zeta_1 + \delta]$ i. e. $\zeta = \xi + i\eta_1$, $\lambda_0 < \xi \leqq \lambda_0 + \delta$. Then

$$|\omega_k(\zeta)| \leqq e^{-2(\xi - \lambda_0)\sqrt{k+1}} (\xi - \lambda_0) |\zeta - \zeta_2| \Big\{ |F(\zeta) - a_0|$$

$$+ \sum_{n=1}^{\nu_0} |a_n| |L_n^{(\alpha)}(-\zeta^2)| + \sum_{n=\nu_0+1}^{k} |a_n| |L_n^{(\alpha)}(-\zeta^2)| \Big\}.$$

Therefore

$$\omega_k(\zeta) = O(e^{-2(\xi - \lambda_0)\sqrt{k+1}} (\xi - \lambda_0)) + O\left(\varepsilon e^{-2(\xi - \lambda_0)\sqrt{k+1}} (\xi - \lambda_0) \sum_{n=\nu_0+1}^{k} n^{-1/2} e^{2(\xi - \lambda_0)\sqrt{n}} \right).$$

The function $\varphi(t) = t^{-1/2} e^{2(\xi - \lambda_0)\sqrt{t}}$ is increasing in the interval $\lambda_0 + 2(\nu_0 + 1)^{-1/2} \leqq \xi \leqq \lambda_0 + \delta$ and we obtain that

$$\sum_{n=\nu_0+1}^{k} n^{-1/2} e^{2(\xi - \lambda_0)\sqrt{n}} < \int_{\nu_0+1}^{k+1} t^{-1/2} e^{2(\xi - \lambda_0)\sqrt{t}} dt$$

$$= 2 \int_{\sqrt{\nu_0+1}}^{\sqrt{k+1}} e^{2(\xi - \lambda_0)t} dt = (\xi - \lambda_0)^{-1} \{ e^{2(\xi - \lambda_0)\sqrt{k+1}} - e^{2(\xi - \lambda_0)\sqrt{\nu_0+1}} \} < (\xi - \lambda_0)^{-1} e^{2(\xi - \lambda_0)\sqrt{k+1}}.$$

That is why

$$e^{-2(\xi - \lambda_0)\sqrt{k+1}} (\xi - \lambda_0) \sum_{n=\nu_0+1}^{k} n^{-1/2} e^{2(\xi - \lambda_0)\sqrt{n}} < 1$$

if $\lambda_0 + 2(\nu_0 + 1)^{-1/2} \leqq \xi \leqq \lambda_0 + \delta$. If $\lambda_0 < \xi \leqq \lambda_0 + 2(\nu_0 + 1)^{-1/2}$ the function $\varphi(t)$ is decreasing and we obtain

$$\sum_{n=\nu_0+1}^{k} n^{-1/2}e^{2(\xi-\lambda_0)\sqrt{n}} = (\nu_0+1)^{-1/2}e^{2(\xi-\lambda_0)\sqrt{\nu_0+1}}$$

$$+\sum_{n=\nu_0+2}^{k} n^{-1/2}e^{2(\xi-\lambda_0)\sqrt{n}} < (\nu_0+1)^{-1/2}e^{2(\xi-\lambda_0)\sqrt{\nu_0+1}}$$

$$+\int_{\nu_0+1}^{k} t^{-1/2}e^{2(\xi-\lambda_0)\sqrt{t}}\,dt = (\nu_0+1)^{-1/2}e^{2(\xi-\lambda_0)\sqrt{\nu_0+1}}$$

$$+2\int_{\sqrt{\nu_0+1}}^{\sqrt{k}} e^{2(\xi-\lambda_0)t}\,dt = (\nu_0+1)^{-1/2}\,e^{2(\xi-\lambda_0)\sqrt{\nu_0+1}}$$

$$+(\xi-\lambda_0)^{-1}\{e^{2(\xi-\lambda_0)\sqrt{k}} - e^{2(\xi-\lambda_0)\sqrt{\nu_0+1}}\}$$

$$<(\nu_0+1)^{-1/2}e^{2(\xi-\lambda_0)\sqrt{\nu_0+1}} + (\xi-\lambda_0)^{-1}\,e^{2(\xi-\lambda_0)\sqrt{k}}.$$

Therefore, in the case under consideration,

$$\omega_k(\zeta) = O(e^{-2(\xi-\lambda_0)\sqrt{k+1}}(\xi-\lambda_0)) + O(\varepsilon).$$

It is not difficult to show that the sequence

$$\{e^{-2(\xi-\lambda_0)\sqrt{k+1}}(\xi-\lambda_0)\}_{k=1}^{\infty}$$

tends uniformly to zero on the interval $\lambda_0 \leqq \xi \leqq \lambda_0+\delta$. Indeed, if $0<\varepsilon<\delta$ is arbitrary and $\lambda_0 \leqq \xi \leqq \lambda_0+\varepsilon$, holds the inequality

$$e^{-2(\xi-\lambda_0)\sqrt{k+1}}(\xi-\lambda_0) \leqq \varepsilon$$

for every $k=1, 2, 3, \ldots$ On the interval $\lambda_0+\varepsilon \leqq \xi \leqq \lambda_0+\delta$ we have

$$e^{-2(\xi-\lambda_0)\sqrt{k+1}}(\xi-\lambda_0) \leqq \delta e^{-2\varepsilon\sqrt{k+1}}$$

and if $k>\delta^2/4\varepsilon^4-1$, we obtain that

$$e^{-2(\xi-\lambda_0)\sqrt{k+1}}(\xi-\lambda_0) \leqq \delta e^{-\delta/\varepsilon} < \delta/(1+\delta/\varepsilon)^{-1} = \varepsilon\delta/(\varepsilon+\delta)^{-1} < \varepsilon.$$

(d) $\zeta \in [\zeta_1+\delta, \zeta_2+\delta]$ i. e. $\zeta=\lambda_0+\delta+i\eta$, where $\operatorname{Im}\{a_1\}-\delta \leqq \eta \leqq \operatorname{Im}\{a_2\}+\delta$. In this case, having Lemma 1 (a) in view, we obtain

$$|\omega_k(\zeta)| \leqq e^{-2\delta\sqrt{k+1}}\,|\zeta-\zeta_1|\,|\zeta-\zeta_2|\,\Bigg\{|F(\zeta)-a_0|$$

$$+\sum_{n=0}^{\nu_0}|a_n|\,|L_n^{(\alpha)}(-\zeta^2)| + \sum_{n=\nu_0+1}^{k}|a_n|\,|L_n^{(\alpha)}(-\zeta^2)|\Bigg\}$$

$$=O\left(e^{-2\delta\sqrt{k+1}} + \varepsilon e^{-2\delta\sqrt{k+1}}\sum_{n=\nu_0+1}^{k} n^{-1/2}e^{2\delta\sqrt{n}}\right)$$

$$=O(e^{-2\delta\sqrt{k+1}}+\varepsilon e^{-2\delta\sqrt{k+1}}\,.\,e^{2\delta\sqrt{k+1}})=O(e^{-2\delta\sqrt{k+1}}+\varepsilon).$$

In a similar way we get that $\omega_k(\zeta)=O(\varepsilon)$ uniformly on the segments $[\zeta_2-\delta,\ \zeta_2]$ and $(\zeta_2,\ \zeta_2+\delta]$. But $\omega_k(\zeta_1)=\omega_k(\zeta_2)=0$ for every $k=1, 2, 3, \ldots$, therefore $\omega_k(\zeta)=O(\varepsilon)$ uniformly on the rectangle R and thus Theorem 5 is proved.

5. Jentzch's theorem for series in Laguerre's polynomials. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and $\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}\neq 0, \infty$. If Z_ν is the set of roots of the polynomial $\sum_{n=0}^{\nu} a_n z^n$ and $Z=\bigcup_{\nu=1}^{\infty} Z_\nu$, every point of the circumference of convergence of the given series is a point of accumulation of the set Z. This statement is known as a theorem of R. Jentzsch [8, p. 92; 9, p. 13]. In this paragraph our purpose is to establish a similar result for series in Laguerre's polynomials.

Lemma 2. *Let $\{a_n\}_{n=0}^{\infty}$ be an arbitrary sequence of complex numbers and define for $\nu=1, 2, 3, \ldots$*

$$s_\nu(z)=\sum_{n=0}^{\nu} a_n L_n^{(\alpha)}(z). \tag{21}$$

Then, if

$$-\overline{\lim_{n\to\infty}}\frac{\ln|a_n|}{2\sqrt{n}}=\lambda_0>-\infty$$

the sequence $\{|s_\nu(z)|^{1/2\sqrt{\nu}}\}_{\nu=1}^{\infty}$ is uniformly bounded on every compact subset of the region $\mathbb{C}-[0, +\infty)$.

Proof. If $-\infty<\eta<\lambda_0$, then

$$a_n=O(e^{-2(\lambda_0-\eta)\sqrt{n}}).$$

Let $K\subset\mathbb{C}-[0, +\infty)$ be compact and $\lambda\geqq\lambda_0-\eta$ be chosen in such a manner that $K\subset\Delta(\lambda)-[0, +\infty)$. From the asymptotic formula (3) it follows that uniformly on K

$$|s_\nu(z)|=O\left(\nu\sum_{n=1}^{\nu} n^{\frac{\alpha}{2}-\frac{1}{4}}\ e^{2(\lambda-\lambda_0+\eta)\sqrt{n}}\right).$$

If $p=\max\left(0,\ \frac{\alpha}{2}-\frac{1}{4}\right)$, we get that $|s_\nu(z)|=O(\nu^{p+2}e^{2(\lambda-\lambda_0+\eta)\sqrt{\nu}})$ and therefore $|s_\nu(z)|^{1/2\sqrt{\nu}}=O(1)$.

Lemma 3. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that*

$$0<-\overline{\lim_{n\to\infty}}\frac{\ln|a_n|}{2\sqrt{n}}=\lambda_0<+\infty. \tag{22}$$

Then, the point $z_0\in\overline{\Delta(\lambda_0)}$ if and only if

(23)
$$\overline{\lim_{\nu\to+\infty}} \frac{\ln|s_\nu(z_0)|}{2\sqrt{n}} \leq 0,$$

where $s_\nu(z)$ is defined by (21).

Proof. (a) Let $z_0 \in \overline{\Delta(\lambda_0)}$ and $\varepsilon>0$ is arbitrary. From the condition (22) and the asymptotic formula (3) it follows that

$$|s_\nu(z_0)| = O\left(\sum_{n=1}^{\nu} e^{-2(\lambda_0-\varepsilon)\sqrt{n}} n^{\frac{\alpha}{2}-\frac{1}{4}} e^{2\lambda_0\sqrt{n}}\right)$$
$$= O\left(\sum_{n=1}^{\nu} n^{\frac{\alpha}{2}-\frac{1}{4}} e^{2\varepsilon\sqrt{n}}\right) = O(\nu^{p+1}e^{2\varepsilon\sqrt{\nu}}),$$

where $p=\max\left(0, \frac{\alpha}{2}-\frac{1}{4}\right)$. Therefore

$$\overline{\lim_{\nu\to+\infty}} \frac{\ln|s_\nu(z_0)|}{2\sqrt{\nu}} \leq \varepsilon.$$

(b) Let the condition (23) hold and ε be an arbitrary positive number. There exists an increasing sequence $\{\nu_k\}_{k=1}^{\infty}$ of positive integers such that

$$|a_{\nu_k}| \geq e^{-2(\lambda_0+\varepsilon)\sqrt{\nu_k}}.$$

From the equality

$$a_{\nu_k} L_{n_k}^{(\alpha)}(z_0) = s_{\nu_k}(z_0) - s_{\nu_k-1}(z_0)$$

it follows that

$$|L_{\nu_k}^{(\alpha)}(z_0)| \leq (|s_{\nu_k}(z_0)| + |s_{\nu_k-1}(z_0)|)/|a_{\nu_k}|,$$

and therefore

$$|L_{\nu_k}^{(\alpha)}(z_0)| = O(e^{2\varepsilon\sqrt{\nu_k}+2(\lambda_0+\varepsilon)\sqrt{\nu_k}}).$$

Without loss of generality we may assume that $z_0 \in \mathbb{C}-[0, +\infty)$. Then, using the asymptotic formula (3) we obtain that

$$\operatorname{Re}\{(-z_0)^{1/2}\} = \lim_{k\to+\infty} \frac{\ln|L_{\nu_k}^{(\alpha)}(z_0)|}{2\sqrt{\nu_k}} \leq \lambda_0 + 2\varepsilon.$$

Theorem 6. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the condition* (22). *If Z is the set of all roots of the partial sums* (21), *every point of the parabola $p(\lambda_0)$ is a point of accumulation of Z.*

Proof. Let us suppose that the statement of the theorem is not true. In such a case, there exist a point $z_0 \in p(\lambda_0)$, a circle $K(z_0; r)$ with center at this point and a positive integer N such that $s_\nu(z)$ does not vanish in $K(z_0; r)$ for every $\nu>N$. We may assume that r is small enough so that $K(z_0; r) \cap [0, +\infty) = \emptyset$. The series (7) is uniformly convergent on every compact subset of the region $G := \Delta(\lambda_0) \cap K(z_0; r)$ and from a theorem of Hurwitz it follows that $f(z) \neq 0$ in G. Let A be an arbitrary compact subset of G and $m=\min_{z\in A}|f(z)|$, $M=\max_{z\in A}|f(z)|$. For every sufficiently large ν holds the inequality $m/2\leq|s_\nu(z)|\leq 2M$. While $0<m\leq M<+\infty$, we get that $\lim_{\nu\to\infty}|s_\nu(z)|^{1/2\sqrt{\nu}}=1$ uniformly on A.

Let $\zeta\in G$, $K(\zeta;\delta)\subset G$ and $\overline{K(w;\varepsilon)}\not\ni 0$ (where $w=f(\zeta)$) be so chosen that for every $\nu\geq\nu_0>N$, $s_\nu\{K(\zeta;\delta)\}\subset K(w;\varepsilon)$. Then, for every $\nu\geq\nu_0$ one can define a single-valued and continuous branch $a_\nu(z)$ of arg $s_\nu(z)$ in the circle $K(\zeta;\delta)$ satisfying also the inequality $|a_\nu(z)|\leq 2\pi$ for every $z\in K(\zeta;\delta)$. If we define for $z\in K(\zeta;\delta)$ and $\nu\geq\nu_0$

$$\psi_\nu(z)=\exp\left(\frac{1}{2\sqrt{\nu}}\{\ln|s_\nu(z)|+ia_\nu(z)\}\right)$$

we get a sequence $\{\psi_\nu(z)\}_{\nu=\nu_0}^{\infty}$ of single-valued and holomorphic branches of the functions $\{s_\nu(z)\}^{1/2\sqrt{\nu}}$ $(\nu\geq\nu_0)$ in the circle $K(\zeta;\delta)$ such that $\lim_{\nu\to\infty}\psi_\nu(z)=1$ for every $z\in K(\zeta;\delta)$.

Let $\varphi_\nu(z)$ $(\nu\geq\nu_0)$ be a single-valued and holomorphic branch of the function $\{s_\nu(z)\}^{1/2\sqrt{\nu}}$ in the circle $K(z_0;r)$. For every $\nu\geq\nu_0$ there exists a constant α_ν such that $|\alpha_\nu|=1$ and $\alpha_\nu\varphi_\nu(z)=\psi_\nu(z)$ if $z\in K(\zeta;\delta)$. From Lemma 2 it follows that the sequence $\{\alpha_\nu\varphi_\nu(z)\}_{\nu=\nu_0}^{\infty}$ is uniformly bounded on every compact subset of $K(z_0;r)$. Moreover, $\lim_{\nu\to\infty}\alpha_\nu\varphi_\nu(z)=\lim_{\nu\to\infty}\psi_\nu(z)=1$ for every $z\in K(\zeta;\delta)$. From Vittali's theorem it follows that $\lim_{\nu\to\infty}\alpha_\nu\varphi_\nu(z)=1$ for every $z\in K(z_0;r)$. Therefore

$$\lim_{\nu\to\infty}|\alpha_\nu\varphi_\nu(z)|=\lim_{\nu\to\infty}|\psi_\nu(z)|=\lim_{\nu\to\infty}|s_\nu(z)|^{1/2\sqrt{\nu}}=1$$

for every $z\in K(z_0;r)$. From Lemma 3 it follows that $K(z_0;r)\subset\overline{\Delta(\lambda_0)}$ which is impossible.

REFERENCES

1. G. Szegö. Orthogonal polynomials. New York, 1959.
2. H. Bateman, A. Erdelyi. Higher trenscendental functions, vol. 1, New York, 1953.
3. O. Perron, *J. reine u. angew. Math.*, **151**, 1921, 63—78.
4. P. Rusev. *Annuaire Univ. Sofia, Faculté des math. et mech.*, **67**, 1972/73.
5. J. Hadamard. *Journ. d. Math.*, (4), **9**, 1893, 171—215.
6. A. Ostrowski. *Sitzber. Preuss. Akad. Wiss., Phys.-Math. Kl.* 1921, 557—565.
7. P. Fatou. *Acta Math.*, **30**, 1906, 335—400.
8. E. Landau. Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. Berlin, 1929.
9. R. Jentzsch. Untersuchungen zur Theorie der Folgen analytischer Funktionen. Inaugural-dissertation, Berlin, 1914.

Centre for Research and Training in Mathematics and Mechanics

Sofia 1000
P. O. Box 373 *Bulgaria*

Received 19. 2. 1974