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BEST CUBATURE FORMULAS

BORISLAV D. BOJANOV

Approximate cubature formulas for double integrals that use the values of the integrand $f(x, y)$ and its (i, l)-partial derivatives, $(i=0, 1, \ldots, r-1;\ l=0, 1, \ldots, s-1)$ at the vertices of a given rectangular domain D are considered. A formula with best estimation of the error in the class $W_q^{r,s}(D)$, $1 \leq q \leq \infty$ is given. The case $q=2$ is studied in detail.

1. Introduction. Let Ω be a given function class and $L(f)$ a functional defined on Ω. Suppose $T(f)$ is an information available about every function from Ω. Let $\mathfrak{S}$ be the set of all admissible methods of approximate evaluation of $L(f)$ using the information $T(f)$ only. Denote by $S(f)$ the approximate value of $L(f)$ given by the method S. A problem of practical interest is to construct a method $S^* \in \mathfrak{S}$ for which

$$\sup_{f\in\Omega} |L(f)-S^*(f)| = \inf_{S\in\mathfrak{S}} \sup_{f\in\Omega} |L(f)-S(f)|.$$

The method S^* is called best for the class Ω relative to the information $T(f)$.

The above statement of the optimization problem seems to be most natural. A. Sard [1] and. S. M. Nikolski [2] initiated a study of best approximation in a sense which is included in the above general formulation. But they derive their formulas under some constraints on the set $\mathfrak{S}$ (for example, linearity of S, exactness for polynomial classes). A complete solution of the problem is given in [3] and [4] for the functional $L(f)=\int_a^b f(x)dx$ and special types of information. The papers [5] and [6] treat the case $L(f)=f(x)$, where x is a fixed point.

The purpose of this paper is to construct the best method of approximation of the integral

$$I(f)=\iint_D f(x, y)\,dxdy,\quad D=[a, a+h]\times[b, b+k]$$

relative to the information

$$\{f^{(i,l)}(a, b),\ f^{(i,l)}(a+h, b),\ f^{(i,l)}(a, b+k),\ f^{(i,l)}(a+h, b+k) \tag{1}$$
$$(i=0, 1, \ldots, r-1;\ l=0, 1, \ldots, s-1)\}.$$

The integrand f is assumed to be from the class $W_q^{rs}(D)$, $1\leq q\leq\infty$. It is the class of all differentiable functions $f(x, y)$ with absolutely continuous $(r-1, l)$, $(i, s-1)$ and $(r-1, s-1)$ derivatives and such that

$$\|f^{(r,s)}\|_{L_q} \leq 1, \quad \|f^{(i,s)}\|_{L_q} \leq 1, \quad \|f^{(r,l)}\|_{L_q} \leq 1$$

for $i=0, 1, \ldots, r-1$; $l=0, 1, \ldots, s-1$. Here $\|\cdot\|_{L_q}$ denotes the norm

$$\{\int\int_D |\cdot|^q \, dxdy\}^{1/q}.$$

2. Preliminaries. An elegant and useful lemma due to S. A. Smoljak [7] (for the proof see also [8]) allows us to search or the best method S^* among the linear methods. Let Ω be a convex centrally symmetrical subset of a linear metric space. Suppose that $L(f)$ is linear and $T(f)\equiv\{L_1(f), L_2(f), \ldots, L_N(f)\}$ where $L_k(f)$, $(k=1, 2, \ldots, N)$ are linear functionals defined on Ω. Denote by Ω_0 the set $\{f\in\Omega;\ L_k(f)=0, (k=1, 2, \ldots, N)\}$.

Lemma 1. (S. A. Smoljak). *If* $\sup\{L(f): f\in\Omega_0\}<\infty$ *then there exist numbers* D_k, $(k=1, 2, \ldots, N)$ *such that*

$$\sup_{f\in\Omega} \left|L(f)-\sum_{k=1}^{N} D_k L_k(f)\right| = \inf_{S\in\mathfrak{S}} \sup_{f\in\Omega} |L(f)-S(f)|.$$

It follows by the proof of the above lemma that

$$R(T)=\inf_{S\in\mathfrak{S}} \sup_{f\in\Omega} |L(f)-S(f)| = \sup_{f\in\Omega_0} L(f). \tag{2}$$

The quantity $R(T)$ is called the best error's estimate in the class Ω.

Let us remind the Taylor formula with integral remainder to which we shall refer repeatedly in the sequel:

$$f(x,y)=\sum_{i=0}^{r-1}\sum_{l=0}^{s-1} \frac{(x-a)^i}{i!}\frac{(y-b)^l}{l!} f^{(i,l)}(a,b) \tag{3}$$

$$+\frac{1}{(s-1)!}\sum_{i=0}^{r-1}\frac{(x-a)^i}{i!}\int_b^y (y-v)^{s-1} f^{(i,s)}(a,v)dv+\frac{1}{(r-1)!}\sum_{l=0}^{s-1}\frac{(y-b)^l}{l!}$$

$$\int_a^x (x-u)^{r-1} f^{(r,l)}(u,b)du+\frac{1}{(r-1)!(s-1)!}\int_a^x\int_b^y (x-u)^{r-1}(y-b)^{s-1} f^{(r,s)}(u,v)du\,dv.$$

3. Main result. Denote by $U_{mp}(t;[\alpha,\beta])$ the polynomial of best L_p approximation of the function zero in the interval $[\alpha,\beta]$, among the polynomials of the type $t^m+a_1t^{m-1}+\cdots+a_m$. Similarly, let $U_{rsp}(u,v;D)$ denote the polynomial of best L_p approximation of the function zero in D by polynomials of the type

$$(a+h-u)^r(b+k-v)^s+\sum_{i=0}^{r-1}\sum_{l=0}^{s-1} a_{il}(a+h-u)^i(b+k-v)^l.$$

In addition, put

$$U_{mp}(t)=U_{mp}(t;[-1,1]), \quad U_{mnp}(t,\tau)=U_{mnp}(1-t, 1-\tau;\ [0,1]\times[0,1]).$$

The main result of this paper is

Theorem 1. *The cubature formula*

$$I(f)\approx\sum_{i=0}^{r-1}\sum_{l=0}^{s-1}\Big\{\frac{(-1)^l}{s!}\frac{h^{i+1}}{(i+1)!}\Big(\frac{k}{2}\Big)^{l+1}[U_{sp}^{(s-l-1)}(1)f^{(i,l)}(a,b+k)$$

$$-U_{sp}^{(s-l-1)}(-1)f^{(i,l)}f(a,b)]$$

$$+\frac{(-1)^i}{r!}\frac{k^{l+1}}{(l+1)!}\Big(\frac{h}{2}\Big)^{i+1}[U_{rp}^{(r-i-1)}(1)f^{(i,l)}(a+h,b)-U_{rp}^{(r-i-1)}(-1)f^{(i,l)}(a,b)]$$

$$+\frac{h^{i+1}}{r!}\frac{k^{l+1}}{s!}[-U_{rsp}^{(r-i-1,\,s-l-1)}(0,0)f^{(i,l)}(a+h,b+k)+U_{rsp}^{(r-i-1,s-l-1)}(0,1)f^{(i,l)}(a+h,b)$$

$$+U_{rsp}^{(r-i-1,s-l-1)}(1,0)f^{(i,l)}(a,b+k)-U_{rsp}^{(r-i-1,s-l-1)}(1,1)f^{(i,l)}(a,b)]\Big\}$$

is best method of approximation of $I(f)$ in the class $W_q^{rs}(D)$ relative to the information (1). *The error $R(T)$ of the best method is*

$$R(T)=(k/2)^{s+1/p}(E_{sp}/s!)\sum_{i=0}^{r-1}h^{i+1}/(i+1)!$$

$$+(h/2)^{r+1/p}(F_{rp}/r!)\sum_{l=0}^{s-1}k^{l+1}/(l+1)!+(k/2)^{S+1/p}(h/2)^{r+1/p}E_{rsp},$$

where $E_{rp}=\|U_{rp}(t)\|_{L_p}$, $E_{sp}=\|U_{sp}(t)\|_{L_p}$, $E_{rsp}=\|U_{rsp}(t;\tau)\|_{L_p}$.

Proof. It is easily seen that the space $W_q^{rs}(D)$ satisfies the assumptions of the lemma 1. So does the set

$$W=\{f\in W_q^{rs}(D):f^{(i,l)}(a,b)=0,\ (i=0,1,\ldots,r-1;\ l=0,1,\ldots,s-1)\}.$$

Then, using (2), we can write

$$R(T)=\inf_{A,B,C}\sup_{f\in W}\Big|I(f)-\sum_{i=0}^{r-1}\sum_{l=0}^{s-1}(A_{il}f^{(i,l)}(a,b+k)+B_{il}f^{(i,l)}(a+h,b)$$
$$+C_{il}f^{(i,l)}(a+h,b+k))\Big|,\tag{4}$$

where A, B and C denote the matrices $\{A_{il}\}$, $\{B_{il}\}$ and $\{C_{il}\}$. By Taylor's formula (3), taking into account that $f\in W$, we calculate

$$I(f)=\sum_{i=0}^{r-1}(h^{i+1}/(i+1)!)\int_b^{b+k}\frac{(b+k-v)^s}{s!}f^{(i,s)}(a,v)dv\tag{5}$$

$$+\sum_{l=0}^{s-1}(k^{l+1}/(l+1)!)\int_a^{a+h}((a+h-u)^r/r!)f^{(r,l)}(u,b)\,du$$

$$+\int_D\int(1/(r!\,s!))(a+h-u)^r(b+k-v)^sf^{(r,s)}(u,v)\,du\,dv,$$

$$f^{(i,l)}(a+h,\, b)=(1/(r-i-1)!)\int_a^{a+h}(a+h-u)^{r-i-1}f^{(r,l)}(u,\, b)du,$$

$$f^{(i,l)}(a,\, b+k)=(1/(s-l-1)!)\int_b^{b+k}(b+k-v)^{s-l-1}f^{(i,s)}(a,\, v)dv,$$

$$f^{(i,l)}(a+h,\, b+k)=(1/(r-i-1)!)\int_a^{a+h}(a+h-u)^{r-i-1}f^{(r,l)}(u,\, b+k)\,du$$

$$+(1/(s-l-1)!)\int_b^{b+k}(b+k-v)^{s-l-1}f^{(i,s)}(a+h,\, v)\,dv$$

$$-\int_D\int(1/((r-i-1)!\,(s-l-1)!))\,(a+h-u)^{r-i-1}(b+k-v)^{s-l-1}f^{(r,s)}(u,\, v)\,du\,dv.$$

For brevity, denote the last double integral by J_{il}. A careful multiple integration by parts in the first two terms shows that

$$f^{(i,l)}(a+h,\, b+k)=\sum_{j=i}^{r-1}(h^{j-i}/(j-i)!)\,f^{(j,l)}(a,\, b+k)$$

$$+\sum_{j=l}^{s-1}(k^{j-l}/(j-l)!)\,f^{(i,j)}(a+h,\, b)-J_{il}.$$

When these quantities are substituted in (4) the following expression for $R(T)$ is obtained after some simple manipulation.

$$R(T)=\inf_{A,B,C}\sup_{f\in W}\Bigg\{(1/s!)\sum_{i=0}^{r-1}(h^{i+1}/(i+1)!)\int_b^{b+k}g_s(A\,;\,v)f^{(i,s)}(a,\, v)dv$$

$$+(1/r!)\sum_{l=0}^{s-1}(k^{l+1}/(l+1)!)\int_a^{a+h}g_r(B\,;\,u)f^{(r,l)}(u,\, b)du$$

$$+(1/(r!\, s!))\int_D\int g_{rs}(C\,;\,u,\, v)f^{(r,s)}(u,\, v)\,du\,dv\Bigg\},$$

where

$$g_s(A\,;\,v)=(b+k-v)^s-\sum_{l=0}^{s-1}A_{il}\,\frac{(i+1)!\,s!}{(s-l-1)!\,h^{i+1}}\,(b+k-v)^{s-l-1},$$

$$g_r(B\,;\,u)=(a+h-u)^r-\sum_{i=0}^{r-1}B_{il}\,\frac{(l+1)!\,r!}{(r-i-1)!\,k^{l+1}}\,(a+h-u)^{r-i-1},$$

$$g_{rs}(C;u,v)=(a+h-u)^r(b+k-v)^s-\sum_{i=0}^{r-1}\sum_{l=0}^{s-1}C_{il}\frac{r!\,s!\,(a+h-u)^{r-i-1}(b+k-v)^{s-l-1}}{(r-i-1)!\,(s-l-1)!}.$$

Now applying Hölder's inequality we get

$$(6)\qquad R(T)\leq\inf_A\frac{1}{s!}\left(\int_b^{b+k}|g_s(A;v)|^p dv\right)^{1/p}\sum_{i=0}^{r-1}h^{i+1}/(i+1)!$$

$$+\inf_B\frac{1}{r!}\left(\int_a^{a+h}|g_r(B;u)|^p du\right)^{1/p}\sum_{l=0}^{s-1}k^{l+1}/(l+1)!$$

$$+\inf_C\frac{1}{r!\,s!}\left(\iint_D|g_{rs}(C;u,v)|^p du dv\right)^{1/p}.$$

Let us define the function $F(x,y)$ by the formula (3) with

$$F^{(i,l)}(a,b)=0,\quad (i=0,1,\dots,r-1;\quad l=0,1,\dots,s-1),$$

$$F^{(i,l)}(a,v)=\|g_s(A;v)\|_{L_p}|g_s(A;v)|^{p/q}\operatorname{sign}g_s(A;v),$$

$$F^{(i,l)}(u,b)=\|g_r(B;u)\|_{L_p}|g_r(B;u)|^{p/q}\operatorname{sign}g_r(B;u),$$

$$F^{(r,s)}(u,v)=\|g_{rs}(C;u,v)\|_{L_p}(C;u,v)|\operatorname{sign}g_{rs}(C;u,v).$$

Clearly, $F\in W$. It is easily verified with the aid of the function F that the inequality in (6) turns into equality for $1<q\leq\infty$. An ε-process, as for example in [2, p. 19, p. 27] shows that the equality holds for $q=1$, too. So

$$R(T)=\frac{1}{s!}\left(\int_b^{b+k}|U_{sp}(v;[b,b+k])|^p dv\right)^{1/p}\sum_{i=0}^{r-1}h^{i+1}/(i+1)!$$

$$+\frac{1}{r!}\left(\int_a^{a+h}|U_{rp}(u;[a,a+h])|^p du\right)^{1/p}\sum_{l=0}^{s-1}k^{l+1}/(l+1)!$$

$$+\frac{1}{r!\,s!}\left(\iint_D|U_{rsp}(u,v;D)|^p du dv\right)^{1/p}.$$

After the transformations $u=ht/2+(2a+h)/2$, $v=k\tau/2+(2b+k)/2$ for the single integrals and $k=a+th$, $v=b+k\tau$ for the double one, we obtain

$$R(T)=(k/2)^{s+1/p}(E_{sp}/s!)\sum_{i=0}^{r-1}h^{i+1}/(i+1)!$$

$$+(h/2)^{r+1/p}(E_{rp}/r!)\sum_{l=0}^{s-1}k^{l+1}/(l+1)!+(k/2)^{s+1/p}(h/2)^{r+1/p}E_{rsp}.$$

Our aim is to find a method S^* with an error equal to $R(T)$. By lemma 1, we have $R(T)=\inf_{S\in\mathfrak{S}}\sup_{f\in W_q^{rs}(D)}|I(f)-S(f)|$.

Let, for convenience $S_0(f)=\sum_{i=0}^{r-1}\sum_{l=0}^{s-1}\frac{h^{i+1}}{(i+1)!}\frac{k^{l+1}}{(l+1)!}f^{(i,l)}(a,b)$.

It is easy to see that the method $S_1=S-S_0$ belongs to $\mathfrak{S}$ iff $S\in\mathfrak{S}$. Then we can write

$$R(T)=\inf_{S_1\in\mathfrak{S}}\sup_{f\in W_q^{rs}(D)}|I(f)-S_0(f)-S_1(f)|. \tag{7}$$

Let $P(t)$ be an abitrary polynomial of degree not greater than $m-1$. After a repeated integration by parts we get

$$\int_\alpha^\beta P(t)f^{(m)}(t)dt=\sum_{i=0}^{m-1}(-1)^{(m-i-1)}(P^{(m-i-1)}(\beta)f^{(m)}(\beta)-P^{(m-i-1)}(\alpha)f^{(m)}(\alpha)). \tag{8}$$

Now, suppose that $Q(t,\tau)$ is a polynomial of degree $(r-1, s-1)$ with respect to the variable t, respectively τ. Applying twice the above formula we can verify that

$$\int_a^{a+h}\int_b^{b+k}Q(t,\tau)f^{(r,s)}(t,\tau)dt\,d\tau \tag{9}$$

$$=\sum_{i=0}^{r-1}\sum_{l=0}^{s-1}(-1)^{s+r-l-i}\{Q^{(r-i-1,s-l-1)}(a+h,b+k)f^{(i,l)}(a+h,b+k)$$

$$-Q^{(r-i-1,s-l-1)}(a,b+k)f^{(i,l)}(a,b+k)-Q^{(r-i-1,s-l-1)}(a+h,b)f^{(i,l)}(a+h,b)$$

$$+Q^{(r-i-1,s-l-1)}(a,b)f^{(i,l)}(a,b)\}.$$

Thus, the methods

$$\bar{S}(\{P_{s-1,i}\}_{i=0}^{r-1},\{P_{r-1,l}\}_{l=0}^{s-1},Q_{r-1,s-1};f) \tag{10}$$

$$=\frac{1}{s!}\sum_{i=0}^{r-1}\frac{h^{i+1}}{(i+1)!}\int_b^{b+k}P_{s-1,i}(v)f^{(i,s)}(a,v)dv+\frac{1}{r!}\sum_{l=0}^{s-1}\frac{k^{l+1}}{(l+1)!}\int_a^{a+h}P_{r-1,l}(u)f^{(r,l)}(u,b)du$$

$$+\frac{1}{r!\,s!}\int\int_D Q_{r-1,s-1}(u,v)f^{(r,s)}(u,v)\,dudv$$

belong to the class $\mathfrak{S}$ if the polynomials $P_{s-1,i}(t)$, $P_{r-1,l}(t)$, $Q_{r-1,s-1}(t,\tau)$, $(i=0,1,\dots,r-1;\ l=0,1,\dots,s-1)$ are of degree, respectively $s-1$, $r-1$ and $(r-1,s-1)$. Then, from (7)

$$R(T)\leq\inf_{\bar{S}}\sup_{f\in W_q^{rs}(D)}|I(f)-S_0(f)-\bar{S}(f)|.$$

But $I(f)-S_0(f)$ is equal to the righthand side of (5), if $f\in W_q^{rs}(D)$. Then, the above inequality becomes

$$R(T)\leq \inf_{\{P_{s-1,i}\},\{P_{r-1,l}\},Q_{r-1,s-1}} \sup_{f\in W_q^{rs}(D)} \{$$

$$\frac{1}{s!}\sum_{i=0}^{r-1}\frac{h^{i+1}}{(i+1)!}\int_b^{b+k}((b+k-v)^s-P_{s-1,i}(v))f^{(i,s)}(a,v)dv$$

$$+\frac{1}{r!}\sum_{l=0}^{s-1}\frac{k^{l+1}}{(l+1)!}\int_a^{a+h}((a+h-u)^r-P_{r-1,l}(u))f^{(r,l)}(u,b)\,du$$

$$+\frac{1}{r!\,s!}\int\int_D((a+h-u)^r(b+k-v)^s-Q_{r-1,s-1}(u,v)f^{(r,s)}(u,v)\,dudv\}.$$

Further, by Hölder's inequality

$$R(T)\leq \inf_{\{P_{s-1,i}\}}\frac{1}{s!}\sum_{i=0}^{r-1}(h^{i+1}/(i+1)!)\left(\int_b^{b+k}|(b+k-v)^s-P_{s-1,i}(v)|^p dv\right)^{1/p}$$

$$+\inf_{\{P_{r-1,l}\}}\frac{1}{r!}\sum_{l=0}^{s-1}(k^{l+1}/(l+1)!\left(\int_a^{a+h}|(a+h-u)^r-P_{r-1,l}(u)|^p du\right)^{1/p}$$

$$+\inf_{Q_{r-1,s-1}}\frac{1}{r!\,s!}\left(\int\int_D|(a+h-u)^r(b+k-v)^s-Q_{r-1,s-1}(u,v)|^p dudv\right)^{1/p}.$$

From the definitions of the polynomials $U_{rp}(t;[a,a+h])$, $U_{sp}(t;[b,b+k])$ and $T_{rsp}(t,\tau;D)$ it follows that

$$R(T)\leq\frac{1}{s!}\left(\int_b^{b+k}|U_{sp}(v;[b,b+k])|^p dv\right)^{1/p}\sum_{i=0}^{r-1}h^{i+1}/(i+1)!$$

$$+\frac{1}{r!}\left(\int_a^{a+h}|U_{rs}(u;[a,a+h])|^p du\right)^{1/p}\sum_{l=0}^{s-1}k^{l+1}/(l+1)!$$

$$+\frac{1}{r!\,s!}\left(\int\int_D|U_{rsp}(u,v;D)|^p dudv\right)^{1/p}.$$

It remains to observe that

$$\left(\int_b^{b+k}|U_{sp}(v;[b,b+k])|^p dv\right)^{1/p}=\left(\frac{k}{2}\right)^{s+1/p}E_{sp},$$

$$\left(\int_a^{a+h} |U_{rp}(u;[a,a+h])|^p du\right)^{1/p}=\left(\frac{h}{2}\right)^{r+1/p}E_{rp},$$

$$\left(\int_D\int |U_{rsp}(u,v;D)|^p dudv\right)^{1/p}=h^{r+1/p}k^{s+1/p}E_{rsp}.$$

Finally,

$$R(T)\leqq(E_{sp}/s!)(k/2)^{s+1/p}\sum_{i=0}^{r-1}h^{i+1}/(i+1)!+(E_{rp}/r!)(h/2)^{r+1/p}\sum_{l=0}^{s-1}k^{l+1}/(l+1)!$$
$$+h^{r+1/p}k^{s+1/p}(E_{rsp}/(r!\,s!))=R(T).$$

The last relation shows that the method

(11) $$S^*(f)=S_0(f)+\bar{S}(\{P_{s-1,i}\},\ \{P_{r-1,l}\},\ Q_{r-1,s-1};f)$$

with

$$P_{r-1,l}(u)=(a+h-u)^r-(-1)^r(U_{rp}(u;[a,a+h]),\quad l=0,1,\ldots,s-1,$$
$$P_{s-1,i}(v)=(b+k-v)^s-(-1)^sU_{sp}(v;[b,b+k]),\quad i=0,1,\ldots,r-1,$$
$$P_{r-1,s-1}(u,v)=(a+h-u)^r(b+k-v)^s-U_{rsp}(u,v;D),$$

is the best one. In what follows we shall find a more explicit form of the best method S^*. First, we observe that

$$U_{sp}^{(s-l-1)}(v;[b,b+k])=(k/2)^{l+1}U_{sp}^{(s-l-1)}(2v/k-(2b+k)/k,$$

(12) $$U_{rp}^{(r-i-1)}(u;[a,a+h]=(h/2)^{i+1}U_{rp}^{(r-i-1)}(2u/h-(2a+h)/h),$$

$$U_{rsp}^{(r-i-1,s-l-1)}(u,v;D)=(-1)^{r+s-l-i}h^{i+1}k^{l+1}U_{rsp}^{(r-i-1,s-l-1)}((a+h-u)/h,$$
$$(b+k-v)/k).$$

Now, (8) and the above relations give

(13) $$\frac{1}{s!}\int_b^{b+k}((b+k-v)^s-(-1)^sU_{sp}(v;[b,b+k]))f^{(i,s)}(a,v)\,dv$$

$$=\frac{1}{r!}\sum_{l=0}^{s-1}(-1)^{s-l-1}\Big[-(-1)^s\left(\frac{k}{2}\right)^{l+1}U_{sp}^{(s-l-1)}(1)f^{(i,l)}(a,b+k)$$

$$-\left((-1)^{s-l-1}\frac{s!}{(l+1)!}k^{l+1}-(-1)^s\left(\frac{k}{2}\right)^{l+1}U_{sp}^{s-l-1)}(-1)\right)f^{(i,l)}(a,b)\Big]$$

$$=-\sum_{l=0}^{s-1}\frac{k^{l+1}}{(l+1)!}f^{(i,l)}(a,b)+\frac{1}{s!}\sum_{l=0}^{s-1}(-1)^{l+1}\left(\frac{k}{2}\right)^{l+1}[U_{sp}^{(s-l-1)}(-1)f^{(i,l)}(a,b)$$
$$-U_{sp}^{(s-l-1)}(1)f^{(i,l)}(a,b+k)].$$

Similarly

$$(14)\qquad \frac{1}{r!}\int_a^{a+h}((a+h-u)^r-(-1)^r U_{rp}(u;[a,a+h]))f^{(r,l)}(u,b)\,du$$

$$=-\sum_{i=0}^{r-1}(h^{i+1}/(i+1)!)f^{(i,l)}(a,b)+\frac{1}{r!}\sum_{i=0}^{r-1}(-1)^{i+1}(h/2)^{i+1}$$

$$\times[U_{rp}^{(r-l-1)}(-1)f^{(i,l)}(a,b)-U_{rp}^{(r-l-1)}(1)f^{(i,l)}(a+h,b)].$$

To calculate the double integral in (10) we apply (9) and the last relation from (12). An easy computation gives

$$(15)\qquad \frac{1}{r!\,s!}\int\int_D((a+h-u)^r(b+k-v)^s-U_{rsp}(u,v;D))f^{(r,s)}(u,v)\,dudv$$

$$=S_0(f)-\frac{1}{r!\,s!}\sum_{i=0}^{r-1}\sum_{l=0}^{s-1}h^{i+1}k^{l+1}\{U_{rsp}^{(r-i-1,s-l-1)}(0,0)f^{(i,l)}(a+h,b+k)$$

$$-U_{rsp}^{(r-i-1,s-l-1)}(1,0)f^{(i,l)}(a,b+k)-U_{rsp}^{(r-i-1,s-l-1)}(0,1)f^{(i,l)}(a+h,b)$$

$$+U_{rsp}^{(r-i-1,s-l-1)}(1,1)f^{(i,l)}(a,b)\}.$$

Now, the insertion of (13), (14) and (15) in (10) together with (11) completes the proof of the theorem.

3. A Special case. In this section we consider in details the function class $W_2^{rs}(D)$. In that case the polynomials $U_{m2}(x)$ reduce to the well-known Legendre polynomials on $[-1, 1]$.

We shall make use of the following result, proved by E. Schats in [9].

Lemma 2

$$U_{rs2}(x,y)=x^r U_{s2}(y;[0,1]+y^s U_{rs}(x;[0,1])-U_{r2}(x;[0,1])U_{s2}(y;[0,1]).$$

As an immediate consequence of theorem 1, we have the following result

Theorem 2. *The cubature formula*

$$I(f)\approx\sum_{i=0}^{r-1}\sum_{l=0}^{s-1}\frac{h^{i+1}}{(i+1)!}\frac{k^{l+1}}{(l+1)!}\frac{\binom{r}{i+1}\binom{s}{l+1}}{\binom{2r}{i+1}\binom{2s}{l+1}}[(-1)^i f^{(i,l)}(a+h,b)$$

$$+(-1)^l f^{(i,l)}(a,b+k)+(-1)^{i+l}f^{(i,l)}(a+h,b+k)+f^{(i,l)}(a,b)]$$

is a best method of approximation of the integral $\int\int_D f(x,y)dxdy$ *for the class* $W_2^{rs}(D)$ *relative to the information* (1). *Here*

$$R(T)=\left(\frac{1}{(2r+1)(2s+1)}\right)^{1/2}\left\{\left(\frac{2^r(r!)^2}{(2r)!}\right)^2\frac{2}{2r+1}+\frac{2}{2s+1}\left(\frac{2^s(s!)^2}{(2s)!}\right)^2\right.$$

$$\left.-\frac{4}{(2r+1)(2s+1)}\left(\frac{2^r(r!)^2}{(2r)!}\right)^2\left(\frac{2^s(s!)^2}{(2s)!}\right)^2\right\}^{1/2}.$$

Proof. Clearly, $U_{s2}(y;\ [0,\ 1]) = U_{s2}(2y-1)/2^s$, $U_{r2}(x;\ [0,1]) = U_{r2}(2x-1)/2^r$. Since

$$U_{r2}^{(r-i-1)}(1) = 2^{i+1}\frac{r!}{(i+1)!}\binom{r}{i+1}\Big/\binom{2r}{i+1},$$

$$U_{r2}^{(r-i-1)}(-1) = (-1)^{i+1}\,2^{i+1}\frac{r!}{(i+1)!}\binom{r}{i+1}\Big/\binom{2r}{i+1},$$

we get by lemma 2

$$U_{rs2}^{(r-i-1,s-l-1)}(0,0) = (-1)^{i+l+1}\cdot\frac{1}{(i+1)!\,(l+1)!}\,\frac{\binom{r}{i+1}\binom{s}{l+1}}{\binom{2r}{i+1}\binom{2s}{l+1}},$$

$$U_{rs2}^{(r-i-1,s-l-1)}(1,\ 0) = (-1)^{l+1}\frac{s!\,r!}{(i+1)!\,(l+1)!}\,\frac{\binom{s}{l+1}}{\binom{2s}{l+1}}\left\{1-\binom{r}{i+1}\Big/\binom{2r}{i+1}\right\},$$

$$U_{rs2}^{(r-i-1,s-l-1)}(0,\ 1) = (-1)^{i+1}\frac{r!\,s!}{(i+1)!\,(l+1)!}\,\frac{\binom{r}{i+1}}{\binom{2r}{i+1}}\left\{1-\binom{s}{l+1}\Big/\binom{2s}{l+1}\right\},$$

$$U_{rs2}^{(r-i-1,s-l-1)}(1,\ 1) = \frac{r!\,s!}{(i+1)!\,(l+1)!}\left\{\frac{\binom{s}{l+1}}{\binom{2s}{l+1}}+\frac{\binom{r}{i+1}}{\binom{2r}{i+1}}-\frac{\binom{r}{i+1}\binom{s}{l+1}}{\binom{2r}{i+1}\binom{2s}{l+1}}\right\}.$$

It remains to insert these values in theorem 1 to get the result. In order to find $R(T)$ we have to calculate E_{r2}, E_{s2} and E_{rs2}. By definition

$$E_{r2} = \left\{\int_{-1}^{1} U_{r2}^2(t)\,dt\right\}^{1/2} = \frac{2^r(r!)^2}{(2r)!}\,(2/(2+1))^{1/2}.$$

Similarly

$$E_{s2} = \frac{2^s\,(s!)^2}{(2s)!}\,(2/(2s+1))^{1/2},$$

$$\int_0^1\int_0^1 U_{rs2}^2(x,y)\,dx\,dy = \int_0^1\int_0^1 \{x^r U_{s2}(y;[0,\ 1]) + y^s\, U_{r2}(x;[0,\ 1])$$

$$-U_{r2}(x;[0,\ 1]\, U_{s2}(y;[0,\ 1])\}^2 dx dy.$$

From the orthogonality of $U_{m2}(t;[0,\ 1])$ to the polynomials of degree $m-1$ or less we have

$$E_{r2} = \left\{\frac{1}{2r+1}\int_0^1 U_{s2}^2(y;[0,\ 1])dy + \frac{1}{2s+1}\int_0^1 U_{r2}^2(x;[0,\ 1])dx\right.$$

$$-\int_0^1 U_{r2}^2(x;[0,1])dx \int_0^1 U_{r2}^2(y;[0,1])dy\Big\}^{1/2}.$$

But $\int_0^1 U_{m2}^2(t;[0,1])dt = \frac{1}{2^{2m+1}}\int_{-1}^1 U_{m2}^2(t)dt, \quad m=1, 2, \ldots$

An elementary calculation completes the proof of the theorem.

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Centre for Research and Training
in Mathematics and Mechanics
1000 Sofia P. O. Box 373

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