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ON THE FIXED POINT THEOREM OF SCHAUDER

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If U is an open acyclic set in an admissible linear topological space E and $\Phi : U \to$ a multivalued acyclic compact map, then Φ has a fixed point.

In earlier papers [5, 6] we considered some generalizations of the well-known Schauder fixed point theorem [8]. They were concerned with multivalued compact acyclic maps of open sets of locally convex linear topological spaces. The purpose of the present paper is to strengthen this result by considering multivalued compact acyclic maps of open sets of some nonlocally convex topological spaces.

1. Preliminaries.

Definition 1. *Let $F: X \to Y$ be a multivalued map of the topological space X in the topological space Y. The map F is called upper semicontinuous at the point $x \in X$ if:*

a) the set $F(x)$ is closed set in the space Y and

b) for every open neighbourhood U of the set $F(x)$ there exists an open neighbourhood V of the point x, such that $F(V) = \cup F(y) \subset U, y \in V$.

The map F is upper semicontinuous if it is upper semicontinuous at every point $x \in X$ (about multivalued maps and their basic properties, see [1]).

We shall use the cohomology functor, defined in the category of all Hausdorff topological spaces — the Grothendieck-Godement cohomology with rational coefficients [2].

If X is a topological space, then $H^*(X) = \{H^i(X)\}_i$ are the cohomologies of the space X with rational coefficients and if $f: X \to Y$ is a single valued continuous map, then $f^* = \{f^i\}: H^*(Y) \to H^*(X)$ is the homomorphism induced by the map f [2].

Definition 2. *Let K be a topological space, K is called an acyclic space if it is connected and $H^i(K) = 0$ for $i \geq 1$.*

For example, every convex (and even starlike) compact in a linear topological Hausdorff space is acyclic.

Definition 3. *A multivalued upper semicontinuous map $F: X \to Y$ is called an acyclic map if for each $x \in X$ the set $F(x)$ is an acyclic compact space.*

Definition 4. *A multivalued map $F: X \to Y$ is called compact if the set $F(X) = \cup F(x)$ has a compact closure in the space Y.*

Let us consider a linear topological Hausdorff space E.

Definition 5. ([3]) *The space E is called admissible if for every compact set $K \subset E$ and each open neighbourhood V of the origin of E there exists a single valued continuous map $\mu: K \to P$ such that:*

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a) P is a finite polytope $(P \subset E)$*;*
b) $\mu(x) - x \in V$ *for every* $x \in K$*;*
c) $\mu(K) = P$.

For example, if E is a locally convex space, it is an admissible one [4, 5]. But there are some nonlocally convex linear topological Hausdorff spaces which are admissible. For example, if m is an infinite cardinal and $0 < p < 1$ then the spaces $l^p m$ are admissible [3].

2. Main theorem. Theorem 1. *Let E be an admissible linear topological Hausdorff space and* $U \subset E$ *be an open acyclic set. If* $\Phi: U \to U$ *is an acyclic compact map, then there exists a point* $x \in X$ *such that* $x \in \Phi(x)$*, i. e. the map* Φ *has a fixed point.*

Remark 2.1. If E is a locally convex space, this theorem follows from the theorem in [6].

Remark 2.2. If E is a Banach space, this theorem follows from the main theorem in [7].

Remark 2.3. Theorem 1 is a generalisation of Schauder's theorem [8, 3].

We shall prove now a more general theorem. Let V be an open set in E for which $\dim H^*(V) < \infty$, and $F: V \to V$ is an acyclic compact map. Let $\Gamma(F) = \{(x, y) \in V \times V \mid y \in F(x)\}$ and let $p_i: \Gamma(F) \to V$ be the map $p_i(x_1, x_2) = x_i$, $i = 1, 2$, for $(x_1, x_2) \in \Gamma(F)$.

The maps p_i have the following properties: a) p_i are single valued continuous maps, and p_1 is a closed map; b) for every $x \in V$ the space $p_1^{-1}(x)$ coincides with the space $\{x\} \times F(x)$.

We apply the Vietoris-Begle theorem [2] to the map $p_1: \Gamma(F) \to V$. From this theorem we obtain that $p_1^*: H^*(V) \to H^*(\Gamma(F))$ is an isomorphism.

Let $F^i = (p_1^i)^{-1} p_2^i$ and $\Lambda(F) = \Sigma(-1)^i \operatorname{tr} F^i$, where $\operatorname{tr} F^i$ is the trace of the linear map F^i. The integer $\Lambda(F)$ is called the Lefschetz number of F.

Theorem 2. *Let E be an admissible linear topological Hausdorff space and let* $V \subset E$ *be an open set, for which* $\dim H^*(V) < \infty$. *If* $\Phi: V \to V$ *is an acyclic compact map, for which* $\Lambda(\Phi) \neq 0$, *then* Φ *has a fixed point.*

Remark 2.4. For a locally convex linear topological Hausdorff space E, theorem 2 has been proved in [6].

Remark 2.5. For a Banach space E, theorem 2 has been proved in [7].

Remark 2.6. Theorem 1 is a corollary of the theorem 2, because in the case when V is an acyclic open set in E and $\Phi: V \to V$ is an acyclic compact map, the Lefschetz number $\Lambda(\Phi) = 1$.

3. Proof of Theorem 2. Let K be the closure of $\Phi(V)$ in the space E. The set K is a compact subset of V, because Φ is a compact map. There exists a starlike open neighbourhood W of the origin of the space E, such that $K + W \subset V$.

Let us consider the compact K and this starlike open neighbourhood W. As the space E is admissible, there exists a single valued continuous map $\mu: K \to P$ such that: a) P is a finite polytope $(P \subset E)$; b) $\mu(x) - x \in W$ for every $x \in K$; c) $\mu(K) = P$.

If $y \in P$ there exists an $x \in K$ such that $\mu(x) = y$ and $y \in x + W \subset V$. Therefore $P \subset V$.

Let $\Phi_1: P \to K$ be the map $\Phi_1 = \Phi | P$ and $\psi = \mu \Phi_1$. It follows that the map ψ is admissible in the sense of [9].

Let $i_K: K \to V$, $i_P: P \to V$ be the imbedings $i_K(x) = x$, $i_P(y) = y$, $x \in K$, $y \in P$. The diagram on fig. 1 is not commutative, but this on fig. 2 is a commutative one.

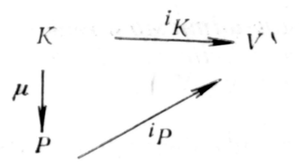

Fig. 1.

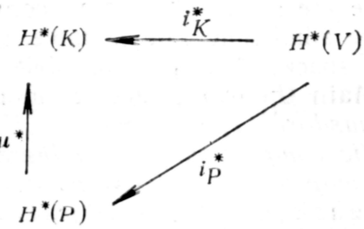

Fig. 2

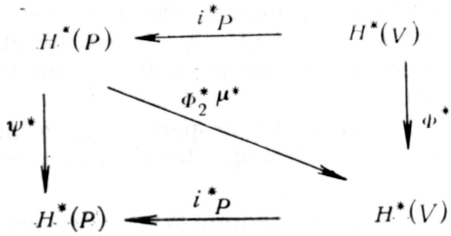

Fig. 3

In order to prove the last statement, we shall prove that the maps i_K and $i_P\mu$ are homotopic. Let us consider the map $F: K \times I \to V$ given by $F(x, t) = tx + (1-t)\mu(x)$ for $x \in K$ and $0 \leq t \leq 1$. The set W is a starlike set with center in the origin, hence the set $x + W$ is a starlike one with center x. It is clear that $F(x, t) \in x + W$ for $0 \leq t \leq 1$. Therefore, the map F is the homotopy between i_K and $i_P\mu$. Using the diagram on fig. 2 it is easy to show that the diagram on fig. 3 is also commutative. Here $\Phi_2: V \to K$ is given by $\Phi_2(x) = \Phi(x)$ for $x \in V$ and by ψ^* we denote the homomorphism $\Phi_1^*\mu^*$. It follows from the last diagram that $\Lambda(\psi^*) = \Lambda(\Phi^*)$.

Applying theorem 1 from [9] we obtain that ψ has a fixed point, i. e. there exists a $x_0 \in P$ such that $x_0 \in \psi(x_0)$. The set $\psi(x_0)$ coincides with the set $\mu\Phi(x_0)$, hence there is a point z_0 belonging to $\Phi(x_0)$ such that $\mu(z_0) = x_0$.

Let us summarize: for every starlike open neighbourhood W of the origin of E we obtain points $x_W \in V$ and $z_W \in K$ such that: a) $z_W \in \Phi(x_W)$, b) $\mu(z_W) = x_W$ c) $x_W - z_W \in W$. But K is a compact space and therefore the sequence $\{z_W\}$ has a subsequence $\{z_{W_\alpha}\}$ which converges to a point $z_1 \in K$. As Φ is an upper semicontinuous map it follows from a) that $z_1 \in \Phi(z_1)$. Thus theorem 2 is proved.

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