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INTEGRABILITY THEOREMS FOR FOURIER SERIES OF POSITIVE FUNCTIONS

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1. Concerning the Fourier coefficients of positive functions, Askey and Boas [1] in 1967 proved the following theorems:

Theorem A. *Let* $G(x)\downarrow 0$ *on* $(0, \pi)$, G *bounded below and* $\int_0^\pi x dG(x)$ *finite, so that* dG *has generalized sine coefficients*

$$b_n = \frac{2}{\pi}\int_0^\pi \sin nx dG(x).$$

If $1<p<\infty$ *and* $1/p<r<1+1/p$ *then* $\{n^{-r}b_n\}\in l^p$ *iff* $t^{r-1-2/p}\int_0^t x dG(x)\in L^p$.

Theorem B. *Let* $F(x)\downarrow$ *on* $(0, \pi)$, F *bounded below and* $\int_0^\pi x^2 dF(x)$ *finite. Let* $a_n = -(2/\pi)\int_0^\pi (1-\cos nx) dF(x)$ *be the generalized cosine coefficient of* dF. *If* $1<p<\infty$ *and* $1/p<r<2+1/p$, *then* $\{n^{-r}a_n\}\in l^p$ *iff* $t^{r-2-2/p}\int_0^t n^2 dF(u)\in L^p$.

Theorem C. *If* $-1/p'<r<1/p$ *and* $\{n^{-r}a_n\}\in l^p$, *where* a_n *are the Fourier coefficients of* dF *with* F *monotone, then* $t^{r-2/p}(F(t)-F(0))\in L^p$.

Theorem D. *If* $-1/p'<r<1/p$, a_n *are the Fourier cosine coefficients of* f *and* $t^{r-2/p}\int_0^t x\,|df(x)|\in L^p$, *then* $\{n^{-r}a_n\}\in l^p$.

The object of this paper is to obtain certain generalizations of these theorems.

Theorem 1. *Let* $G(x)$ *satisfy the hypotheses of Theorem A. If* $1<p<\infty$ *and* $\lambda(x)$ *is a positive function such that*

(i) $x^{1+\delta}\lambda(x)\downarrow$ *for some small* $\delta>0$,

(ii) $x^{p+1-\delta}\lambda(x)\uparrow+\infty$ *for some small* $\delta>0$ *as* $x\to\infty$, *then* $\{\lambda(n)^{1/p}b_n\}\in l^p$ *iff*

$$\lambda(\pi/t)^{1/p}t^{-1-2/p}\int_0^t x dG(x)\in L^p. \tag{1.1}$$

Theorem 2. *Let* $F(x)$ *satisfy the hypotheses of Theorem B. If* $1<p<\infty$ *and* $\lambda(x)$ *is a positive function such that*

(i)′ $x^{1+\delta}\lambda(x)\downarrow$ *for some small* $\delta>0$,

(ii)′ $x^{2p+1-\delta}\lambda(x)\uparrow\infty$ *for some small* $\delta>0$, as $x\to\infty$, *then* $\{\lambda^{1/p}(n)a_n\}\in l^p$ *iff* $\lambda^{1/p}(\pi/t)t^{-2-2p}\int_0^t x^2 dF(x)\in L^p$.

Theorem 3. *If a_n are the Fourier coefficients of dF with F monotonic and $\mu(x)$ is a positive function such that*

(a) $x^{1+\delta-p}\mu(x)\downarrow 0$ *for some small* $\delta>0$, $x\to\infty$,

(b) $x^{1-\delta}\mu(x)\uparrow\infty$ *for some small* $\delta>0$, $x\to\infty$ *and* $\{\mu^{1/p}(n)a_n\}\in l^p$, *then* $\mu^{1/p}(\pi/t)t^{-2/p}(F(t)-F(0))\in L^p$.

Theorem 4. *If a_n are the Fourier coefficients of f and if* $\mu^{1/p}(\pi/t)\times t^{-2/p}\int_0^t x\,|df(x)|\in L^p$, *then* $\{\mu^{1/p}(n)a_n\}\in l^p$, *where* $\mu(x)$ *satisfies the same conditions as in Theorem 3.*

Taking $\lambda(x)=x^{-rp}$, $1/p<r<1+1/p$ in Theorem 1 we derive Theorem A. Similarly all other theorems of Askey and Boas can be deduced from our theorems.

2. The following lemmas are needed for the proof of our theorems.

Lemma 1. [4] *Let $\lambda(n)$ be a positive monotonic decreasing sequence such that* $\sum_{k=n}^{\infty}\lambda_k=O(n\lambda(n))$, $n\to\infty$. *Let* $S_n=\sum_{k=1}^{n}a_k$, $a_k\geq 0$ *and* $\sum_{n=1}^{\infty}\lambda(n)\,(na_n)^p<\infty$, $p>1$, *then* $\sum_{n=1}^{\infty}\lambda(n)S_n^p<\infty$ *and* $\sum_{n=1}^{\infty}\lambda(n)S_n^p\leq K\sum_{n=1}^{\infty}\lambda(n)(na_n)^p$, *where K is a positive constant.*

Lemma 2 [1]. *If* $G(x)\downarrow$, $\int_0^{\pi}x\,|dG(x)|<\infty$ *and b_n are the generalized sine coefficients of dG, then* $\left|\sum_{k=1}^{n}{}' b_k\right|\geq Kn^2\int_0^{\pi/n}x\,|dG(x)|$, *where in Σ' the last term is multiplied by* $1/2$.

Lemma 3. *Suppose $\psi(x)$ is a positive function such that*

(i) $[x^{1+\delta}\psi(x)]^{-1}$ *is an increasing function when x increases from zero to ∞ for sufficiently small* $\delta>0$.

(ii) *there exists a number $k>1$ for which* $[x^k\psi(x)]^{-1}$ *is a decreasing function. Let* $F(x)=\int_0^x f(t)\,dt, f(t)\geq 0$. *Then* $\int_0^{\pi}\psi(x)F(x)^p dx\leq K\int_0^{n}\psi(x)(xf(x))^p dx$ *where* $1\leq p<\infty$.

This is a particular case of a Lemma of Chen [2].

Lemma 4. *Let $x^{1-\delta}\varphi(x)$ be a positive increasing function for some small $\delta>0$. Let* $G(x)=\int_x^{\pi}g(t)\,dt$, $g(t)\geq 0$ *where $g(t)$ is integrable in* $0<\varepsilon\leq t\leq\pi$, *then* $\int_0^{\pi}\varphi(x)G^p(x)dx\leq K\int_0^{\pi}\varphi(x)(xg(x))^p dx$, $1\leq p<\infty$.

This is a generalization of a lemma of Hardy [3], [2].

Proof of lemma 4. Putting $R(x)=\int_0^x\varphi(t)\,dt$, one has $\int_{\varepsilon}^{\pi}\varphi(x)G^p(x)dx$

$$=[R(x)\,G^p(x)]_{\varepsilon}^{\pi}+p\int_{\varepsilon}^{\pi}R(x)G^{p-1}(x)g(x)dx=-R(\varepsilon)G(\varepsilon)^p+p\int_{\varepsilon}^{\pi}R(x)g(x)G^{p-1}(x)dx.$$

Further $R(x)=\int_0^x t^{1-\delta}\varphi(t)t^{\delta-1}dt\leq x\varphi(x)/\delta$ and for $t\geq 1$

$$t^{-1}txg(x)G^{p-1}(x)\leq t^{-1}\max\,(G^p(x),\ (txg(x))^p)\leq t^{-1}G^p(x)+t^{p-1}(xg(x))^p.$$

Hence $p\int^{\pi}R(x)g(x)G^{p-1}(x)dx\leq(p/\delta)\int^{\pi}xg(x)\varphi(x)G(x)^{p-1}dx$

$$\leq (p/\delta t)\int_\varepsilon^\pi \varphi(x)G(x)^p dx + (pt^{p-1}/\delta)\int_\varepsilon^\pi \varphi(x)(xg(x))^p\, dx.$$

Thus $(1-p/t\delta)\int_\varepsilon^\pi \varphi(x)G^p(x)dx \leq -R(\varepsilon)G^p(\varepsilon)+(pt^{p-1}/\delta\int_\varepsilon^\pi \varphi(x)(xg(x))^p dx.$

Now $R(\varepsilon)G(\varepsilon)^p \leq K\varepsilon\varphi(\varepsilon)G(\varepsilon)^p$ and $\{\int_\varepsilon^\pi g(x)dx\}^p \leq \int_\varepsilon^\pi \varphi(x)(xg(x)^p dx$

$$\times\{\int_\varepsilon^\pi x^{-p'}\varphi(x)^{-p'/p}dx\}^{p/p'} \leq [K\varepsilon^{-1}/\varphi(\varepsilon)]\int_\varepsilon^\pi \varphi(x)(xg(x)^p\, dx$$

so that $G^p(\varepsilon)R(\varepsilon) \leq K\int_\varepsilon^\pi \varphi(x)(xg(x))^p\, dx.$

Thus $(1-p/t\delta)\int_\varepsilon^\pi \varphi(x)G^p(x)dx \leq K\int_\varepsilon^\pi \varphi(x)(xg(x))^p dx.$

Choosing t large enough one has the inequality

$$\int_\varepsilon^\pi \varphi(x)G^p(x)dx \leq \int_\varepsilon^\pi \varphi(x)(xg(x))^p dx,$$

and for $\varepsilon\to 0$,

$$\int_0^\pi \varphi(x)G(x)^p dx \leq \int_0^\pi \varphi(x)(xg(x))^p dx.$$

This proves Lemma 4.

Lemma 5. *If $\varphi(x)$ increases and is bounded with $\varphi(+0)=0$, then $\int_0^\pi \psi(u)\varphi^p(u)\, du<\infty$ iff $\int_0^\pi \psi(u)\, u^{sp}(\int_u^\pi x^{-s}d\varphi(x))^p du<\infty$, where $s>0$, $p>1$ and*

(i) *$x^{1-\delta+sp}\,\psi(x)$ is a positive increasing function for some small $\delta>0$.*

(ii) *$(x^{1+\delta}\psi(x))^{-1}$ is increasing for some small $\delta>0$ and tends to zero as $x\to 0$.*

This generalizes a lemma of Askey and Boas [1].

Proof. Suppose that $\int_0^\pi \psi(u)\varphi^p(u)du<\infty$. Then

$$\int_0^\pi \psi(u)\, u^{sp}(\int_u^\pi x^{-s}d\varphi(x))^p du \leq K\int_0^\pi \psi(u)u^{sp}du + K\int_0^\pi \psi(u)\varphi^p(u)du$$

$$+K\int_0^\pi \psi(u)u^{sp}(\int_u^\pi x^{-s-1}\varphi(x)dx)^p du \leq K+K\int_0^\pi \psi(u)u^{sp}(u^{-s}\varphi(u))^p du \leq K,$$

by virtue of Lemma 4 and the hypotheses of Lemma 5.

Now suppose that $\int_0^\pi \psi(u)u^{sp}(\int_u^\pi x^{-s}d\varphi(x))^p du<\infty$. Put $X(y)=\int_y^\pi x^{-s}\, d\varphi(x)$, then $\varphi(u)=-\int_0^u x^s dX(x)=-x^sX(x)|_0^u+s\int_0^u x^{s-1}X(x)dx$, provided $\lim_{x\to 0}x^s X(x)$ exists.

Since $\int_0^\pi \psi(u)u^{sp}X^p(u)du<\infty$ we find that $\int_{y/2}^y \psi(u)u^{sp}X(u)^p du$ is bounded. But $\int_{y/2}^y \psi(u)u^{sp}X^p(u)du \geq X^p(y)\int_{y/2}^y u^{sp}\psi(u)du \geq KX^p(y)\,y^{sp+1}\psi(y).$

Thus $y^{sp+1}\psi(y)X^p(y)$ is bounded as $y\to 0$ and hence $y^sX(y)\to 0$ as $y\to 0$ so that $\varphi(u)=-u^sX(u)+s\int_0^u x^{s-1}X(x)dx$ and therefore

$$\int_0^\pi \psi(u)\varphi(u)^p du \le K\int_0^\pi \psi(u)u^{sp}X^p(u)du$$

$$+K\int_0^\pi \psi(u)(\int_0^u x^{s-1}X(x)dx)^p du \le K+K\int_0^\pi \psi(u)(u^sX(u))^p du<\infty,$$

by virtue of the hypotheses and Lemma 3. This proves Lemma 5.

3. Proof of Theorem 1.

Necessity. We have by Lemmas 1 and 2

$$\int_0^\pi \lambda(\pi/u)\,u^{-p-2}(\int_0^u x\,|dG(x)|)^p du=\sum_{n=1}^\infty \int_{\pi/(n+1)}^{\pi/n}\lambda\left(\frac{\pi}{u}\right)u^{-p-2}(\int_0^u x\,|dG(x)|)^p du$$

$$\le K\sum_{n=1}^\infty n^p\lambda(n)(\int_0^{\pi/n} x|dG(x)|)^p \le K\sum_{n=1}^\infty \lambda(n)n^{-p}(\sum_{k=1}' |b_k|)^p \le K\sum_{n=1}^\infty |b_n|^p\lambda(n)<\infty.$$

Sufficiency. Setting $\|a_n\|=(\sum_{n=1}^\infty |a_n|^p)^{1/p}$ one has

$$\|\lambda^{1/p}(n)b_n\|=K\|\lambda^{1/p}(n)\int_0^{1/n}\sin nx\,dG(x)+\lambda^{1/p}(n)\int_{1/n}^\pi \sin nx\,dG(x)\|$$

$$\le K\|\lambda^{1/p}(n)n\int_0^{\pi/n} x|dG(x)|\,\|+K\|\lambda^{1/p}(n)\int_{1/n}^\pi |dG(x)|\,\|$$

$$\le K\{\sum_{n=2}^\infty \int_{\pi/n}^{\pi/(n-1)}\lambda(\pi/t)t^{-p-2}(\int_0^t x\,|dG(x)|)^p dt\}^{1/p}+K\{\sum_{n=2}^\infty \lambda(n)(\int_{1/n}^{\pi/(n-1)}+\int_{\pi/(n-1)}^{\pi})^p\}^{1/p}+K$$

$$\le K\{\int_0^\pi \lambda(\pi/t)t^{-p-2}(\int_0^t x|dG(x)|)^p\,dt\}^{1/p}+K\{\sum_{n=2}^\infty \lambda(n)\int_{1/n}^{\pi/(n-1)}|dG(x)|)^p\}^{1/p}$$

$$+K\{\sum_{n=1}^\infty \lambda(n)(\int_{\pi/(n-1)}^\pi |dG(x)|)^p\}^{1/p}+K.$$

Since $\int_0^\pi \lambda(\pi/t)t^{-p-2}(\int_0^t x|dG(x)|)^p dt=\sum_{n=3}^\infty \int_{\pi/(n-1)}^{\pi/(n-2)}\lambda(\pi/t)t^{-p-2}(\int_0^t x\,dG(x))^p dt$

$$\ge \sum_{n=3}^\infty[(n-2)/\pi]^p\int_{\pi/(n-1)}^{\pi/(n-2)}t^{-2}\lambda(n-1)(\int_{1/n}^t x|dG(x)|)^p dt \ge K\sum_{n=3}^\infty \lambda(n)n^p(\int_{1/n}^{\pi/(n-1)}x|dG(x)|)^p,$$

we have in view of the hypothesis

$$\|\lambda^{1/p}(n)b_n\|\le K+K\{\sum_{n=2}^\infty \lambda(n)(\int_{\pi/(n-1)}^\pi |dG(x)|)^p\}^{1/p}$$

$$\le K+K\{\sum_{n=2}^\infty \int_{\pi/n}^{\pi/(n-1)}\lambda(\pi/t)t^{-2}(\int_t^\pi dG(x)|)^p dt\}^{1/p}=K+K\{\int_0^\pi \lambda(\pi/t)t^{-2}(\int_t^\pi |dG(x)|)^p\,dt\}^{1/p}.$$

Now putting in Lemma 5 $\varphi(t)=\int_0^t x|dG(x)|$, $s=1$, $\psi(t)=\lambda(\pi/t)t^{-p-2}$, we find that the condition (1.1) is equivalent to the convergence of the above integral. Thus $\|\lambda^{1/p}(n)b_n\|<\infty$. This completes the proof of Theorem 1.

4. Proof of Theorem 2. It is similar to that of Theorem 1 and can be omitted.

5. Proof of Theorem 3. Since $\{\mu^{1/p}(n)a_n\}\in l^p$ we have in view of condition (b)

$$\sum_{n=1}^{\infty} n^{-1}|a_n|\leq\Big(\sum_{n=1}^{\infty}\mu(n)|a_n|^p\Big)^{1/p}\Big(\sum_{n=1}^{\infty}\mu^{-p'/p}(n)n^{-p'}\Big)^{1/p'}<\infty .$$

Hence $\Sigma a_n \sin nx/n$ is the Fourier series of a function F so that $\Sigma a_n \cos nx$ is the Fourier-Stieltjes series of dF. Suppose that F is an increasing function. Now

$$\frac{a_0}{2}+\sum_{k=1}^{n} a_k=(2/\pi)\int_0^{\pi}\frac{\sin(n+1/2)x\,dF(x)}{2\sin(x/2)}$$

$$=(1/\pi)\int_0^{\pi}\sin nx\cot(x/2)dF+(1/\pi)\int_0^{\pi}\cos nx\,dF(x)=L_n+M_n,\ \text{say}.$$

But $\sum_{n=1}^{\infty}\mu(n)n^{-p}|L_n+M_n|^p\leq\sum_{n=1}^{\infty}\mu(n)n^{-p}\Big(\sum_{k=1}^{n}|a_k|\Big)^p+K\leq K\sum_{n=1}^{\infty}\mu(n)|a_n|^p+K<\infty$,

by virtue of condition (a) and Lemma 1. Since $M_n=O(1)$ it follows that $\sum_{n=1}^{\infty}\mu(n)n^{-p}|M_n|^p<\infty$ and therefore $\sum_{n=1}^{\infty}\mu(n)n^{-p}|L_n|^p<\infty$. Writing $dG(x)$ $=-\cot(x/2)dF(x)$ we have $L_n=-(1/\pi)\int_0^{\pi}\sin x\,dG(x)$. Applying Theorem 1 we have $\mu^{1/p}(\pi/t)t^{-2/p}\int_0^t x\,dG(x)\in L^p$. Since $F(t)-F(0)=\int_0^t dF(x)\leq K\int_0^t x|dG(x)|$, $0\leq t\leq\pi/4$ we have $\mu^{1/p}(\pi/t)t^{-2/p}(F(t)-F(0))\in L^p$. This proves Theorem 3.

6. Proof of Theorem 4. We are given that $\mu^{1/p}(\pi/t)t^{-2/p}\Big(\int_0^t x|df(x)|\Big)\in L^p$ and $na_n=-(2/\pi)\int_0^{\pi}\sin nt\,df(t)$.

Since the sufficiency part of Theorem 1 is true even when $\int_0^t x\,dG(x)$ is replaced by $\int_0^t x|dG(x)|$ with $G(x)$ not necessarily monotonic, we apply Theorem 1 to conclude that $\{\mu^{1/p}(n)a_n\}\in l^p$. This proves Theorem 4.

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