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APPROXIMATION OF CONVEX FUNCTIONS BY ALGEBRAIC POLYNOMIALS IN HAUSDORFF METRIC

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Let $E_n(f)_r$ be the best approximation of the function f by means of algebraic polynomials of degree n with respect to the Hausdorff metric in the interval [0, 1], and let K_M be the set of all convex functions in the interval [0, 1] for which $\sup_{x\in[0,1]}|f(x)|\leq M$. It is proved that

$$\sup_{f\in K_M} E_n(f)_r\leq c\,\frac{\ln(c+M)}{n},$$

where c is an absolute constant. For arbitrary functions the order of $E_n(f)_r$ is $\ln n/n$, as it has been shown by Bl. Sendov (1969).

In this paper we consider the order of approximation of convex functions by means of algebraic polynomials with respect to Hausdorff distance. In the theory of approximation of functions the Hausdorff distance was introduced by Bl. Sendov [1, 2].

We shall denote by $r(f, g)$ the Hausdorff distance between the functions f and g bounded in the interval $[a, b]$. Let H_n denote the set of all algebraic polynomials of degree n.

Consider the best approximation $E_n(f)_r$ of the function f bounded in the interval $[a, b]$ by algebraic polynomials of degree n with respect to the Hausdorff distance

$$E_n(f)_r=\inf_{P\in H_n} r(f, P).$$

In [1] (see also [2]) Bl. Sendov obtains the following basis result: *For every function f bounded in the interval $[a, b]$*

$$E_n(f)_r\leq c\frac{\ln n}{n}$$

is valid, where c is a constant depending only on the interval $[a, b]$ and $M=\sup_{x\in[a,b]}|f(x)|$.

The order $\ln n/n$ is exact in the class of all bounded functions [1].

On the other hand, it is known [3] that from

$$E_n(f)_r=O(n^{-1-\varepsilon}),\qquad \varepsilon>0,$$

it follows that we have the same order for the best uniform approximation $E_n(f)$:

$$E_n(f)=\inf_{P\in H_n}\ \sup_{x\in[a,b]}|f(x)-P(x)|=O(n^{-1-\varepsilon}).$$

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In this connection the question can be asked of finding the classes of functions for which the Hausdorff approximation is essentially better than the uniform approximation and is also better than $\ln n/n$.

Bl. Sendov [4] states the hypothesis that one such class is the class of functions convex in the interval $[a, b]$. He supposed that for this class the order of Hausdorff's best approximation is $O(1/n)$. In this paper we prove this statement. More exactly we prove the following

Theorem 1. *Let K_M denote the class of all functions f convex in the interval* $[0, 1]$ *for which*

$$\sup_{x\in[0,1]} |f(x)| \leq M.$$

Then

$$\sup_{f\in K_M} E_n(f)_r \leq c\frac{\ln(e+M)}{n},$$

where c is an absolute constant.

The proof of Theorem 1 is cumbersome and we shall need some lemmas and definitions.

1. We shall use the one-sided Hausdorff distance from the function g to the continuous function f

$$h(g,f)=\sup_{x\in[a,b]}\inf_{t\in[a,b]}\max\{|t-x|,\ |f(t)-g(x)|\}.$$

We have

$$r(f,g)=\max\{h(f,g),\ h(g,f)\}. \tag{1}$$

Denote if needed in $h(f,g)$ and $r(f,g)$ also the interval $[a, b]$: for example $h(f,g;[a,b])$, $r(f,g;[a,b])$. The next lemma follows from the definition of $h(f,g;[a,b])$ and $r(f,g;[a,b])$:

Lemma 1. *Let f and g be continuous functions in the intervals* $[a, c]$ *and* $[c, b]$. *Then*

$$h(f,g;[a,b])\leq\max\{h(f,g;[a,c]),\ h(f,g;[c,b])\},$$

$$r(f,g;[a,b])\leq\max\{r(f,g;[a,c]),\ r(f,g;[c,b])\}.$$

Lemma 2. *Let f and g be continuous functions in the interval* $[a, b]$ *and g be monotone in* $[a, b]$. *Then*

$$h(g,f;[a,b])\leq\max\{|b-a|;\ \min_{y\in[a,b]}\max\{|a-y|,\ |g(a)-f(y)|\};$$

$$\min_{y\in[a,b]}\max\{|b-y|,\ |g(b)-f(y)|\}\}.$$

Proof. We can suppose that g is monotone increasing in $[a, b]$. Let

$$\min_{y\in[a,b]}\max\{|a-y|,\ |g(a)-f(y)|\}=\alpha,$$

$$\min_{y\in[a,b]}\max\{|b-y|,\ |g(b)-f(y)|\}=\beta,$$

$$\min_{y\in[a,b]} f(y)=f(\underline{y})=A;\quad \max_{y\in[a,b]} f(y)=f(\bar{y})=B.$$

We have

$$g(a)\geq A-\alpha,\quad g(b)\leq B+\beta. \tag{2}$$

Let $x\in[a,\ b]$ be arbitrary. In view of (2) $g(x)\in[A-\alpha,\ B-\beta]$. If $g(x)\in[A,B]$, since f is continuous function in $[a,b]$, there exists a point y_x such that $g(x)=f(y_x)$. In this case

$$\min_{y\in[a,b]} \max\{|x-y|,\ |g(x)-f(y)|\}\leq|x-y_x|\leq|b-a|. \tag{3}$$

If $g(x)\in[A-\alpha,\ A]$, then

$$\min_{y\in[a,b]} \max\{|x-y|,\ |g(x)-f(y)|\}\leq\max\{|a-\underline{y}|,\alpha\} \tag{4}$$

and if $g(x)\in[B,\ B+\beta]$, then

$$\min_{y\in[a,b]} \max\{|x-y|,\ |g(x)-f(y)|\}\leq\max\{|b-\bar{y}|,\beta\}. \tag{5}$$

From (3)—(5) follows Lemma 2.

Lemma 3. *Let f and g be continuous functions in the interval $[a,b]$ and f be monotone in $[a,b]$. Then*

$$h(f,g)\leq\max\{h(g,f),\ \min_{y\in[a,b]}\max\{|a-y|,|f(a)-g(y)|\},$$
$$\min_{y\in[a,b]}\max\{|b-y|,\ |f(b)-g(y)|\}\}.$$

Proof. We can suppose that f is monotone increasing in $[a,b]$. Let $\alpha=h(g,f)$. If $\alpha\geq|b-a|$, then Lemma 3 follows from Lemma 2. Let $\alpha<|b-a|$ and $a+\alpha\leq x\leq b-\alpha$. Let us suppose that

$$\beta=\min_{y\in[a,b]}\max\{|x-y|,\ |f(x)-g(y)|\}>\alpha.$$

There are two cases:

a) $g(y)>f(x)$ for $y\in[x-\alpha,\ x+\alpha]$;

b) $g(y)<f(x)$ for $y\in[x-\alpha,\ x+\alpha]$.

Since f is monotone increasing, in the first case we have

$$\min_{y\in[a,b]}\max\{|x-\alpha-y|,\ |g(x-\alpha)-f(y)|\}\geq\beta>\alpha,$$

in the second

$$\min_{y\in[a,b]}\max\{|x+\alpha-y|,\ |g(x+\alpha)-f(y)|\}\geq\beta>\alpha$$

which contradicts $\alpha=h(g,f)$.

Hence

$$\max_{a+\alpha\leq x\leq b-\alpha}\ \min_{y\in[a,b]}\max\{|x-y|,\ |f(x)-g(y)|\}\leq\alpha. \tag{6}$$

In particular

$$\min_{y\in[a,b]} \max\{|a+\alpha-y|,\ |f(a+\alpha)-g(y)|\}\le\alpha,\tag{7}$$
$$\min_{y\in[a,b]} \max\{|b-\alpha-y|,\ |f(b-\alpha)-g(y)|\}\le\alpha.$$

If we apply Lemma 1 twice we obtain

$$h(f,g;\ [a,b])\le \max\{h(f,g;\ [a,a+\alpha]),\tag{8}$$
$$h(f,g;\ [a+\alpha,\ b-\alpha]),\ h(f,g;\ [b-\alpha,\ b])\}.$$

From Lemma 2 and (7) it follows

$$h(f,g;\ [a,\ a+\alpha])\le\max\{\alpha,\ \alpha,\ \min_{y\in[a,b]}\max\{|a-y|,\ |f(a)-g(y)|\}\},\tag{9}$$
$$h(f,g;\ [b-\alpha,\ b])\le\max\{\alpha,\ \alpha,\ \min_{y\in[a,b]}\max\{|b-y|,\ |f(b)-g(y)|\}\}.$$

From (6), (8) and (9) follows Lemma 3.

Lemma 4. *Let f_i, $i=1,\ldots,m$, be monotone increasing continuous functions in the interval $[a,b]$ and g_i, $i=1,\ldots,m$, are functions such that $h(g_i,f_i)\le\delta_i$, $i=1,\ldots,m$. If $\mu_i\ge0$, $i=1,\ldots,m$,* then

$$h\left(\sum_{i=1}^{m}\mu_i g_i,\ \sum_{i=1}^{m}\mu_i f_i\right)\le\max\left\{\sum_{i=1}^{m}\mu_i\delta_i,\ \max_i\ \delta_i\right\}.$$

Proof. Denote $\delta=\max_i \delta_i$. Let $x\in[a,b]$. First we shall suppose that $x\in[a+\delta,\ b-\delta]$. From the monotony of f_i and the definition of $h(.,.)$ follow the inequalities

$$f_i(x-\delta_i)\le g_i(x)+\delta_i,\tag{10}$$
$$f_i(x+\delta_i)\ge g_i(x)-\delta_i.$$

From (10) we obtain

$$\sum_{i=1}^{m}\mu_i f_i(x-\delta)\le\sum_{i=1}^{m}\mu_i f_i(x-\delta_i)\le\sum_{i=1}^{m}\mu_i g_i(x)+\sum_{i=1}^{n}\mu_i\delta_i,\tag{11}$$
$$\sum_{i=1}^{m}\mu_i f_i(x+\delta)\ge\sum_{i=1}^{m}\mu_i f_i(x+\delta_i)\ge\sum_{i=1}^{m}\mu_i g_i(x)-\sum_{i=1}^{m}\mu_i\delta_i.$$

If the point x is such that $x+\delta>b$ (or $x-\delta<a$), we obtain similarly

$$\sum_{i=1}^{m}\mu_i f_i(b)\ge\sum_{i=1}^{m}\mu_i g_i(x)-\sum_{i=1}^{m}\mu_i\delta_i,\tag{12}$$

or

$$\sum_{i=1}^{m}\mu_i f_i(a)\le\sum_{i=1}^{m}\mu_i g_i(x)+\sum_{i=1}^{m}\mu_i\delta_i.$$

Since the function $f=\sum_{i=1}^{m}\mu_i f_i$ is monotone increasing, the lemma follows from (11), (12) and the definition of $h(g,f)$.

The absolute constants in what follows will be denoted by c_i, $i=1, 2, \ldots$. Denote

$$\tau(x_0;\ x)=\begin{cases}0 & \text{if}\quad -1\leq x\leq x_0,\\ M(x-x_0)/(1-x_0) & \text{if}\quad x_0\leq x\leq 1.\end{cases}$$

Lemma 5. *Let $n\geq 1$ be an integer positive number and let $x_0\geq 0$. There exists an algebraic polynomial p of degree n such that*

$$h(p,\tau(x_0;\ x);\ [-1,\ 1])\leq c_1\frac{\ln(e+M)}{n}.$$

Proof. We shall suppose that $x_0\leq 1-1/n$. If $x_0\geq 1-1/n$ the polynomial $Q(x)\equiv 0$ satisfies $h(Q,\tau)\leq 1/n$.

Let us set $x=\cos u$ and let

$$q_{m,r}(u)=\mu\int_{-\pi}^{\pi}\left(\frac{\sin m\frac{t}{2}}{m\sin\frac{t}{2}}\right)^{2r}\tau(x_0;\ \cos(u+t))dt.$$

If m and r are natural numbers, $q_{m,r}(\arccos x)$ is algebraic polynomia of degree $(m-1)r$. The constant μ is defined by the condition

$$\mu\int_{-\pi}^{\pi}\left(\frac{\sin m\frac{t}{2}}{m\sin\frac{t}{2}}\right)^{2r}dt=1. \tag{13}$$

From (13) we obtain

$$\mu^{-1}=2\int_{0}^{\pi}\left(\frac{\sin m\frac{t}{2}}{m\sin\frac{t}{2}}\right)^{2r}dt\geq 2\int_{0}^{\pi/m}\left(\frac{\sin m\frac{t}{2}}{m\sin\frac{t}{2}}\right)^{2r}dt\geq\frac{2\pi}{m}\left(\frac{2}{\pi}\right)^{2r}$$

or

$$0\leq\mu\leq\frac{m}{2\pi}\left(\frac{\pi}{2}\right)^{2r}. \tag{14}$$

Let us set

$$r=[\ln((e+M)(1-x_0)^{-2}],\ m=\left[\frac{n}{r}\right],\ \delta=e\frac{\pi^2}{2m},\ p(x)=q_{m,r}(\arccos x). \tag{15}$$

We shall assume in what follows that $\delta\leq\pi/8$. This inequality is satisfied if $n\geq 50r$. There exists an algebraic polynomial of degree zero such that $h(Q,\tau(x_0;x))\leq 1$ (it is sufficient to set $Q\equiv 0$). Since $0\leq x_0\leq 1-1/n$, then if $n\leq 50r$ we have $n\leq 50r\leq 50[\ln(e+M)+2\ln n]$ and there exists c_2 such that in this case

Therefore we assume that $50r \leq n$. The algebraic polynomial $p(x)$ is of degree at most n. We shall estimate $h(p, \tau)$. Let $\theta > 0$ be such that $x_0 - \theta = \cos(u_0 + \delta)$, $x_0 = \cos u_0$, $u_0 \in [0, \pi/2]$. In order to obtain an estimation for $h(p, \tau)$ we shall consider two cases:

a) $x < x_0 - \theta$. In this case $x = \cos u$, $u \geq u_0 + \delta$, $u \in [0, \pi]$ and $\tau(x_0; \cos(u + t)) = 0$ for $|t| \leq \delta$.

Since

$$(16) \qquad p(x) - \tau(x_0; x) = \mu \int_0^{\pi} \left(\frac{\sin m \frac{t}{2}}{m \sin \frac{t}{2}} \right)^{2r} [\tau(x_0; \cos(u+t)) - 2\tau(x_0; \cos u) + \tau(x_0; \cos(u-t))] dt$$

we have

$$(17) \qquad \min_{y \in [-1, 1]} \max \{ |x - y|, |p(x) - \tau(x_0; y)| \} \leq |p(x) - \tau(x_0; x)|$$

$$= \left| \mu \int_{\delta}^{\pi} \left(\frac{\sin m \frac{t}{2}}{m \sin \frac{t}{2}} \right)^{2r} [\tau(x_0; \cos(u+t)) - 2\tau(x_0; \cos u) + \tau(x_0; \cos(u-t))] dt \right| = \varphi_\delta(u).$$

b) $x \geq x_0 - \theta$. For every function f and every x we have for $u \in [-\delta, \delta]$

$$(18) \qquad 2 \min_{|t-x| \leq \delta} (f(t) - f(x)) \leq f(x+u) - 2f(x) + f(x-u) \leq 2 \max_{|t-x| \leq \delta} (f(t) - f(x)).$$

From (18) follows that for every φ we have

$$(19) \qquad \min_{|u-\varphi| \leq \delta} (\tau(x_0; \cos \varphi) - \tau(x_0; \cos u)) \leq \mu \int_0^{\delta} \left(\frac{\sin m \frac{t}{2}}{m \sin \frac{t}{2}} \right)^{2r} [\tau(x_0; \cos(u+t)) - 2\tau(x_0; \cos u) + \tau(x_0; \cos(u-t))] dt \leq \max_{|u-\varphi| \leq \delta} (\tau(x_0; \cos \varphi) - \tau(x_0; \cos u)).$$

Using the continuity of τ we obtain from (19) that there exists a point ξ_u such that

$$(20) \qquad |\xi_u - u| \leq \delta,$$

$$(21) \qquad \tau(x_0; \cos \xi_u) - \tau(x_0; \cos u) = \mu \int_0^{\delta} \left(\frac{\sin m \frac{t}{2}}{m \sin \frac{t}{2}} \right)^{2r} [\tau(x_0; \cos(u+t)) - 2\tau(x_0; \cos u) + \tau(x_0; \cos(u-t))] dt.$$

Using (16) and (21) we obtain

$$(22)\qquad \min_{y\in[-1,1]} \max\{|x-y|,\ |p(x)-\tau(x_0;\ y)|\}$$

$$\leq \max\{|\cos u-\cos\xi_u|,\ |p(x)-\tau(x_0;\ \cos u)+\tau(x_0;\ \cos u)-\tau(x_0;\cos\xi_u)|\}$$

$$\leq \max\Big\{|\cos u-\cos\xi_u|,\ \Big|\mu\int_\delta^\pi\left(\frac{\sin m\frac{t}{2}}{m\sin\frac{t}{2}}\right)^{2r}[\tau(x_0;\ \cos(u+t))$$

$$-2\tau(x_0;\ \cos u)+\tau(x_0;\ \cos(u-t))]dt\Big|\Big\}=\max\{|\cos u-\cos\xi_u|,\ \varphi_\delta(u)\}.$$

Let us estimate $|\cos u-\cos\xi_u|$ in this case. Since $x\geq x_0-\theta$, $0\leq u\leq u_0+\delta$ we have

$$(23)\qquad |\cos u-\cos\xi_u|\leq\delta\,|\sin z_u|,$$

where $0\leq z_u\leq u_0+2\delta$.

We shall consider again two cases:

b1) $x_0<\cos\pi/4$. In this case

$$|\cos u-\cos\xi_u|\leq\delta=e\pi/2m\leq c_3\frac{\ln((e+M)(1-x_0)^{-2})}{n}\leq c_4\frac{\ln(e+M)}{n}.$$

b2) $x_0\geq\cos\pi/4$. Since we can assume that $\delta\leq\pi/8$, we have $u_0+2\delta=\arccos x_0+2\delta\leq\pi/2$ and

$$|\sin z_u|\leq|\sin(u_0+2\delta)|\leq\sin u_0+\sin 2\delta\leq\sqrt{1-x_0^2}+2\delta.$$

From (23) we obtain

$$(24)\qquad |\cos u-\cos\xi_u|\leq\ \delta(\sqrt{1-x_0^2}+2\delta)$$

$$\leq\sqrt{1-x_0^2}\frac{e\pi}{2n}\ln((e+M)(1-x_0)^{-2}+c_5\left(\frac{\ln((e+M)(1-x_0)^{-2})}{n}\right)^2.$$

As before, we can consider only these n, for which $\frac{\ln(e+M)}{n}\leq 1$. Since $x_0\leq 1-1/n$ we obtain

$$(25)\qquad \left(\frac{\ln((e+M)(1-x_0)^{-2})}{n}\right)^2\leq c_6\,\frac{\ln(e+M)}{n}.$$

Furthermore we have (and this is the crucial point in the proof)

$$(26)\qquad \sup_{0\leq x_0\leq 1}\ \sqrt{1-x_0^2}\,\frac{\ln((e+M)(1-x_0)^{-2})}{n}\leq c_7\,\frac{\ln(e+M)}{n}.$$

From (24)—(26) we obtain

$$|\cos u-\cos\xi_u|\leq c_8\frac{\ln(e+M)}{n}.$$

Consequently in the two cases b1) and b2) we have

$$|\cos u - \cos \xi_u| \leq c_9 \frac{\ln (e+M)}{n}.$$

Therefore in the case b) we have

$$\min_{y \in [-1,1]} \max \{ |x-y|, |p(x)-\tau(x_0; y)| \} \leq \max \{c_9 \frac{\ln (e+M)}{n}, \varphi_\delta(u)\}. \tag{27}$$

Let us estimate now $\varphi_\delta(u)$ in the two cases. Since

$$|\tau(x_0; \cos (u+t)) - \tau(x_0; \cos u)| \leq \frac{M}{1-x_0} |\cos (u+t) - \cos u| \leq \frac{tM}{1-x_0}$$

using (14) and (15) we obtain

$$\varphi_\delta(u) = \frac{m}{2\pi} \left(\frac{\pi}{2} \right)^{2r} \int_\delta^\pi \left(\frac{\pi}{mt} \right)^{2r} \frac{2tM}{1-x_0} \, dt \tag{28}$$

$$\leq \frac{\pi}{2} \left(\frac{\pi^2}{2m\delta} \right)^{2r} \frac{M\delta}{(1-x_0)(2r-2)} \leq c_{10} \frac{(1-x_0) \ln ((e+M)(1-x_0)^{-2})}{n} \leq c_{11} \frac{\ln (e+M)}{n}.$$

The estimations (17), (27) and (28) give us for every $x \in [-1, 1]$:

$$\min_{y \in [-1,1]} \max \{ |x-y|, |p(x)-\tau(x_0; y)| \} \leq c_1 \frac{\ln (e+M)}{n}.$$

Hence

$$h(p, \tau) \leq c_1 \frac{\ln (e+M)}{n} = c_1 \frac{\ln (e+M)}{n} \tag{29}$$

and the lemma is proved.

R e m a r k. Let us mention that from (29) and the definition of $h(.,.)$ it follows that in the interval $[-1, x_0 - c_1 \frac{\ln (e+M)}{n}]$ we have

$$|p(x)| \leq c_1 \frac{\ln (e+M)}{n}.$$

The following lemma can be obtained easily from Lemma 2 in [4], but we shall give the full proof.

L e m m a 6. *Let $M>0$ be given. For every natural number $n>0$ there exists an algebraic polynomial $s_n(x)$ of n-th degree such that*

1) $|s_n(x)| \leq 1/n$ *for* $0 \leq x \leq 1 - 2\left(\frac{\ln 2n(e+M)}{n} \right)^2$,
2) $s_n(x) \geq -1/n$ *for* $x \in [0, 1]$,
3) $s_n(1) \geq M$.

P r o o f. Let

$$T_n(x) = \frac{1}{2} \{ (x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n \}$$

be the Chebyshev polynomial of degree n.

Let us consider the algebraic polynomial

$$s_n(x)=\frac{1}{n}\,T_n\left(\frac{1+\alpha}{1-\alpha}\,x\right),\qquad \alpha=\left(\frac{\ln 2n(e+M)}{n}\right)^2.$$

Obviously $|s_n(x)|\leq 1/n$ for $0\leq x\leq(1-\alpha)/(1+\alpha)$, or, since $(1-\alpha)/(1+\alpha)\geq 1-2\alpha$, we have $|s_n(x)|\leq 1/n$ for $0\leq x\leq 1-2\alpha$.

In the interval $[(1-\alpha)/(1+\alpha),\ 1]$ the function $s_n(x)$ is monotone increasing. We have

$$s_n(1)\geq\frac{1}{2n}\left(\frac{1+\alpha}{1-\alpha}+\sqrt{\left(\frac{1+\alpha}{1-\alpha}\right)^2-1}\right)^n=\frac{1}{2n}\left(\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)^n$$

$$\geq\frac{1}{2n}\,e^{2n\sqrt{\alpha}}=\frac{1}{2n}\exp\left(\ln 2n(e+M)\right)\geq M.$$

This proves the lemma.

2. Lemma 7. *Let f be a monotone convex function in the interval $[0,\ 1]$, $0\leq f(x)\leq M$ for $x\in[0,\ 1]$, $f(x)=0$ for $x\in[0,\ 2c_1\frac{\ln(e+M)}{n}]$, where c_1 is the constant from Lemma 5. Then*

$$E_n(f)_r\leq c_{13}\frac{\ln(e+M)}{n}.$$

Proof. For every $\varepsilon>0$ there exists a linear combination $\varphi(x)=\sum_{i=1}^{m}a_i\tau(x_i;\ x)$ with $a_i\geq 0$, $\sum_{i=1}^{m}a_i\leq 1$, where $\tau(x_i;\ x)$ is the function defined before the Lemma 5, such that

$$\max_{x\in[0,\ 1]}|\varphi(x)-f(x)|<\varepsilon. \tag{30}$$

From Lemma 5 it follows that there exist algebraic polynomials $p_i(x)$, $i=1,\dots,m$, of degree n such that

$$h(p_i,\ \tau(x_i;\ x);\ [0,\ 1])\leq c_1\frac{\ln(e+M)}{n}.$$

Using Lemma 4 we obtain that

$$h\Big(\sum_{i=1}^{m}a_ip_i,\ \varphi;\ [0,\ 1]\Big)\leq c_1\frac{\ln(e+M)}{n}. \tag{31}$$

From (30) and (31) we have

$$h\Big(\sum_{i=1}^{m}a_i\,p_i,\ f\Big)\leq h\Big(\sum_{i=1}^{m}a_ip_i,\ \varphi\Big)+h(\varphi,\ f)\leq c_1\frac{\ln(e+M)}{n}+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we obtain that there exists an algebraic polynomial Q of degree n such that

$$h(Q,\ f)\leq c_1\frac{\ln(e+M)}{n}. \tag{32}$$

If we have, moreover,

$$(33)\qquad \min_{y\in[0,1]} \max\{|x-y|, |f(x)-Q(y)|\}\le c_1\frac{\ln(e+M)}{n} \quad \text{for } x=0 \text{ and } x=1,$$

using Lemma 3 we obtain from (32) and (33) $h(f, Q)\le c_1 \ln(e+M)/n$ and consequently $r(f, Q)=\max\{h(f, Q), h(Q, f)\}\le c_1 \ln(e+M)/n$ and the lemma is proved.

Since $f(x)=0$ for $x\in[0, 2c_1 \ln(e+M)/n]$, from (32) it follows that (33) holds for $x=0$.

Let us mention now that for every function f, bounded in the interval $[0, 1]$, we have

$$E_n(f)_r\le 1.$$

Indeed let us set

$$m=\inf_{x\in[0,1]} f(x),\quad M=\sup_{x\in[0,1]} f(x).$$

In this case the linear function $g(x)=m+(M-m)x$ satisfies $r(f, g)\le 1$.

Consequently we can consider only the case $\ln(e+M)\le n$ (or $c_{15}\ln(e+M)\le n$). In this case there exists c_{12} such that

$$(34)\qquad 2\left(\frac{\ln 2n(e+M)}{n}\right)^2\le c_{12}\frac{\ln(e+M)}{n}.$$

If (33) does not hold true for $x=1$, it follows that

$$\max_{y\in[0,1]} Q(y)<f(1)-c_1\frac{\ln(e+M)}{n}.$$

In this case we add to Q the polynomial as_n, where s_n is the polynomial from Lemma 6 and a, $0\le a\le 1$, is chosen so that

$$(35)\qquad \max_{y\in[1-c_{12}\ln(e+M)/n,\,1]} \{Q(y)+as_n(y)\}=f(1)-c_1\frac{\ln(e+M)}{n}$$

(c_{12} is the constant from (34)).

We can do this since $s_n(1)\ge M$ and $Q(x)\ge -c_1\ln(e+M)/n$ for $x\in[0, 1]$.

Let us estimate in this case $h(Q+as_n, f)$. For $x\in[0, 1-c_{12}\ln(e+M)/n]$, using Lemma 6, (32) and the choice of a, we obtain

$$(36)\qquad \max_{y\in[0,1]} \{|x-y|, |Q(x)+as_n(x)-y(x)|\}$$

$$\le \max_{y\in[0,1]} \{|x-y|, |Q(x)-f(y)|\}+\frac{1}{n}\le c_1\frac{\ln(e+M)}{n}+\frac{1}{n}.$$

If $x\in[1-c_{12}\ln(e+M)/n, 1]$, then $Q(x)-1/n\le Q(x)+as_n(x)\le f(1)$. From (32) it follows that there exists a point y_x such that

$$|x-y_x|\le c_1\frac{\ln(e+M)}{n},\quad |Q(x)-f(y_x)|\le c_1\frac{\ln(e+M)}{n}.$$

Consequently, there exists y_x such that

$$|x-y_x|\le c_1\frac{\ln(e+M)}{n},$$

(37)
$$f(y_x)-c_1\frac{\ln(e+M)}{n}-\frac{1}{n}\le Q(x)+as_n(x)\le f(1).$$

Since f is a continuous function there exists a point y_x^* for which

(38)
$$|x-y_x^*|\le c_1\frac{\ln(e+M)}{n}+c_{12}\frac{\ln(e+M)}{n}$$

and

(39)
$$|Q(x)+as_n(x)-f(y_x^*)|\le c_1\frac{\ln(e+M)}{n}+\frac{1}{n}.$$

From (37)—(39) it follows that we have

(40)
$$\min_{y\in[0,1]}\max\{|x-y|,\ |Q(x)+as_n(x)-f(y)|\}\le(c_1+c_{12})\frac{\ln(e+M)}{n}+\frac{1}{n}.$$

The inequalities (36) and (40) give

(41)
$$h(Q+as_n,\ f)\le(c_1+c_{12})\frac{\ln(e+M)}{n}+\frac{1}{n}.$$

From (33) for $x=0$ and Lemma 6 we obtain

(42)
$$\min_{y\in[0,1]}\max\{|y|,\ |f(0)-Q(y)-as_n(y)|\}\le c_1\frac{\ln(e+M)}{n}+\frac{1}{n}.$$

The inequalities (35), (41), (42) and Lemma 3 give

$$r(Q+as_n,\ f)\le c_{13}\frac{\ln(e+M)}{n}.$$

The lemma is proved.

Let us now prove Theorem 1. Let f be convex function in the interval [0, 1]. We suppose that $\min_{x\in[0,1]} f(x)=0$, $\max_{x\in[0,1]} f(x)=M$.

Let x_0 be such that $f(x_0)=0$. We may assume that $x_0\le 1/2$.

Let us set

$$c_{14}=\max\{c_{13},\ 2c_1\},\qquad c_{14}\frac{\ln(e+M)}{n}=a_n.$$

As in the proof of Lemma 7 we can consider only these n for which $a_n\le 1/8$.

We consider the continuous convex function f_n defined in the following way:

$$f_n(x)=\begin{cases} f(x) & \text{for } x\le x_0,\\ 0 & \text{for } x_0\le x\le x_0+2a_n,\\ f(x-a_n) & \text{for } x_0+2a_n\le x\le 1-1/n,\\ f(1) & \text{for } x=1,\\ \text{continuous and linear in the interval } [1-1/n,\ 1]. \end{cases}$$

Obviously

(43) $$r(f, f_n) \leqq 2a_n + 1/n.$$

We set

$$\theta_n = x_0 + a_n,$$

$$g_{1n}(x) = \begin{cases} f_n(x) & \text{for} \quad x \leqq \theta_n, \\ 0 & \text{for} \quad x \geqq \theta_n, \end{cases}$$

$$g_{2n}(x) = f_n(x) - g_{1n}(x).$$

The functions g_{in}, $i=1, 2$, are monotone convex functions, $0 \leqq g_{in}(x) \leqq M$ for $x \in [0, 1]$, g_{1n} is monotone decreasing and g_{2n} is monotone increasing in the interval $[0, 1]$.

Furthermore

(44) $$g_{1n}(x) = 0 \quad \text{for} \quad x \in [\theta_n - a_n, \ 1] \supset [1 - 2c_1 \frac{\ln (e+M)}{n}, \ 1],$$
$$g_{2n}(x) = 0 \quad \text{for} \quad x \in [0, \ \theta_n + a_n] \supset [0, \ 2c_1 \frac{\ln (e+M)}{n}].$$

Therefore for the functions g_{1n} and g_{2n} we can apply Lemma 7 and we obtain that there exist algebraic polynomials p_{1n} and p_{2n} such that

(45) $$r(p_{in}, g_{in}) \leqq c_{13} \frac{\ln (e+M)}{n}, \quad i = 1, 2.$$

From (43) and (44) and the definition of the Hausdorff distance it follows that

(46) $$|p_{1n}(x)| \leqq a_n \quad \text{for} \quad x \in [\theta_n, \ 1],$$
$$|p_{2n}(x)| \leqq a_n \quad \text{for} \quad x \in [0, \ \theta_n].$$

From (44)—(46) and Lemma 1 it follows

(47) $$r(f_n, p_{1n} + p_{2n}) \leqq \max \{r(f_n, p_{1n} + p_{2n}; [0, \theta_n]),$$
$$r(f_n, p_{1n} + p_{2n}; [\theta_n, 1])\} \leqq \max \{r(g_{1n}, p_{1n}; [0, \theta_n])$$
$$+ a_n, r(g_{2n}, p_{2n}; [\theta_n, 1]) + a_n\} \leqq 2a_n.$$

The inequalities (43) and (47) give

$$r(f, p_{1n} + p_{2n}) \leqq r(f, f_n) + r(f_n, p_{1n} + p_{2n}) \leqq c_1 \frac{\ln (e+M)}{n}.$$

Theorem 1 is proved.

Problem. We state the following problem:

Find the functions f from K_1 such that

$$\sup_{g \in K_1} E_n(f)_r = E_n(f)_r.$$

in other words the functions from K_1 with worst approximation by algebraic polynomials in Hausdorff's distance in the interval [0, 1].

We have the hypothesis that the function $f(x)=2|x|^{-1/2}$ is the unique function from K_1 with this property.

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