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ON RATIONAL APPROXIMATION OF FUNCTIONS WITH UNBOUNDED VARIATION

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V. A. Popov (1975) proved the following estimations for the rational Hausdorff approximation and the rational local approximation of functions with bounded variation

$$R_n(f;\alpha)=O(\ln\ln n/n),$$

$$|f(x)-q_n(x)|\leqq\omega(f,x;O(\ln\ln n/n))+O(1/n),\ x\in[0,1],$$

where $\omega(f,x;\delta)$ is the local modulus of continuity of the function f in the point $x\in[0,1]$

In this note it is shown that these estimations are valid for wider classes of functions which include functions with unbounded variation.

For the class ot all functions f with variation $\leqq V$ in the interval $[0,1]$ V. A. Popov [1] has obtained the following estimation for the best approximation of the function f by means of rational functions of degree n in the Hausdorff metric with parameter $\alpha>0$:

$$R_n(f;\alpha)\leqq C\max\{\frac{1}{\alpha n}\ln(V\alpha(\ln n)\ln(\alpha nV)),\ \frac{1}{\alpha n}\} \tag{1}$$

for $\alpha nV\geqq e$, where C is an absolute constant.

Consequently

$$R_n(f;\alpha)=O(\ln\ln n/n). \tag{2}$$

On the other hand (see [1])

$$|f(x)-g(x)|\leqq\omega(f;x,\ \alpha r(f,g;\alpha))+r(f,g;\alpha), \tag{3}$$

where $r(f,g;\alpha)$ (see [2]) denote the Hausdorff distance with parameter α between the functions f and g; $\omega(f;x,\delta)=\sup_{|y-x|\leqq\delta}|f(y)-f(x)|$ is the local modulus of continuity of the function f at the point x.

From (1) and (3) one can obtain the following local estimation: for every function f bounded in $[0,1]$ there exists a rational function $q_n(x)$ of degree n, such that for each $x\in[0,1]$

$$|f(x)-q_n(x)|\leqq\omega(f;x,\ C(V)\ln\ln n/n)+C(V)/n, \tag{4}$$

where $C(V)$ is a constant depending only on the variation $V=V_0^1(f)$ of the function f in $[0,1]$.

Let us mention that the order of approximation in the estimations (2), (4) is better than the corresponding order in the respective estimations for polynomial approximations. By now it is not known whether the orders in (2) and (4) are exact.

For functions, bounded in the interval $[0,1]$, V. A. Popov [3] has introduced the following characteristic

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(5) $$\varkappa(f;n)=\sup_{0\le x_0\le x_1\le\ldots\le x_n\le 1}\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|.$$

A similar characteristic has been introduced by Z. A. Čanturia in [4]

One may obtain the following properties:

$\varkappa(f;n)\le\varkappa(f;n+1)\le V_0^1(f)$, $\varkappa(f;p.n)\le p.\varkappa(f;n)$, $\lim_{n\to\infty}\varkappa(f;n)=V_0^1(f)$. Moreover in most cases $\varkappa(f;n)=O(n)$.

It is not difficult to see that the variation of f in the right hand side of (1) can be replaced by $\varkappa(f;2n)$. Therefore

(6) $$R_n(f;a)\le C\max\left\{\frac{1}{an}\ln[a\varkappa(f;n)(\ln n)\ln(an\varkappa(f;n))],\ \frac{1}{an}\right\},$$

for $an\varkappa(f;n)\ge e$.

From (6) it follows that the estimations (2), (4) are valid for all functions f, with $\varkappa(f;n)=O(\ln^s n)$.

Our purpose is to characterize in a better way the functions for which (2) and (4) are valid. We establish some relations between $\varkappa(f;n)$ and other constructive characteristics.

Denote by $\omega(f;\delta)$ the modulus of continuity of the function f. We have (see [4])

(7) $$\varkappa(f;n)\le 2n\omega(f;1/n).$$

V. A. Popov [5] introduced the moduli

$$\nu_k(f;\delta)=\inf_{\varphi\in V}\ \sup_{|\varphi(x+kh)-\varphi(x)|\le\delta}|\Delta_h^k f(x)|,\quad k=1,2,3,\ldots,$$

where V is the class of all functions, with variation ≤ 1, monotone in [0, 1] as usual

$$\Delta_h^k f(x)=\sum_{l=0}^{k}(-1)^{k+l}\binom{k}{l}f(x+lh).$$

One may obtain, using the moduli $\nu_k(f;\delta)$, (see [5]), direct and converse theorems for spline approximation with free knots.

For $k=1$

(8) $$\nu_1(f;\delta)=\inf_{\varphi\in V}\ \sup_{|\varphi(x+h)-\varphi(x)|\le\delta}|f(x+h)-f(x)|.$$

Let us denote by $E_n^0(f)$ the best uniform approximation of the function f by means of all step-functions with $n-1$ jumps, continuous either on the right or on the left at each $x\in[0,1]$.

If $f\in C[0,1]$, then

(9) $$\nu_1(f;1/n)=2E_n^0(f).$$

For every bounded function f

(10) $$\nu_1(f;1/n)\le 2E_n^0(f).$$

Let us mention that if we impose additionally in the definition (8) of $\nu_1(f;\delta)$ that every function $\varphi\in V$ is contionuous either on the right or on the left at each $x\in[0,1]$, then (9) holds for every f.

The characteristics $\varkappa(f;n)$ and $\nu_1(f;\delta)$ are mutually connected. If f is continuous (see [3]), then

(11) $$\frac{1}{2}\, n\nu_1(f;1/n) \leqq \varkappa(f;n) \leqq 3n\nu_1(f;1/n).$$

The right inequality may be not valid for f not continuous. In this case the following lemma in helpful.

Lemma 1. *Let f be bounded in* [0, 1]. *Then*

(12) $$\varkappa(f;n) \leqq \sum_{k=1}^{n} \nu_1(f;1/k)$$

and therefore

(13) $$\varkappa(f;n) \leqq 2\sum_{k=1}^{n} E_n^0(f).$$

Proof. Let $\varphi_k(x)$, $k=1,2,\ldots,n$ are arbitrary functions from the class V, i. e. for each $k=1,2,\ldots,n$ $\varphi_k(x)$ is monotone and $V_0'(\varphi_k) \leqq 1$.

Let us consider an arbitrary sum:

$$\sigma = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|, \quad 0 \leqq x_0 \leqq x_1 \leqq \cdots \leqq x_n \leqq 1.$$

We shall prove that

(14) $$\sigma \leqq \sum_{k=1}^{n} \sup_{|\varphi_k(x+h)-\varphi_k(x)| \leqq 1/k} |f(x+h) - f(x)|.$$

From (14) and the definitions (5) and (8) for $\varkappa(f;n)$ and $\nu_1(f;\delta)$ the inequality (12) follows immedeately.

For any k, $1 \leqq k \leqq n$ let us consider the point sets:

$$A_s = \{x \in [0,1]:\ \varphi_k(0) + (s-1)/k \leqq \varphi_k(x) < \varphi_k(0) + s/k\}, \quad s = 1, 2, \ldots, k-1,$$

$$A_k = \{x \in [0,1]: \varphi_k(0) + (k-1)/k \leqq \varphi_k(x) \leqq \varphi_k(0) + 1\}.$$

They have the following properties:

1. $A_s \subset [0,1]$ and A_s is an interval, a point or empty set ($\emptyset$),
2. $A_s \cap A_t = \emptyset$ for $s \neq t$,
3. $\bigcup_{s=1}^{k} A_s = [0,1]$,
4. If $x+h$, $x \in A_s$, then $|\varphi_k(x+h) - \varphi_k(x)| \leqq 1/k$.

Let us consider

$$\sigma = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{j=1}^{n} a_j,$$

where $a_j \geqq a_{j+1}$ for $j = 1, 2, 3, \ldots, n-1$. Evidently there exist at least $n-k+1$ intervals $[x_{i-1}, x_i]$, such that $[x_{i-1}, x_i] \subset A_{s_i}$ for some s_i. Therefore taking into account property 4. of A_s we get for $j = k, k+1, \ldots, n$

$$a_j \leqq \sup_{|\varphi_k(x+h)-\varphi_k(x)| \leqq 1/k} |f(x+h) - f(x)|.$$

So $a_j \leqq \sup_{|\varphi_j(x+h)-\varphi_j(x)| \leqq 1/j} |f(x+h) - f(x)|$ for $j = 1, 2, \ldots, n$.

These inequalities imply (14). Thus lemma 1 is proved.

Bl. Sendov [2] has introduced the modulus of non-monotonicity of a function f for the purposes of Hausdorff approximations (see also [6]) as follows:

$$\mu(f;\delta)=\frac{1}{2}\sup_{|x_1-x_2|\leq\delta}\{\sup_{x_1\leq x\leq x_2}[|f(x_1)-f(x)|+|f(x_2)-f(x)|]-|f(x_1)-f(x_2)|\}. \tag{15}$$

Evidently $\mu(f;\delta)\leq\omega(f;\delta)$. The following lemma gives a relation between $\varkappa(f;n)$ and $\mu(f;\delta)$.

Lemma 2. *Let f be a bounded function in* $[0, 1]$. *Then*

$$\varkappa(f;n)\leq\varkappa(f;4)+8\sum_{k=1}^{n}\mu(f;1/k). \tag{16}$$

Proof. First we prove that

$$\varkappa(f;4m)\leq\varkappa(f;2m)+4m\mu(f;1/m). \tag{17}$$

Let $p\geq 2m+1$ and $0\leq z_1\leq z_2\leq\cdots\leq z_p\leq 1$ are arbitrary points chosen in $[0, 1]$. It is evident, that there exist points z_{j-1}, z_j, z_{j+1}, so that

$$|z_{j-1}-z_{j+1}|\leq 1/m.$$

Consider now an arbitrary sum for $\varkappa(f;4m)$:

$\sigma=\sum_{s=1}^{4m}|f(x_s)-f(x_{s-1})|$, $0\leq x_0\leq x_1\leq\cdots\leq x_{4m}\leq 1$. From $4m+1\geq 2m+1$, taking into account the above statement it follows that there exist x_{j-1}, x_j, x_{j+1} so that $|x_{j-1}-x_{j+1}|\leq 1/m$. From the definition (15) of $\mu(f;\delta)$ we get

$$|f(x_j)-f(x_{j-1})|+|f(x_{j+1})-f(x_j)|\leq|f(x_{j+1})-f(x_{j-1})|+2\mu(f;1/m).$$

Therefore

$$\sigma\leq\{\sum_{s=1}^{j-1}|f(x_s)-f(x_{s-1})|+|f(x_{j+1})-f(x_{j-1})|+\sum_{s=j+2}^{4m}|f(x_s)-f(x_{s-1})|\}$$
$$+2\mu(f;1/m).$$

The expression on the right hand side is a sum for $\varkappa(f;4m-1)$. One can estimate this sum in a similar way, using a sum for $\varkappa(f;4m-2)$ and so on. The inequality (17) is obtained with $2m$ iterations of the above process.

Let n be an arbitrary integer and q such that $2^q\leq n\leq 2^{q+1}$. Setting $m=2^i$ in (17), we get

$$\varkappa(f;2^{i+2})\leq\varkappa(f;2^{i+1})+2^{i+2}\mu(f;1/2^i). \tag{18}$$

After adding the inequalities (18) for $i=2, 3,\ldots, q-1$ and taking into account that $\varkappa(f;n)$ and $\mu(f;\delta)$ are monotone functions, we get

$$\varkappa(f;n)\leq\varkappa(f;2^{q+1})\leq\varkappa(f;4)+8\sum_{j=1}^{q-1}2^{j-1}\mu(f;1/2^j)$$

$$\leq\varkappa(f;4)+8\sum_{k=2}^{2^{q-1}}\mu(f;1/k)\leq\varkappa(f;4)+8\sum_{k=1}^{n}\mu(f;1/k),$$

which proves lemma 2.

Using (7), (12), (13) and (16) we get from (6):

Theorem. *Consider the class of all functions f, which satisfy at least one of the conditions*

$$\varkappa(f;n)\leqq C\ln^s n,\ \omega(f;\delta)\leqq C\delta\ln^s\frac{1}{\delta},\ \nu_1(f;\delta)\leqq C\delta\ln^s\frac{1}{\delta},$$

$$E_n^0(f)\leqq C\frac{\ln^s n}{n},\ \mu(f;\delta)\leqq C\delta\ln^s\frac{1}{\delta},$$

where C and s are positive constants and $\sup_{0\leqq x\leqq 1}|f(x)|\leqq M$.

There exist constants $C_1(\alpha)$ *and* C_2, *depending on C, s, M so that for every function f of the above class*

$$R_n(f;\alpha)\leqq C_1(\alpha)\ln\ln n/n$$

and there exists a rational function $q_n(x)$ *of degree n such that*

$$|f(x)-q_n(x)|\leqq\omega(f;x,\ C_2\ln\ln n/n)+C_2/n,\ x\in[0,1].$$

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