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CONVOLUTIONS FOR THE RIGHT-INVERSE LINEAR OPERATORS OF THE GENERAL LINEAR DIFFERENTIAL OPERATOR OF THE FIRST ORDER

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It is given an explicit expression for a nontrivial convolution of an arbitrary linear right-inverse operator of the general nonsingular linear differential operator of the first order $D=\alpha(t)\frac{d}{dt}+\beta(t)$ with continuous coefficients $\alpha(t)>0$ and $\beta(t)$ in an interval $0\leqq t<T$.

In [1] we gave a direct Mikusiński-type construction of an operational calculus, connected with the general linear differential operator of the first order

$$D=\alpha(t)\frac{d}{dt}+\beta(t) \tag{1}$$

with continuous coefficients $\alpha(t)>0$ and $\beta(t)$ in a half-open interval $[0, T)$. Exactly speaking, this is an operational calculus for the simplest right-inverse linear operator of D in the space $C[0, T)$ of the continuous in $[0, T)$ complex-valued functions, namely for the operator

$$Lf(t)=\frac{1}{\varrho(t)}\int_0^t \varrho(\tau)f(\tau)\frac{d\tau}{\alpha(\tau)} \text{ with } \varrho(t)=\exp\left(\int_0^t \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right). \tag{2}$$

Now we will extend this approach to an arbitrary linear right-inverse operator of D.

Theorem 1. *Each linear right-inverse operator L of D has the form*

$$Lf(t)=\frac{1}{\varrho(t)}F(f)+\frac{1}{p(\tau)}\int_0^t \varrho(\tau)f(\tau)\frac{d\tau}{\alpha(\tau)}, \tag{3}$$

where $F(f)$ is a linear functional in $C[0, T)$. Conversely, if $F(f)$ is an arbitrary linear functional in $C[0, T)$, then the operator (3) is a linear right-inverse operator of (1).

Proof. Let L be an arbitrary linear right-inverse operator of D in $C[0, T)$, i. e. it is defined for each $f(t)\in C[0, T)$ and the function $y=Lf(t)$ satisfies, the equation

$$\alpha(t)\frac{dy}{dt}+\beta(t)y=f(t).$$

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It is evident that the initial value $y(0)=Lf(t)|_{t=0}=F(f)$ is a linear functional in $C[0, T)$. Then L should have the form (3). It is not more difficult to verify that each oparator of the form (3) is a right-inverse operator of (1).

The base of a direct algebraic Mikusiński-type approach to an operational calculus for L is the notion of convolution for a linear operator, which maps a linear space into itself (see [2], p. 106). Applied to our operator (3), this definition states:

A bilinear, commutative, associative operation $f*g$ in $C[0, T)$ $(*: C[0, T) \times C[0, T) \to C[0, T))$ is called a convolution for L in $C[0, T)$ iff it satisfies the convolution property

$$L(f*g)=(Lf)*g. \tag{4}$$

We shall show the existence of a convolution for each linear right-inverse operator (3) of (1) with a continuous linear functional $F(f)$ with respect to the almost uniform convergence, not in the whole space $C[0, T)$, but in the subspace $C^{(1)}[0, T)$ of the continuously differentiable functions of $C[0, T)$.

Firstly, we will show this for the corresponding linear right-inverse operators of the differentiation operator in $C^{(1)}(0, T)$.

Theorem 2. *If*

$$Lf(t)=F(f)+\int_0^t f(\tau)\,d\tau \tag{5}$$

is a linear right-inverse operator of the differentiation operator $\frac{d}{dt}$ *in* $C^{(1)}[0, T)$ *with continuous linear functional* $F(f)$, *then the operation*

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau+F_x\left\{\frac{\partial}{\partial x}\int_0^{x-t} f(x-\tau)g(t+\tau)\,d\tau\right\}, \tag{6}$$

where ihe subscipt x *of* F *denotes that the linear functional* F *is applied on the corresponding expression as a function of* x, *and* t *is considered as a parameter, is a convolution for* L *in* $C^{(1)}[0, T)$.

Proof. The bilinearity of (6) is evident. It is also evident that $f, g \in C^{(1)}[0, T)$ imply $f*g \in C^{(1)}[0, T)$. The commutativity is also easy for verification. More difficult is the verification of the associativity of (6). By a tedious algebra, using the proposed continuity of $F(f)$ with respect to the almost uniform convergence, it is possible to verify the identity

$$(f*g)*h=f*(g*h)$$

directly for arbitrary $f, g, h \in C^{(1)}[0, T)$, but it is easier to verify it at first for functions of the form $f(t)=e^{\alpha t}$, $g(t)=e^{\beta t}$ and $h(t)=e^{\gamma t}$ with pairwise different α, β and γ. We have

$$e^{\alpha t}*e^{\beta t}=\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}+F_x\left\{\frac{\alpha e^{\alpha x+\beta t}-\beta e^{\alpha t+\beta x}}{\alpha-\beta}\right\}. \tag{7}$$

Then

$$(e^{\alpha t} * e^{\beta t}) * e^{\gamma t} = [(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)]^{-1}\{-(\beta-\gamma)\, e^{\alpha t}-(\gamma-\alpha)e^{\beta t}-(\alpha-\beta)e^{\gamma t}$$
$$+\alpha\,[(\alpha-\beta)e^{\gamma t}+(\gamma-\alpha)e^{\beta t}]\,F(e^{\alpha x})+\beta\,[(\beta-\gamma)e^{\alpha t}+(\alpha-\beta)e^{\gamma t}]\,F(e^{\beta x})$$
$$+\gamma\,[(\gamma-\alpha)\,e^{\beta t}+(\beta-\gamma)\,e^{\alpha t}]\,F(e^{\gamma t})-\alpha\beta(\alpha-\beta)\,e^{\gamma t}\,F(e^{\alpha x})\,F(e^{\beta x})$$
$$-\beta\gamma\,(\beta-\gamma)\,e^{\alpha t}\,F(e^{\beta x})\,F(e^{\gamma x})-\gamma\alpha\,(\gamma-\alpha)\,e^{\beta t}\,F(e^{\gamma x})\,F(e^{\alpha x})\}.$$

From the symmetry of this expression with respect to α, β, γ, it follows

$$(e^{\alpha t} * e^{\beta t}) * e^{\gamma t} = e^{\alpha t} * (e^{\beta t} * e^{\gamma t}).$$

The linear combinations of the functions of the form $e^{\alpha t}$ are dense in $C^{(1)}[0, T)$. We can find sequences of such linear combinations $f_n(t)$, $g_n(t)$, $h_n(t)$, converging to $f(t)$, $g(t)$, $h(t)$ with respect to the almost uniform convergence and consisting of members with different exponentials. To prove (7) it remains only to let $n\to\infty$ in $(f_n * g_n) * h_n = f_n * (g_n * h_n)$.

The convolution property (4) of (6) follows directly from the associativity, if we note that $Lf(t)=\{1\} * f$.

Example 1. Let $Ef(t)=f(0)$. Then $Lf(t)=f(0)+\int_0^t f(\tau)d\tau$ and the convolution (6) for L takes the form

$$(f * g)(t)=\int_0^t f(t-\tau)\,g(\tau)\,d\tau-\left(\int_0^t f(t-\tau)\,g(\tau)d\tau\right)'+f(t)g(0)+f(0)g(t).$$

Example 2. Let $T<\infty$ and $F(f)=-\int_0^T f(\tau)d\tau$. Then

$$Lf(t)=-\int_t^T f(\tau)\,d\tau$$

and

$$(f * g)(t)=\int_t^T f(T+t-\tau)\,g(\tau)\,d\tau.$$

Now we proceed to the general operator (3). We shall show that the operator (3) is similar to an operator of the form (5). To this end we shall introduce the function

$$\varphi(t)=\int_0^t \frac{d\tau}{\alpha(\tau)}$$

and denote $\tilde{T}=\varphi(T)$. It well may happen $\tilde{T}=+\infty$.

The function $\varphi(t)$ is strictly increasing in $[0, T)$ and hence there exists the inverse function $\varphi^{-1}(x)$, defined in $[0, \tilde{T})$.

Theorem 3. *If* $\lambda: C^{(1)}[0, T)\to C^{(1)}[0, \tilde{T})$ *is the linear map*

$$\lambda : f(t) \to \widetilde{f}(x) = \varrho[\varphi^{-1}(x)] f[\varphi^{-1}(x)],$$

then the similarity relation

$$L = \lambda^{-1} \widetilde{L} \lambda \tag{9}$$

is fulfilled for

$$\widetilde{L} \widetilde{f}(x) = F\left\{\frac{1}{\varrho(t)} \widetilde{f}[\varphi(t)]\right\} + \int_0^x \widetilde{f}(x)\, dx. \tag{10}$$

The proof reduces to a simple verification. Let us note that the inverse map λ^{-1} is given by $\lambda^{-1} : \widetilde{f}(x) \to \widetilde{f}[\varphi(t)]/\varrho(t)$.

Now, using the similarity relation (9) and the expression $\widetilde{\widetilde{f} * \widetilde{g}}$ of the form (6) for the convolution of (10), we can assert that $f * g = \lambda^{-1}[(\lambda f) \widetilde{*} (\lambda g)]$ is a convolution for L in $C^{(1)}[0, T)$ (see [2], p. 110). Thus we have proved the following

Theorem 4. *The operation*

$$(f * g)(t) = \frac{1}{\varrho(t)} \int_0^t \widetilde{f}[\varphi(t) - \varphi(\tau)]\, \widetilde{g}[\varphi(\tau)]\, d\varphi(\tau)$$

$$+ \frac{1}{\varrho(t)} F_x \left\{ \frac{\alpha(x)}{\varrho(x)} \frac{\partial}{\partial x} \int_0^{\varphi(x) - \varphi(t)} \widetilde{f}[\varphi(x) - \varphi(\tau)]\, \widetilde{g}[\varphi(t) + \varphi(\tau)]\, d\varphi(\tau) \right\} \tag{11}$$

is a convolution for (3) *in* $C^{(1)}[0, T)$.

After we have the explicit expressions (6) and (11), they should be studied in order to characterize the eventual divisors of zero. We may expect to prove analogons of the Titchmarsh theorem for the ordinary convolution only for some linear functionals $E(f)$. It seems that this is not an easy question. In order to construct a quotient ring of $C^{(1)}[0, T]$ with respect to the convolution (11) we need to know sufficiently many elements, which are not divisors of zero for the convolution (11). It is evident that especially the element $L = \{1/\varrho(t)\}$ is not a divisor of zero. For an algebraic treating of differential equations of the form

$$P(D)\, y = f(t)$$

we need to know only for which numbers μ the function

$$L^2 - \mu L = \frac{1}{\varrho(t)} F\left(\frac{1}{\varrho(t)}\right) + \frac{\varphi(t)}{\varrho(t)} - \frac{\mu}{\varrho(t)}$$

is a divisor of zero for (11) or, what is the same, the eigenvalues of L.

As far as we know, the problem for developing of an operational calculus for an arbitrary linear right-inverse operator of the differentiation operator is posed for the first time by L. Berg in his book [3].

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