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ORDER OF BEST HAUSDORFF POLYNOMIAL APPROXIMATION OF CERTAIN FUNCTIONS

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The aim of this paper is to estimate the best approximation with algebraic polynomials in Hausdorff distance of the functions

$$\varphi_\alpha(x)=|x|^\alpha,$$

$$\psi_\alpha(x)=|x|^\alpha \operatorname{sgn} x,$$

$$\theta_\alpha(x)=(1-x^2)^\alpha,$$

for $0<\alpha<1$, on the interval $[-1, 1]$. While the best uniform polynomial approximation of φ_α, ψ_α and θ_α depends essentially upon α, the best Hausdorff polynomial approximation can be estimated in order by n^{-1}, for φ_α and ψ_α or by n^{-2} for θ_α, for every $0<\alpha<1$.

1. Hausdorff distance between continuous functions. Let f and g be two continuous functions on the interval Δ, i. e. $f, g \in C_\Delta$. The Hausdorff distance (or H-distance) between f and g is defined by

$$r(f, g)=\max\{\max_{x\in\Delta}\min_{\xi\in\Delta}[|x-\xi|, |f(x)-g(\xi)|], \max_{x\in\Delta}\min_{\xi\in\Delta}[|x-\xi|, |f(\xi)-g(x)|]\}.$$

The definition and some properties of H-distance in the general case can be found in [1], but the properties used in the paper will be given here without proof; as usual the functions considered, are continuous.

Lemma 1. *If $f, g\in C_\Delta$ and for every $x_0\in\Delta$ there exist x_1, $x_2\in\Delta$ for which*

$$\max[|x_0-x_1|, |f(x_0)-g(x_1)|]\leq\delta$$

and

$$\max[|x_0-x_2|, |f(x_2)-g(x_0)|]\leq\delta,$$

then $r(f, g)\leq\delta$.

Lemma 2. *If f, f_1, f_2, $g\in C_\Delta$ and*

$$f_1(x)\leq f(x)\leq f_2(x),\ x\in\Delta,$$

$$f_1(x)\leq g(x)\leq f_2(x),\ x\in\Delta,$$

then

$$r(f, g)\leq\max[r(f, f_1), r(f, f_2)].$$

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The relation between the uniform distance

$$R(f, g)=\max_{x\in\Delta}|f(x)-g(x)|$$

of two continuous functions and the Hausdorff distance is given by the inequality

$$(1)\qquad r(f, g)\leq R(f, g)\leq r(f, g)+\omega(r(f, g)),$$

where $\omega(\delta)$ is the modulus of continuity of any of the functions f or g.

Let H_n be the set of algebraic polynomials of degree not higher than n. The best Hausdorff approximation of $f\in C_\Delta$ by element of H_n is denoted by

$$E_{n,r}(f)=\inf_{P\in H_n} r(f, P).$$

The following proposition holds [1], [2]:

For every function $f\in C_\Delta$ we have

$$(2)\qquad E_{n,r}(f)=O(\ln n/n).$$

Furthermore, as it is shown in [3], this order is exact for absolute continuous functions too.

The following proposition is proved in [4]:

If f is an analytic function in the circle $|z|<1$, $|f(z)|\leq M$ for $|z|\leq 1$ and $f(z)$ takes real values in the interval $[-1, 1]$, then

$$(3)\qquad E_{n,r}(f)=O((\ln n/n)^2)$$

and this estimate can not be improved in the class considered.

In this paper we shall prove that for the functions φ_α and ψ_α the estimate (2) can be improved by $\ln n$ and the estimate (3) for the function θ_α can be improved by $(\ln n)^2$.

2. Oscillating polynomials. In order to determine the exact order of uniform approximation of $|x|$ with algebraic polynomials in the interval $[-1, 1]$, S. N. Bernstein used the polynomials, that he called oscillating polynomials [5]. We give here the definition and some properties of these polynomials, required further in the paper.

Definition [5]. The polynomial

$$P(x)=A_0x^{\alpha_0}+A_1x^{\alpha_1}+\cdots+A_nx^{\alpha_n}$$

is called an oscillating polynomial in $[0, 1]$ corresponding to the sequence of non-negative degrees $\alpha_0<\alpha_1<\cdots<\alpha_n$, if it reaches its maximal absolute value in $n+1$ points of this interval. The number n is called order of the oscillating polynomial.

We denote by $\|P\|$ the uniform norm of P for the interval $[0, 1]$,

$$\|P\|=\max_{0\leq x\leq 1}|P(x)|.$$

The following oscillating polynomials will be used

$$(4)\qquad T_{2k}(x)=\cos(2k\arccos x)=(-1)^k\{1-2k^2x^2+\cdots+(-1)^k2^{2k-1}x^{2k}\},$$

$$(5)\qquad T_{2k+1}(x)=\cos((2k+1)\arccos x)$$

$$=(-1)^k(2k+1)\left\{x-\frac{2}{3}k(k+1)x^3+\cdots+(-1)^k 2^{2k}x^{2k+1}\right\}$$

for which $\|T_{2k}\|=\|T_{2k+1}\|=1$.

It is not difficult to prove, that the oscillating polynomial coefficients have alternating signs and the adjacent extrema of the oscillating polynomial have opposite signs [5].

S. N. Bernstein proved the following theorems [5]:

Theorem 1. *If* $P(x)=\sum_{i=0}^{n}A_i x^{\alpha_i}$ *is an oscillating polynomial and* $Q(x)=\sum_{i=0}^{n}B_i x^{\alpha_i}$ *is another polynomial of* x *of the same degrees and if* $P(x)$ *and* $Q(x)$ *have one common coefficient* $A_{i_0}=B_{i_0}$ *(where* $\alpha_{i_0}>0$*) then*

$$\|P\|<\|Q\|.$$

Theorem 2. *If for two oscillating polynomials*

$$P(x)=x^{\alpha_0}+A_1x^{\alpha_1}+\cdots+A_nx^{\alpha_n},$$

$$Q(x)=x^{\alpha_0}+B_1x^{\beta_1}+\cdots+B_nx^{\beta_n}$$

the inequalities $0<\alpha_0<\beta_1<\alpha_1<\beta_2<\cdots<\beta_n<\alpha_n$, *hold, then* $\|Q\|<\|P\|$.

We shall prove two other propositions for the oscillating polynomials:

Lemma 3. *If* $P(x)=x^{\alpha_0}-\sum_{i=1}^{n}a_i x^{\alpha_i}$ *is an oscillating polynomial with degrees* $0<\alpha_0<\alpha_1<\cdots<\alpha_n$, *then*

$$v(x)=\sum_{i=1}^{n}a_i x^{\alpha_i}\geqq 0$$

for every $x\in[0,\xi_1]$, *where* ξ_1 *is the very left local extremum of* $P(x)$ *in the interval* $(0,1]$.

Proof. Denote by

$$0<\xi_1<\xi_2<\cdots<\xi_n<\xi_{n+1}=1$$

the points, in which $|P(x)|=\|P\|$. It is evident, that

(6) $$P(\xi_k)=(-1)^{k-1}\|P\| \quad \text{for } k=1,2,3,\ldots,n+1$$

and

(7) $$P'(\xi_k)=0 \quad \text{for } k=1,2,3,\ldots,n.$$

Denote further by $u(x)$ the polynomial

(8) $$u(x)=xv'(x)-\alpha_0 v(x)=\sum_{i=1}^{n}b_i x^{\alpha_i}.$$

According to Descartes' rule, $u(x)$ has no more than $n-1$ positive zeroes. On the other hand from (6) and (7) we obtain

$$(-1)^{k-1}u(\xi_k)>0\,;\; k=1, 2, 3, \dots, n,$$

and therefore $u(x)$ has at least $n-1$ positive zeroes in the interval (ξ_1, ξ_n).

Hence, $u(x)>0$ in the interval $(0, \xi_1)$, as $u(\xi_1)>0$. But then $v(x)>0$ for $x\in(0, \xi_1)$ too. Indeed, if we suppose, that there exists a point $\eta\in(0, \xi_1)$ for which $v(\eta)=0$, then there exists a point $\eta_1\in(0, \xi_1)$ for which $v(\eta_1)>0$, and $v'(\eta_1)<0$. But from the latter inequalities and (8), it follows that $u(\eta_1)<0$, which is impossible. Thus the lemma is proved.

Lemma 4. *If $s\neq m$, P and Q are oscillating polynomials of the kind*

$$P(x)=a_0x^s+x^m+a_1x^{\alpha_1}+\cdots+a_nx^{\alpha_n},$$
$$Q(x)=x^m+b_1x^{\alpha_1}+\cdots+b_nx^{\alpha_n},$$

then

$$\|P\|>(1+2^{s/|m-s|})^{-1}\|Q\|.$$

Proof. We shall use S. N. Bernstein's method from [5]. Obviously for every $0\leqq\mu\leqq1$ and $x\in[0, 1]$ the inequality

$$\left|a_0\left(\frac{x}{1+\mu}\right)^s+\left(\frac{x}{1+\mu}\right)^m+a_1\left(\frac{x}{1+\mu}\right)^{\alpha_1}+\cdots+a_n\left(\frac{x}{1+\mu}\right)^{\alpha_n}\right|\leqq\|P\|$$

holds, or

$$|a_0x^s+(1+\mu)^{s-m}x^m+\cdots+a_n(1+\mu)^{s-\alpha_n}x^{\alpha_n}|\leqq(1+\mu)^s\|P\|.$$

Taking into account the form of P, we obtain from the last inequality

$$|(1+\mu)^{m-s}-1|\,|x^m+a_1'x^{\alpha_1}+\cdots+a_n'x^{\alpha_n}|\leqq(1+(1+\mu)^s)\|P\|$$

and according to Theorem 1

$$\|P\|>|(1+\mu)^{m-s}-1|\,(1+(1+\mu)^s)^{-1}\|Q\|.$$

In order to prove the lemma it is sufficient to put $\mu=2^{1/|m-s|}-1$.

Using Theorems 1 and 2, S. N. Bernstein proved the following

Lemma 5. *For the oscillating polynomial*

$$R_k(x)=x+a_1x^2+a_2x^4+\cdots+a_kx^{2k}$$

the inequalities

$$\frac{1}{2(1+\sqrt{2})}\,\frac{1}{2k-1}<\|R_k\|<\frac{1}{2k+1}$$

hold.

Further we shall prove the following proposition

Lemma 6. *For the oscillating polynomial*

$$U_k(x)=x+b_1x^2+b_2x^3+\cdots+b_kx^{k+1}$$

the following inequalities hold

$$\frac{1}{2}(k+1)^{-2}<\|U_k\|<(k+1)^{-2}.$$

P r o o f. From (4) we obtain

$$v_k(x)=T_{2k+2}(\sqrt{x})=(-1)^{k+1}\{1-2(k+1)^2x+\cdots+(-1)^{k+1}2^{2k+1}x^{k+1}\}$$

and therefore

$$\| U_k \|<\frac{1}{2}(k+1)^{-2}\| v_k(x)-(-1)^{k+1}\|\leqq(k+1)^{-2}.$$

On the other hand

$$\| U_k \|>\frac{1}{2}(k+1)^{-2}\| v_k \|=\frac{1}{2}(k+1)^{-2}.$$

Indeed, let us assume the contrary, that the polynomial

$$U_k(x)-(-1)^k\frac{1}{2}(k+1)^{-2}v_k(x)=c_1+c_2x^2+c_3x^3+\cdots+c_{k+1}x^{k+1} \tag{9}$$

has at least $k+1$ positive zeroes, since $v_k(x)$ reaches its maximal absolute value in $k+2$ points, where the values of $v_k(x)$ have alternating signs. But on the other hand the polynomial (9) has $k+1$ terms and as regards Descartes' rule, it has no more than k positive zeroes. The obtained contadiction proves our proposition, so the lemma proof is completed.

3. Auxiliary propositions. For every positive integer n, $0<\alpha<1$ and $x\geqq0$ we define the functions

$$g_{\alpha,n}(x)=\max[0,\ x^\alpha-x^{\alpha-1}/n]$$

and

$$h_{\alpha,n}(x)=\min[2x^\alpha,\ x^\alpha+x^{\alpha-1}/n].$$

It is evident, that $g_{\alpha,n}$ and $h_{\alpha,n}$ are continuous and have derivatives for every $x>0$ with the exception of $x=1/n$.

L e m m a 7. *For every positive integer n and $0<\alpha<1$ there exists an even polynomial $P_{\alpha,n}\in H_n$ for which the inequalities*

$$g_{\alpha,n}(x)\leqq P_{\alpha,n}(x)\leqq h_{\alpha,n}(x)$$

hold for every $x\in[0,1]$.

P r o o f. Without any restriction, we can assume that α is a rational number and $\alpha=1-p/q$, where p and q are integers, $p<q$. Consider the oscillating polynomial

$$S(x)=x^q+a_1x^{2q+p}+a_2x^{4q+p}+\cdots+a_kx^{2kq+p}.$$

In accordance with Theorem 2, if we denote by Q the oscillating polynomial

$$Q(x)=x^q+b_1x^{3q}+b_2x^{5q}+\cdots+b_kx^{(2k+1)q}$$

then $\| Q\|>\| S\|$. On the other hand, according to Theorem 1 and (5) we have

$$Q(x)=\frac{(-1)^k}{2k+1}T_{2k+1}(x^q)$$

i. e. $\| Q\|=1/(2k+1)$ and hence

(10) $$\|S\|<1/(2k+1).$$

Denote by ξ_1 the very left local extremum of $S(x)$ in the interval $(0, 1]$, then from Lemma 3 we obtain

(11) $$0\leqq x^p V_k(x^{2q})=-a_1x^{2q+p}-\cdots-a_kx^{2kq+p}\leqq x^q$$

for every $x\in[0, \xi_1]$.

From (10) and (11) we get

$$x^p|x^{q-p}-V_k(x^{2q})|\leqq\|S\|$$

or

(12) $$|x^\alpha-V_k(x)|\leqq x^{\alpha-1}\|S\| \text{ for } x\in(0, 1]$$

and

(13) $$0\leqq V_k(x)\leqq x^\alpha \text{ for } x\in[0, x_1],\ x_1=\xi_1^q.$$

Since

$$x_1^\alpha-V_k(x_1)=x_1^{\alpha-1}\|S\|$$

or

$$V_k(x_1)=x_1^\alpha-x_1^{\alpha-1}\|S\|=(x_1-\|S\|)\,x_1^{\alpha-1}>0$$

it follows that

(14) $$x_1>\|S\|.$$

Let n be an arbitrary positive integer. Denote $P_{\alpha,n}(x)=V_k(x)$, $k=[n/2]$ From (12), (13) and (14) it follows that

$$P_{\alpha,n}(x)\geqq 0 \text{ for } x\in[0, \|S\|]$$

and

$$P_{\alpha,n}(x)\geqq x^\alpha-x^{\alpha-1}\|S\|\geqq 0 \text{ for } x\in[\|S\|, 1]$$

or according to (10) $P_{\alpha,n}(x)\geqq \max[0, x^\alpha-x^{\alpha-1}\|S\|]\geqq g_{\alpha,n}(x)$ for $x\in[0, 1]$.

On the other hand, again from (12), (13) and (14), we get

$$P_{\alpha,n}(x)\leqq 2x^\alpha \text{ for } x\in[0, \|S\|]$$

and

$$P_{\alpha,n}(x)\leqq x^\alpha+x^{\alpha-1}\|S\|\leqq 2x^\alpha \text{ for } x\in[\|S\|, 1],$$

or, according to (10)

$$P_{\alpha,n}(x)\leqq\min[2x^\alpha, x^\alpha+x^{\alpha-1}\|S\|]\leqq h_{\alpha,n}(x).$$

Thus the lemma is proved.

Lemma 8. *For every positive integer n and $0<\alpha<1$ there exists an odd polynomial $P^*_{\alpha,n}\in H_n$ for which the inequalities*

$$g_{\alpha,n}(x)\leqq P^*_{\alpha,n}(x)\leqq h_{\alpha,n}(x)$$

are satisfied for every $x\in[0, 1]$.

The proof is similar to that of the previous lemma, but here the following oscillating polynomial should be considered

$$S(x)=x^q+a_1x^{p+q}+a_2x^{3q+p}+\cdots+a_kx^{(2k-1)q+p}.$$

All other reasonings remain unchanged.

Lemma 9. *For every positive integer* n *and* $0<\alpha<1$ *there exists the polynomial* $U_{\alpha,n}\in H_n$ *for which*

$$g_{\alpha,n^2}(x)\leq V_{\alpha,n}(x)\leq h_{\alpha,n^2}(x)$$

for every $x\in[0, 1]$.

The proof is similar to that of Lemma 7. In this case the oscillating polynomial

$$S(x)=x^q+a_1x^{p+q}+a_2x^{2p+q}+\cdots+a_{k-1}x^{(k-1)p+q}$$

is considered. According to Theorem 2, if we denote by Q the oscillating polynomial

$$Q(x)=x^q+a_1x^{2q}+\cdots+a_{n-1}x^{nq}$$

then $\|S\|<\|Q\|$. On the other hand, according to Lemma 6, $\|Q\|<n^{-2}$ and consequently

$$\|S\|<n^{-2}.$$

The further considerations are similar to those of Lemma 7.

Lemma 10. *For every positive integer* n *and* $0<\alpha<1$ *there is no even polynomial* $P\in H_n$ *for which the inequality*

$$|x^\alpha-P(x)|\leq 2^{-1-1/\alpha}x^{\alpha-1}n^{-1} \tag{15}$$

is satisfied for every $x\in(0, 1]$.

Proof. Let us assume the contrary, that such a polynomial exists. Without any restrictions, we can suppose again that α is a rational number and $\alpha=1-p/q$, where p and q are integers, $p<q$. Then replacing x by x^q and taking into account that P is an even polynomial, from (15) we obtain

$$M=\max_{0\leq x\leq 1}\left|x^q-\sum_{k=0}^{[n/2]}c_kx^{2kq+p}\right|\leq 2^{-1-1/\alpha}n^{-1}. \tag{16}$$

Denote by $S(x)$ the oscillating polynomial

$$S(x)=a_0x^p+x^q+a_1x^{2q+p}+\cdots+a_mx^{2mq+p},\ m=[n/2],$$

then according to Theorems 1 and 2, Lemma 4 and (5), we get

$$M\geq\|S\|>(1+2^{p/(q-p)})^{-1}(2m+1)^{-1}\|T_{2m+1}\|$$

or

$$M>(1+2^{1/\alpha-1})^{-1}(n+1)^{-1}\geq(2+2^{1/\alpha})^{-1}n^{-1}\geq 2^{-1-1/\alpha}n^{-1}.$$

The last inequality contradicts (16), which proves the lemma.

L e m m a 11. *For every positive integer* n *and* $0<\alpha<1$ *there is no odd polynomial* $P\in H_n$ *for which the inequality*

$$|x^{\alpha}-P(x)|\leq 2^{-1-1/\alpha(1-\alpha)}x^{\alpha-1}n^{-1}$$

is satisfied for every $x\in(0, 1]$.

P r o o f. Assume the contrary, that such an odd polynomial does exist. Denote $\alpha=1-p/q$, where p and q, $p<q$, are integers, i. e.

$$M=\max_{0\leq x\leq 1}\left|x^{q}-\sum_{k=0}^{[n/2]}c_k x^{(2k+1)q+p}\right|\leq 2^{-1-1/\alpha(1-\alpha)}n^{-1}. \tag{17}$$

If we denote by $S(x)$ the oscillating polynomial

$$S(x)=a_0x^p+x^q+a_1x^{q+p}+a_2x^{3q+p}+\cdots+a_mx^{(2m+1)q+p},\ m=[n/2],$$

then according to Theorems 1 and 2 and twice using Lemma 4, we get

$$M\geq\|S\|>\frac{1}{2}(1+2^{p/(q-p)})^{-1}(1+2^{(q+p)/p})^{-1}\|T_{2m+1}\|$$

or

$$M>\frac{1}{4}(1+2^{-1+1/\alpha})^{-1}(1+2^{1+1/(1-\alpha)})^{-1}n^{-1},$$

which leads to

$$M>2^{-1-1/\alpha(1-\alpha)}n^{-1}.$$

The last inequality contradicts (17). Thus the lemma is proved.

L e m m a 12. *For every positive integer* n *and* $0<\alpha<1$ *there does not exist any polynomial* $P\in H_n$ *for which the inequality*

$$|x^{\alpha}-P(x)|\leq 2^{-3-1/\alpha(1-\alpha)}x^{\alpha-1}n^{-2}$$

holds for every $x\in(0, 1]$.

P r o o f. Just in the same way as in the two previous lemmas, we assume the contrary, that there exists such a polynomial. Denote again $\alpha=1-p/q$, where $p<q$ are positive integers, i. e.

$$M=\max_{0\leq x\leq 1}\left|x^{q}-\sum_{k=0}^{n}c_k x^{kq+p}\right|\leq 2^{-3-1/\alpha(1-\alpha)}n^{-2}. \tag{18}$$

If we denote by $S(x)$ the oscillating polynomial

$$S(x)=a_0x^p+x^q+a_1x^{q+p}+a_2x^{2q+p}+\cdots+a_nx^{nq+p},$$

then according to Theorems 1 and 2 and Lemma 4

$$M>\|S\|>\frac{1}{2}(1+2^{p/(q-p)})^{-1}(1+2^{(q+p)/p})^{-1}\|U_{k-1}\|, \tag{19}$$

where U_{k-1} is the oscillating polynomial

$$U_{k-1}(x)=x^q+a_2x^{2q}+\cdots+a_nx^{nq}.$$

Then from Lemma 6 and (19), we obtain

$$M>\frac{1}{8}(1+2^{-1+1/\alpha})^{-1}(1+2^{1+1/(1-\alpha)})^{-1}n^{-2}$$

or

$$M>2^{-3-1/\alpha(1-\alpha)}n^{-2},$$

which contradicts (18). So the lemma is completely proved.

Lemma 13. *If for a given positive integer n and $0<\alpha<1$ there is no polynomial $P\in H_n$, for which*

$$|x^\alpha-P(x)|\leqq 2\lambda x^{\alpha-1};\ x\in(0, 1],$$

then the best Hausdorff approximation $E_{n,r}(x^\alpha)$ of x^α with algebraic polynomials on the interval $[0, 1]$ satisfies the inequality

$$E_{n,r}(x^\alpha)>\lambda.$$

Proof. Assume the contrary, that

$$E_{n,r}(x^\alpha)\leqq\lambda.$$

Then, $P\in H_n$ exists, for which

$$P(x)\geqq(x-\lambda)^\alpha-\lambda \quad \text{for} \quad x\in[\lambda, 1],$$
$$P(x)\geqq-\lambda \quad \text{for} \quad x\in[0, \lambda]$$

and

$$P(x)\leqq(x+\lambda)^\alpha+\lambda \quad \text{for} \quad x\in[0, 1-\lambda],$$
$$P(x)\leqq 1+\lambda \quad \text{for} \quad x\in[1-\lambda, 1].$$

Consequently, on the one hand

$$x^\alpha-P(x)\leqq x^\alpha-(x-\lambda)^\alpha+\lambda=\alpha\lambda(x-\theta\lambda)^{\alpha-1}+\lambda<2\lambda x^{\alpha-1} \quad \text{for} \quad x\in[\lambda, 1]$$

and

$$x^\alpha-P(x)\leqq x^\alpha+\lambda\leqq\lambda x^{\alpha-1}+\lambda<2\lambda x^{\alpha-1} \quad \text{for} \quad x\in[0, \lambda].$$

On the other hand

$$x^\alpha-P(x)\geqq x^\alpha-(x+\lambda)^\alpha-\lambda=-\alpha\lambda(x+\theta\lambda)^{\alpha-1}-\lambda>2\lambda x^{\alpha-1} \quad \text{for} \quad x\in[0, 1-\lambda]$$

and

$$x^\alpha-P(x)\geqq x^\alpha-1-\lambda\geqq(1-\lambda)x^\alpha-1-\lambda\geqq-\lambda x^{\alpha-1}-\lambda>-2\lambda x^{\alpha-1}$$

for $x\in[1-\lambda, 1]$.

So we obtain, that

$$|x^\alpha-P(x)|<2\lambda x^{\alpha-1} \quad \text{for} \quad x\in(0, 1],$$

which is a contradiction to the assumption of the lemma and thus proves it.

4. Estimates for the best Hausdorff approximation. It can be checked directly that

$$\max_{0\le x\le 1}\min_{0\le \xi\le 1}\max[|x-\xi|,\ |x^\alpha-g_{\alpha,n}(\xi)|]$$

$$=\max_{0\le x\le 1}\min_{0\le \xi\le 1}\max[|x-\xi|,\ |\xi^\alpha-g_{\alpha,n}(x)|]\le n^{-1}.$$

Further we calculate

$$\lambda_\alpha=\max_{0\le x\le 1}\min_{0\le \xi\le 1}\max[|x-\xi|,\ |x^\alpha-h_{\alpha,n}(\xi)|]$$

$$=\max_{0\le x\le 1}\min_{0\le \xi\le 1}\max[|x-\xi|,\ |\xi^\alpha-h_{\alpha,n}(x)|].$$

It follows from the monotony, that λ_α is defined by the equation $\lambda_\alpha=\delta=h_{\alpha,n}(1/n)-(1/n+\delta)^\alpha$ or $\delta n^\alpha+(1+n\delta)^\alpha=2$.

Therefore we have $(1+n\delta)^\alpha<2$ or $\lambda_\alpha=\delta<(2^{1/\alpha}-1)n^{-1}<2^{1/\alpha}n^{-1}$.

If we take into account, that $g_{\alpha,n}(0)=h_{\alpha,n}(0)$ and use lemmas 1, 2, 7, 8, 10, 11 and 13, we obtain:

Theorem 3. *For the best approximations with algebraic polynomials relative to the H-distance on the interval* $[-1,1]$ *of the functions* $\varphi_\alpha(x)=|x|^\alpha$ *and* $\psi_\alpha(x)=|x|^\alpha\operatorname{sgn}x$, *the inequalities*

$$2^{-2-1/\alpha}n^{-1}<E_{n,r}(\varphi_\alpha)<2^{1/\alpha}n^{-1}$$

and

$$2^{-2-1/\alpha(1-\alpha)}n^{-1}<E_{n,r}(\psi_\alpha)<2^{1/\alpha}n^{-1}$$

hold.

Similarly, using in addition lemmas 9 and 12, we get

Theorem 4. *For the best approximations with algebraic polynomials relative to Hausdorff distance on the interval* $[0,1]$ *of the function* x^α *the following inequalities hold*

$$2^{-4-1/\alpha(1-\alpha)}n^{-2}E_{n,r}(x^\alpha)<2^{1/\alpha}n^{-2}.$$

From Theorem 4 we obtain directly

Theorem 5. *For the best approximation with algebraic polynomials relative to Hausdorff distance on the interval* $[-1,1]$ *of the function* $\theta_\alpha(x)=(1-x^2)^\alpha$, $0<\alpha<1$, *the inequality*

$$E_{n,r}(\theta_\alpha)<2^{3+1/\alpha}n^{-2}. \tag{20}$$

holds.

Indeed, if we replace x by $1-x^2$ in the polynomial $U_{n-1,\alpha}$ of Lemma 9, its degree will double and it will become even. From this change follows that the Hausdorff distance between the transformed functions will at most double, because of the inequality

$$|(1-x^2)-(1-\xi^2)|=|(x-\xi)(x+\xi)|\le 2|x-\xi|$$

for $x,\xi\in[-1,1]$.

In [6] the particular case in (13) is considered for $\alpha=1/2$ and the constant to the right is 5 instead of 32.

The order of approximation in Theorems 3 and 4 is exact, but the constants depending upon α to the right and to the left differ considerably. The question is still open for narrowing this difference and studying the asymptotics of $E_{n,r}(\varphi_\alpha)$, $E_{n,r}(\psi_\alpha)$ and $E_{n,r}(\theta_\alpha)$.

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