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HERMITE FUNCTIONS OF SECOND KIND

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The system $\{K_n(z)\}_{n=0}^{\infty}$ of Hermite functions of second kind is defined as a second solution of the difference equation for Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$. We consider the asymptotic properties of Hermite functions of second kind, the convergence of the series $\sum_{n=0}^{\infty} b_n K_n(z)$ and expansions of analytic functions in series of Hermite functions of second kind.

1. Definition and some properties of Hermite polynomials. The system of Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$ is a system of polynomials orthogonal on the whole real axis $(-\infty, \infty)$ with respect to the weight function $\exp(-x^2)$. More precisely, the system $\{H_n(z)\}_{n=0}^{\infty}$ is uniquely determined by the conditions

$$(1.1)\qquad \int_{-\infty}^{\infty} \exp(-x^2)\, H_m(x)\, H_n(x)\, dx = \sqrt{\pi}2^n n!\delta_{mn},\quad m, n = 0, 1, 2, \ldots$$

provided that the coefficient of z^n in $H_n(z)$ is positive [1, (5.5.1)].

The system of Hermite polynomials can also be defined by means of a formula of Rodrigues type, namely [1, (5.5.3)]

$$(1.2)\qquad \exp(-z^2)H_n(z) = (-1)^n \{\exp(-z^2)\}^{(n)},\quad n = 0, 1, 2, \ldots$$

From (1.2) one can derive without difficulty that the polynomial $H_{2n}(z)$ is even, and $H_{2n+1}(z)$ is odd.

It is well-known that the system of Hermite polynomials is a solution of the difference equation

$$(1.3)\qquad y_{n+1} - 2zy_n + 2ny_{n-1} = 0.$$

One of the most convenient ways to prove this last property may be the following: denote by γ an arbitrary circumference with center at the point z and then using the integral formulas for the derivatives of an analytic function we get easily that for $n = 1, 2, 3, \ldots$

$$\exp(-z^2)\{H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z)\}$$
$$= (-1)^{n+1} n!(2\pi i)^{-1} \int_{\gamma} \frac{\partial}{\partial \zeta} \left\{ -\frac{\exp(-\zeta^2)}{(\zeta - z)^{n+1}} \right\} d\zeta = 0.$$

The asymptotic properties of Hermite polynomials have been investigated by many authors and with different methods. A full account of the results with the corresponding bibliographic references is given in [1,8.22, 8.23].

Put $\lambda_{2n}=\Gamma(2n+1)/\Gamma(n+1)$ and $\lambda_{2n+1}=(2n+1)^{-1/2}\Gamma(2n+1)/\Gamma(n+2)$. Then the asymptotic behaviour of $H_n(z)$ for $n\to+\infty$ and z bounded can be characterized by the following formula [1, Theorem (8.22.7)]

$$(1.4)\qquad H_n(z)=\lambda_n \exp(z^2/2)\{\cos[\sqrt{2n+1}\,.\,z-(n\pi)/2]\sum_{k=0}^{p-1}u_k(z)(2n+1)^{-k}$$
$$+\sin[\sqrt{2n+1}\,.\,z-(n\pi)/2]\sum_{k=0}^{p-1}v_k(z)(2n+1)^{-k-1/2}+h_{n,p}(z)\},$$

where $u_k(z)$, $v_k(z)$ $(k=0, 1, 2, \ldots)$ are polynomials depending only on k and $\{h_{n,p}(z)\}$, $n=0, 1, 2, \ldots$; $p=1, 2, 3, \ldots$ are entire functions such that $h_{n,p}(z) =O\{n^{-p}\exp(\sqrt{2n+1}\,.\,|\operatorname{Im} z|)\}$ uniformly on every compact subset K of the complex plane $\mathbb{C}$.

One can derive from (1.4) simpler asymptotic formulas for Hermite polynomials provided z is real or $z\in\mathbb{C}\setminus(-\infty, \infty)$. First of all from Stirlings formula it follows that $\lambda_n=\sqrt{2}(2n/e)^{n/2}[1+O(1/\sqrt{n})]$. Then putting $p=1$ in (1.4) and supposing that $z\in\mathbb{C}\setminus(-\infty, \infty)$ we get easily the following formula

$$(1.5)\qquad H_n(z)=\sqrt{2}\exp(z^2/2)(2n/e)^{n/2}\cos[\sqrt{2n+1}\,.\,z-(n\pi)/2][1+h_n(z)]$$

where $\{h_n(z)\}_{n=0}^{\infty}$ are complex functions analyitic in the open set $\mathbb{C}\setminus(-\infty, \infty)$ and such that $h_n(z)=O(1/\sqrt{n})$ uniformly on every compact set $K\subset\mathbb{C}\setminus(-\infty, \infty)$.

If $z=x$ is real, by the same argument we get the formula

$$(1.6)\qquad H_n(x)=\sqrt{2}\exp(x^2/2)(2n/e)^{n/2}\{\cos[\sqrt{2n+1}\,.\,x-(n\pi)/2]+g_n(x)\}$$

where $g_n(x)=O(1\sqrt{n})$ uniformly on every compact subset of the real axis $(-\infty, \infty)$.

Let us note that the formulas (1.5) and (1.6) are given by E. Hille in his paper [2].

2. Definition and elementary properties of Hermite functions of second kind. The system of Hermite functions of second kind $\{K_n(z)\}_{n=0}^{\infty}$ is defined by the equalities

$$(2.1)\qquad K_n(z)=-\int_{-\infty}^{\infty}\frac{\exp(-t^2)H_n(t)}{t-z}\,dt,\quad n=0, 1, 2,\ldots$$

provided that $z\in\mathbb{C}\setminus(-\infty, \infty)$. It is clear that for every $n=0, 1, 2, \ldots, K_n(z)$ is a complex function analytic in the open set $\mathbb{C}\setminus(-\infty, \infty)$.

Using Rodrigues formula (1.2) one can derive from (2.1) after integration by parts another integral representation for Hermite functions of second kind namely

$$(2.2)\qquad K_n(z)=(-1)^{n+1}n!\int_{-\infty}^{\infty}\frac{\exp(-t^2)}{(t-z)^{n+1}}\,dt.$$

As it was mentioned above, the system $\{K_n(z)\}_{n=0}^{\infty}$ is a (second) solution of the difference equation (1.3). A proof of this fact can be given following the same idea we used for Hermite polynomials. Indeed, for $n=1, 2, 3, \ldots$ from (2.2) we get after some algebra that

$$K_{n+1}(z)-2zK_n(z)+2nK_{n-1}(z)=(-1)^{n+1}n!\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left\{\frac{\exp(-t^2)}{(t-z)^{n+1}}\right\}dt=0.$$

In view of future applications we shall give another integral representation of Hermite functions of second kind based on the relation between Hermite and Laguerre polynomials. The Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with parameter α can be defined by the corresponding Rodrigues formula [1, (5.1.5)].

(2.3) $$n!\, z^{\alpha} \exp(-z) L_n^{(\alpha)}(z) = \{z^{n+\alpha} \exp(-z)\}^{(n)} \quad (n=0, 1, 2, \ldots).$$

Then, using the fact that $H_{2n}(z)$ is an even polynomial, the relation $H_{2n}(z)=(-1)^n 2^{2n} n!\, L_n^{(-1/2)}(z^2)$ [1, (5.6.1)] and the formula (2.3) for $\alpha=-1/2$, we get

$$K_{2n}(z) = -2z \int_0^{\infty} \frac{\exp(-t^2) H_{2n}(t)}{t^2-z^2}\, dt = (-1)^{n+1} 2^{2n+1} n!\, z \int_0^{\infty} \frac{\exp(-t^2) L_n^{(-1/2)}(t^2)}{t^2-z^2}\, dt$$

$$=(-1)^{n+1} 2^{2n} n!\, z \int_0^{\infty} \frac{t^{-1/2} \exp(-t) L_n^{(-1/2)}(t)}{t-z^2}\, dt = (-1)^{n+1} 2^{2n} z \int_0^{\infty} \frac{\{t^{n-1/2} \exp(-t)\}^{(n)}}{t-z^2}\, dt.$$

After an integration by parts it follows that

(2.4) $$K_{2n}(z) = (-1)^{n+1} 2^{2n} n!\, z \int_0^{\infty} \frac{t^{n-1/2} \exp(-t)}{(t-z^2)^{n+1}}\, dt.$$

In similar way using the relation $H_{2n+1}(z)=(-1)^n 2^{2n+1} n!\, z\, L_n^{(1/2)}(z^2)$ [1, (5.6.1)], we can derive the representation

(2.5) $$K_{2n+1}(z) = (-1)^{n+1} 2^{2n+1} n! \int_0^{\infty} \frac{t^{n+1/2} \exp(-t)}{(t-z^2)^{n+1}}\, dt.$$

3. Asymptotic formulas for Hermite functions of second kind. In this section following the idea of Laplace's method we derive asymptotic formulas for Hermite functions of second kind $\{K_n(z)\}_{n=0}^{\infty}$ for $n \to +\infty$ and z belonging to an arbitrary compact subset of $\mathbb{C} \setminus (-\infty, \infty)$.

Let A be a compact subset of the upper half-plane $H^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For $z \in H^+$ and $n=0, 1, 2, \ldots$ we define

(3.1) $$\tau_n(z) = -i\sqrt{(n+1)/2}\, [1 - z^2/(2n+2)]^{1/2} + z/2.$$

The function $\tau_n(z)$ satisfies the equation

(3.2) $$2t^2 - 2zt + (n+1) = 0,$$

which is equivalent to the equation $(\partial/\partial t)\{\exp(-t^2)/(t-z)^{-n-1}\} = 0$.

For $\tau_n(z)$ holds the following asymptotic formula

(3.3) $$\tau_n(z) = -i\sqrt{(n+1)/2} + z/2 + iz^2/\sqrt{32(n+1)} + O(1/n\sqrt{n}),$$

uniformly on every compact subset of H^+ and in particular on the set A.

From (3.3) it follows that $\operatorname{Im} \tau_n(z) < 0$ for every sufficiently large n and every $z \in A$. Then from Cauchy's integral theorem we get that for every $z \in A$ and every sufficiently large n holds the equality

(3.4) $$K_n(z) = (-1)^{n+1} n! \int_{l_n(z)} \frac{\exp(-\zeta^2)}{(\zeta - z)^{n+1}}\, d\zeta,$$

where $l_n(z)$ is the straight line parallel to the real axis and passing through the point $\tau_n(z)$. Putting $\zeta = t + \tau_n(z)\, (-\infty < t < \infty)$ in (3.4) we get that

$$K_n(z)=(-1)^{n+1}n!\frac{\exp[-\tau_n^2(z)]}{[\tau_n(z)-z]^{n+1}}\int_{-\infty}^{\infty}\frac{\exp[-2\tau_n(z).t-t^2]}{\{1+t/[\tau_n(z)-z]\}^{n+1}}dt$$

If we define

$$I_n(z)=\int_{-\infty}^{\infty}\frac{\exp[-2\tau_n(z).t-t^2]}{\{1+t/[\tau_n(z)-z]\}^{n+2}}dt, \tag{3.5}$$

then

$$K_n(z)=(-1)^{n+1}n!\frac{\exp[-\tau_n^2(z)]}{[\tau_n(z)-z]^{n+1}}I_n(z). \tag{3.6}$$

From (3.3) it follows that

$$\tau_n^2(z)=-(n+1)/2-iz/\sqrt{(n+1)/2}+z^2/2+O(1/\sqrt{n}). \tag{3.7}$$

Now we shall establish the asymptotic formula

$$[\tau_n(z)-z]^{n+1}=(-i)^{n+1}[(n+1)/2]^{(n+1)/2}\exp[-iz\sqrt{(n+1)/2}][1+O(1/\sqrt{n})]. \tag{3.8}$$

Indeed

$$[\tau_n(z)-z]^{n+1}=(-i\sqrt{(n+1)/2})^{n+1}[1-iz(\sqrt{2n+2}-z^2/4(n+1)+O(1/n^2)]^{n+1}$$

and having in view that

$$(n+1)\log[1-iz/\sqrt{2n+2}-z^2/4(n+1)+O(1/n^2)]=(n+1)\{-iz/\sqrt{2n+2}$$

$$-z^2/4(n+1)+O(1/n^2)-\frac{1}{2}[-iz/\sqrt{2n+2}-z^2/4(n+1)O(1/n^2)]^2$$

$$+O(1/n\sqrt{n})\}=-iz\sqrt{(n+1)/2}+O(1/\sqrt{n})$$

we get (3.8).

From (3.7) and (3.8) follows that

$$\exp[-\tau_n^2(z)].[\tau_n(z)-z]^{-n-1} \tag{3.9}$$

$$=i^{n+1}[(n+1)/2]^{-(n+1)/2}\exp[(n+1)/2\quad -z^2/2+iz\sqrt{2n+2}][1+O(1/\sqrt{n})].$$

Using (3.9), (3.6) and Stirling's formula we can write

$$K_n(z)=(-i)^{n+1}2\sqrt{\pi}(2n/e)^{n/2}\exp[-z^2/2+iz\sqrt{2n+2}][1+O(1/\sqrt{n})]I_n(z).$$

But $\exp(iz\sqrt{2n+2})=\exp(iz\sqrt{2n+1})[1+O(1/\sqrt{n})]$, therefore

$$K_n(z)=(-i)^{n+1}2\sqrt{\pi}(2n/e)^{n/2}\exp[-z^2/2+iz\sqrt{2n+1}][1+O(1/\sqrt{n})]I_n(z). \tag{3.10}$$

It remains to investigate the asymptotic behaviour of $I_n(z)$ if $n\to+\infty$ and $z\in A$. We shall see that

$$\lim_{n\to+\infty}I_n(z/)=\sqrt{\pi/2} \tag{3.11}$$

uniformly on A.

We mentioned that the function $\tau_n(z)$ satisfies the equation (3.2). Having this fact in view we get that

$$I_n(z)=\int_{-\infty}^{\infty}\frac{\exp[-2\tau_n(z).t-t^2]}{\{1+2\tau_n(z)t/[2\tau_n^2(z)-2z\tau_n(z)]\}^{n+1}}dt \tag{3.12}$$

$$=\int_{-\infty}^{\infty}\frac{\exp[-2\tau_n(z).t-t^2]}{[1-2\tau_n(z)t/(n+1)]^{n+1}}dt=\int_{-\infty}^{\infty}\lambda_n(t,z)\exp(-t^2)\,dt,$$

where

(3.13) $$\lambda_n(t,z)=\exp[-2\tau_n(z).t][1-2\tau_n(z)t/(n+1)]^{-n-1}.$$

We shall prove that for every $T>0$,

(3.14) $$\lim_{n\to+\infty}\lambda_n(t,z)=\exp(-t^2)$$

uniformly for $(t,z)\in[-T,T]\times A$. Indeed,

$$\log\lambda_n(t,z)=-2\tau_n(z).t-(n+1)\log\{1-\frac{2\tau_n(z).t}{n+1}\}$$

$$=-2\tau_n(z).t-(n+1)\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu}\{-\frac{2\tau_n(z).t}{n+1}\}^\nu=\frac{2t^2\tau_n^2(z)}{n+1}+\sum_{\nu=3}^{\infty}\frac{2\,t^\nu}{\nu}\{\frac{\tau_n(z)}{n+1}\}^\nu$$

and taking into consideratation the asymptotic formulas (3.1) and (3.7) we get that $\log\lambda_n(t,z)=-t^2+O(1/\sqrt{n})$ uniformly for $(t,z)\in[-T,T]\times A$.

Since A is a compact set, there exists a positive a, such that $|\operatorname{Re} z|=|\operatorname{Re}(x+iy)|=|x|\leq a$ for $z\in A$. Now we shall prove that for any $\delta>0$ there exists a $N=N(\delta)$ such that

(3.15) $$|\lambda_n(t,z)|\leq(1+\delta)\exp(2a|t|)$$

for every $n>N$ and every $(t,z)\in(-\infty,\infty)\times A$.

If $-4a\leq t\leq 4a$, then from (3.14) it follows that the inequality (3.15) is satisfied for every sufficiently large n and $(t,z)\in[-4a,4a]\times A$. Let us suppose that $|t|\geq 4a$ and put $2\tau_n(z)=\xi_n(z)+i\eta_n(z)$ i. e. $\xi_n(z)=2\operatorname{Re}\tau_n(z)$ and $\eta_n(z)=2\operatorname{Im}\tau_n(z)$. From the asymptotic formula (3.3) we get that $\xi_n(z)=\operatorname{Re} z+O(1/\sqrt{n})$ and $\eta_n(z)=-\sqrt{2n+2}+\operatorname{Im} z+O(1/\sqrt{n})$, uniformly on A. Therefore, $|\xi_n(z)|\geq 2a$ and $|\eta_n(z)|\geq\sqrt{(n+1)/2}$ for every sufficiently large n and every $z\in A$. Then, $|\exp[-2\tau_n(z)t]|=|\exp[-\xi_n(z)t]|\leq\exp(|\xi_n(z)t|)\leq\exp(2a|t|)$ and

$$|1-\frac{2\tau_n(z)t}{n+1}|^2=\{1-\frac{\xi_n(z)t}{n+1}\}^2+\{\frac{\eta_n(z)t}{n+1}\}^2=1-\frac{2\xi_n(z)t}{n+1}+\frac{\xi_n^2(z)t^2}{(n+1)^2}+\frac{\eta_n{}^2(z)t^2}{(n+1)^2}$$

$$\geq 1-\frac{2a|t|}{n+1}+\frac{t^2}{2(n+1)}=1+\frac{|t|(|t|-4a)}{n+1}\geq 1,$$

i. e. in this case ($|t|\geq 4a$) we have the inequality $|\lambda_n(t,z)|\leq\exp(2a|t|)$ for every sufficiently large n and every $z\in A$.

In view of (3.12), (3.14) and (3.15) it is not difficult to prove that

$$\lim_{n\to+\infty}I_n(z)=\int_{-\infty}^{\infty}\exp(-2t^2)dt=\sqrt{\pi/2},$$

uniformly with respect to $z\in A$. Then from (3.12) we get finally the asymptotic formula for Hermite functions of secone kind in the upper half-plane H^+ namely

(3.16) $$K_n(z)=(-i)^{n+1}\pi\sqrt{2}(2n/e)^{n/2}\exp[-z^2/2+iz\sqrt{2n+1}][1+k_n(z)],$$

where $\{k_n(z)\}_{n=0}^{\infty}$ are complex functions analytic in the half-plane H^+ and $\lim_{n\to+\infty}k_n(z)=0$ uniformly on every compact set $A\subset H^+$.

In order to get an asymptotic formula for the functions $\{K_n(z)\}_{n=0}^{\infty}$ in the lower half-plane $H^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ we make use of the relation $K_n(z) = \overline{K_n(\bar{z})}$, $n = 0, 1, 2, \dots$, which is satisfied for every $z \in \mathbb{C} \setminus (-\infty, \infty)$. In particular, if $\operatorname{Im} z < 0$ from (3.16) and this last relation follows that

$$K_n(z) = i^{n+1} \pi\sqrt{2}(2n/e)^{n/2} \exp[-z^2/2 - iz\sqrt{2n+1}][1 + k_n^*(z)] \tag{3.17}$$

where $k_n^*(z) = \overline{k_n(\bar{z})}$. Therefore, $\{k_n^*(z)\}_{n=0}^{\infty}$ are complex functions analytic in the half-plane H^- and $\lim_{n\to\infty} k_n^*(z) = 0$ uniformly on every compact subset of H^-. In other words, (3.17) is an asymptotic formula for Hermite functions of second kind in the half-plane $\operatorname{Im} z < 0$.

4. Convergence of series in Hermite functions of second kind. The asymptotic formulas for Hermite functions of second kind can be used to give a full solution of the problem for the region and the mode of convergence of a series of the kind

$$\sum_{n=0}^{\infty} b_n K_n(z) \tag{4.1}$$

with arbitrary complex coefficients. The proofs of the results are in principle the same as the proofs of the classic Abel — Cauchy — Hadamard theorems for power series. That is why we shall confine us only to the formulation of the corresponding statements.

First of all, if τ is an arbitrary real number, with $H^+(\tau)$ we denote the half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > \tau\}$ and with $H^-(\tau)$ the half-plane $\{z \in \mathbb{C} : \operatorname{Im} z < \tau\}$. In particular $H^+(0) = H^+$ and $H^-(0) = H^-$ are respectively the upper and the lower half-planes.

Theorem 1 (Abel). *If the series (4.1) is convergent at a point $z_0 \in \mathbb{C} \setminus (-\infty, \infty)$, it is absolutely uniformly convergent on every compact set $A \subset H^+(\tau_0) \cup H^-(-\tau_0)$ where $\tau_0 = |\operatorname{Im} z|$.*

Theorem 2. (Cauchy — Hadamard). *Let $\{b_n\}_{n=0}^{\infty}$ be an arbitrary sequence of complex numbers and*

$$\tau_0 = \max\{0, \overline{\lim}_{n\to+\infty} (2n+1)^{-1/2} \ln |(2n/e)^{n/2} b_n|\}$$

Then: (a) if $\tau_0 = +\infty$, the series (4.1) is divergent for every $z \in \mathbb{C} \setminus (-\infty, \infty)$; (b) if $0 < \tau_0 < +\infty$, the series (4.1) is absolutely uniformly convergent on every compact set $A \subset H^+(\tau_0) \cup H^-(-\tau_0)$ and diverges at every point z such that $|\operatorname{Im} z| < \tau_0$; (c) if $\tau_0 = 0$, the series (4.1) is absolutely uniformly convergent on every compact set $A \subset H^+ \cup H^- = \mathbb{C} \setminus (-\infty, \infty)$.

Remark. We say that a series $\sum_{n=0}^{\infty} f_n(z)$ of complex functions is absolutely uniformly convergent on a set $E \subset \mathbb{C}$, if the series $\sum_{n=0}^{\infty} |f_n(z)|$ is uniformly convergent on E.

5. General solution of the difference equation for Hermite polynomials. Formula of Christoffel — Darboux. As a next application of the asymptotic formulas for Hermite functions of second kind and also for Hermite polynomials, we shall prove that the general solution of the difference equation (1.3) has the form

(5.1) $$\{aH_n(z)+bK_n(z)\}_{n=0}^{\infty}$$

where a, b are arbitrary complex constants. In other words we have to show that the solutions $\{H_n(z)\}_{n=0}^{\infty}$ and $\{K_n(z)\}_{n=0}^{\infty}$ of the equation (1.3) are linearly independant for every $z \in \mathbb{C} \setminus (-\infty, \infty)$.

Let us suppose that for some $z \in H^+$ and for every sufficiently large n holds the equality

(5.2) $$aH_n(z)+bK_n(z)=0.$$

Using the asymptotic formulas (1.5) and (3.16) for Hermite polynomials and Hermite functions of second kind, from (5.2) we get that

$$a\sqrt{2}\exp(z^2/2)\cos[\sqrt{2n+1}\,.\,z-(n\pi)/2][1+h_n(z)]$$
$$+b(-i)^{n+1}\pi\sqrt{2}\exp[-z^2/2+iz\sqrt{2n+1}][1+k_n(z)]=0.$$

Multiplying the last equality by $(-i)^n\exp[z^2/2+iz\sqrt{2n+1}]$, we can write it in the following form

(5.3) $$a\exp z^2\,.\,\{\exp[2iz\sqrt{2n+1}-n\pi i]+1\}[1+h_n(z)]$$
$$+b(-i)^{2n+1}\pi\sqrt{2}\exp(2iz\sqrt{2n+1})[1+k_n(z)]=0.$$

Since $\operatorname{Im} z>0$, $\lim_{n\to\infty}\exp(2iz\sqrt{2n+1})=0$ and moreover $\lim_{n\to\infty}h_n(z)=0$, $\lim_{n\to\infty}k_n(z)=0$. Then from (5.3) follows that $a\exp z^2=0$ and therefore $a=0$. From the asymptotic formula (3.16) we can conclude that $K_n(z)\neq 0$ for every sufficiently large n and then from (5.2) we get that $b=0$.

Let the sequence $\{I_n\}_{n=0}^{\infty}$ be defined by the equalities

(5.4) $$I_n=\int_{-\infty}^{\infty}\exp(-x^2)\{H_n(x)\}^2dx \quad (n=0, 1, 2, \ldots).$$

Then from (1.1) we have

(5.5) $$I_n=\sqrt{\pi}\,2^n\,n! \quad (n=0, 1, 2, \ldots).$$

Dividing the difference equation (1.3) by $2I_n$, we can write it in the following canonical form

(5.6) $$k_n\,y_{n+1}-\frac{z}{I_n}\,y_n+k_{n-1}\,y_{n-1}=0$$

where $k_n=(2I_n)^{-1}$. From (5.6) we derive in the usual way the corresponding formula of Christoffel — Darboux type, namely

(5.7) $$\frac{1}{\zeta-z}=\sum_{n=0}^{\nu}\frac{1}{I_\nu}H_n(z)\,K_n(\zeta)+\frac{\Delta_{\nu+1}(z,\zeta)}{\zeta-z}$$

where

(5.8) $$\Delta_{\nu+1}(z,\zeta)=k_\nu\{H_\nu(z)K_{\nu+1}(\zeta)-H_{\nu+1}(z)K_\nu(\zeta)\}$$
$$=\frac{1}{2I_\nu}\{H_\nu(z)K_{\nu+1}(\zeta)-H_{\nu+1}(z)K_\nu(\zeta)\}.$$

6. A necessary condition for an analytic function to be represented by a series in Hermite functions of second kind. One of the most important problems connected with Hermite functions of second kind $\{K_n(z)\}_{n=0}^{\infty}$ is the problem of expansion of analytic functions in series of these functions. We shall see now that not every analytic function $f(z)$ can be represented by a series of the kind

$$f(z)=\sum_{n=0}^{\infty} b_n K_n(z). \tag{6.1}$$

Theorem 3. *Let* $0\leq\tau_0<+\infty$ *and* $f(z)$ *be a complex function analytic in the half-plane* $H^+(\tau_0)$. *If* $f(z)$ *is represented in* $H^+(\tau_0)$ *by the series (6.1), then for every positive* ε, $f(z)=O(|z|)$ *for* $z\to\infty$ *in the region defined by the inequalities* $\operatorname{Re} z^2<-(\tau_0+\varepsilon)^2$ *and* $\operatorname{Im} z>\tau_0$.

Proof. From Theorem 2 it follows that $\overline{\lim}_{n\to\infty}(2n+1)^{-1/2}\ln|(2n/e)^{n/2}b_n|\leq\tau_0$. Therefore, if ε is positive, there exists a $B=B(\varepsilon)$ such that

$$|b_n|\leq B(e/2n)^{n/2}\exp[(\tau_0+\varepsilon/2)\sqrt{2n+1}] \tag{6.2}$$

for $n=0, 1, 2, \ldots$

Let $z^2=\xi+i\eta$, $\operatorname{Im} z>\tau_0$ and $\operatorname{Re} z^2<-(\tau_0+\varepsilon)^2$, then from the integral representation (2.4) we get that

$$|K_{2n}(z)|\leq 2^{2n}n!\,|z|\int_0^\infty \frac{t^{n-1/2}\exp(-t)}{[(t-\xi)^2+\eta^2]^{(n+1)/2}}dt\leq 2^{2n}n!\,|z|\int_0^\infty \frac{t^{n-1/2}\exp(-t)}{(t-\xi)^{n+1}}dt$$

$$\leq 2^{2n}n!\,|z|\int_0^\infty \frac{t^{n-1/2}\exp(-t)}{\{t-[i(\tau_0+\varepsilon)]^2\}^{n+1}}dt.$$

Having again (2.4) in view we can write

$$|K_{2n}(z)|\leq(-1)^{n+1}|z|[i(\tau_0+\varepsilon)]^{-1}K_{2n}[i(\tau_0+\varepsilon)].$$

Then from the asymptotic formula (3.16) for Hermite functions of second kind follows that

$$|K_{2n}(z)|\leq L|z|(4n/e)^n\exp[-(\tau_0+\varepsilon)\sqrt{4n+1}] \tag{6.3}$$

where L is a constant.

On the same way we get the corresponding inequality for the functions $K_{2n+1}(z)$ namely

$$|K_{2n+1}(z)|\leq M[(4n+2)/e]^{n+1/2}\exp[-(\tau_0+\varepsilon)\sqrt{4n+3}] \tag{6.4}$$

where M is a constant.

Using (6.1), (6.2), (6.3) and (6.4) we can write that

$$|f(z)|\leq\sum_{n=0}^{\infty}|b_{2n}|\cdot|K_{2n}(z)|+\sum_{n=0}^{\infty}|b_{2n+1}|\,|K_{2n+1}(z)|$$

$$\leq BL|z|\sum_{n=0}^{\infty}\exp[-(\varepsilon/2)\sqrt{4n+1}]+BM\sum_{n=0}^{\infty}\exp[-(\varepsilon/2)\sqrt{4n+3}]=O(|z|)$$

and thus Theorem 3 is proved.

7. Inequalities for Hermite polynomials. The asymptotic formula (1.4) gives the behaviour of $H_n(z)$ as a function of n provided that z belongs to a compact subset of the complex plane. It is important to study the asymptotic properties of $H_n(z)$ as a function of both variables n and z. Some results have been established in this direction under the condition that between n and z exists a suitable relation [1, Theorem 8.22.9].

In the general case if the variables n and z are independant, the solution of the problem of the asymptotic behaviour of $H_n(z)$ is much more difficult. Instead of asymptotic formulas in this case it is possible only to get some inequalities for Hermite polynomials. An example of such an inequality is the following one

$$H_n(x)=O\{\exp(x^2/2)n^{1/6}(2n/e)^{n/2}\} \quad (-\infty<x<\infty), \tag{7.1}$$

which can be obtained from the asymptotic formula for the sequence of the maximums of the functions $\{\exp(-x^2)|H_n(x)|^2\}$ on the interval $(-\infty, \infty)$ [1, (8.91.10)].

In view of future applications we shall establish here an inequality for Hermite polynomials in special domains of the complex plane. This inequality can be regarded as an O-asymptotic formula for $H_n(z)$ as a function of n and z.

Theorem 4. *For every $\tau>0$ there exists a positive constant $B(\tau)$ such that*

$$|(e/2n)^{n/2}\exp[-z^2-\tau\sqrt{2n+1}]H_n(z)|\leq B(\tau) \tag{7.2}$$

for every $n=0, 1, 2, \ldots$ and arbitrary z with $|\operatorname{Im} z|\leq\tau$.

Proof. If n is an even positive integer, we have to show that the sequence of functions defined as follows

$$E_n(z)=(e/4n)^n\exp(-z^2-\tau\sqrt{4n+1})H_{2n}(z) \quad (n=0, 1, 2, \ldots) \tag{7.3}$$

is uniformly bounded on the closed strip $\overline{S(\tau)}=\{z\in\mathbb{C}:|\operatorname{Im} z|\leq\tau\}$.

We shall use ane integral representation of Hermite polynomials with even index, namely [1, (5.6.4)]

$$H_{2n}(z)=(-1)^n\pi^{-1/2}2^{2n+1}\exp z^2 . \int_0^\infty \exp(-t^2) . t^{2n}\cos(2zt)\,dt.$$

Having in view this representation, from (7.3) we get that

$$E_n(z)=2(-1)^n\pi^{-1/2}(e/n)^n\exp(-\tau\sqrt{4n+1})\int_0^\infty \exp(-t^2)t^{2n}\cos(2zt)\,dt.$$

It is not difficult to show that the inequality $|\cos(2zt)|\leq\exp(2\tau t)$ holds for every $z\in\overline{S(\tau)}$ and $t\in[0_1, +\infty)$. Therefore

$$E_n(z)=O\{(e/n)^n\exp(-\tau\sqrt{4n+1})\int_0^\infty \exp(-t^2+2\tau t) . t^{2n}\,dt\}$$

$$=O\{2^{-n}(e/n)^n\exp(-\tau\sqrt{4n+1})\int_0^\infty \exp(-t^2/2+\tau\sqrt{2} . t)t^{2n}\,dt\}.$$

Let $D_\nu(z)$ be the parabolic cylinder function with parameter ν[3, p. 117, (4)]. Taking into consideration the integral representation [3, p. 119, (3)] of $D_\nu(z)$, we can write that

$$E_n(z)=O\{2^{-n}(e/n^n\exp(-\tau\sqrt{4n+1})\Gamma(2n+1)D_{-(2n+1)}(-\tau\sqrt{2})\}.$$

Then using Stirling's formula and the asymptotic formula [4, p. 123, (5)] for the function $D_\nu(z)$, we get finally

$$E_n(z)=O\{\exp[\tau(\sqrt{4n+2}-\sqrt{4n+1})]\}=O(1).$$

In the case of odd n we use the corresponding integral representation of $H_{2n+1}(z)$, namely [1, (5.6.4)]

$$H_{2n+1}(z)=(-1)^n \pi^{-1/2} 2^{2n+2} \exp z^2 \int_0^\infty \exp(-t^2) t^{2n+1} \sin(2zt)\, dt.$$

8. Expansion of analytic functions defined by integrals of Cauchy type in series of Hermite functions of second kind. The formula (5.7) of Christoffel — Darboux gives us a (finite) expansion of the Cauchy kernel $(\zeta-z)^{-1}$ in terms of Hermite polynomials and Hermite functions of second kind. Therefore one can expect that analytic functions defined by integrals of Cauchy type can be represented by series in Hermite functions of second kind. It is our purpose in this section to show that this is really the fact and as a first example of this kind we shall prove the following

Theorem 5. *Let $F(t)$ be a complex function defined and measurable on the interval $(-\infty, \infty)$ and satisfy the following conditions:*

(a) $\int_{-T}^{T} |F(t)|\, dt < +\infty$ *for every* $T>0$;

(b) $F(t)=O[|t|^{-\alpha} \exp(-t^2/2)]$ *if* $|t| \to +\infty$, for some $\alpha>1$.

Then the function

$$f(z)=\int_{-\infty}^{\infty} \frac{F(t)}{z-t}\, dt \tag{8.1}$$

can be represented in the "region" $\mathbb{C}\setminus(-\infty, \infty)$ *by a series of the kind (6.1) with coefficients*

$$b_n=\frac{1}{I_n} \int_{-\infty}^{\infty} H_n(t)\, F(t)\, dt \quad (n=0, 1, 2, \ldots). \tag{8.2}$$

Proof. From the condition (a) and the asymptotic formula (1.6) follows that for every $T>0$,

$$\int_{|t|\leq T} |H_n(t)\, F(t)|\, dt = O[(2n/e)^{n/2}].$$

Using the inequality (7.1) we get that for every $T>0$,

$$\int_{|t|\geq T} |H_n(t)\, F(t)|\, dt = O[n^{1/6}(2n/e)^{n/2} \int_{|t|\geq T} |t|^{-\alpha}\, dt] = O[n^{1/6}(2n/e)^{n/2}].$$

Therefore,

$$\int_{-\infty}^{\infty} |H_n(t)\, F(t)|\, dt = O[n^{1/6}(2n/e)^{n/2}] \tag{8.3}$$

for $n\to+\infty$.

In the formula (5.7) of Christoffel — Darboux we replace ζ with z, z with t, multiply with $F(t)$, integrate over the interval $(-\infty, \infty)$ and get that for every $z \in \mathbb{C}\setminus(-\infty, \infty)$

$$f(z)=\sum_{n=0}^{\nu} b_n K_n(z)+\int_{-\infty}^{\infty} \frac{A_{\nu+1}(t, z) F(t)}{z-t}\, dt = \sum_{n=0}^{\nu} b_n K_n(z) \tag{8.4}$$

$$+\frac{1}{2I_\nu}\int_{-\infty}^{\infty}\{H_\nu(t)\,K_{\nu+1}(z)-H_{\nu+1}(t)\,K_\nu(z)\}\frac{F(t)}{z-t}\,dt,$$

where the coefficients b_n, $n=0,1,2,\ldots$ are given by the equalities (8.2).

From (5.5), (8.3), the asymptotic formulas (3.16), (3.17) and Stirling's formula we get that

$$\frac{1}{2I_\nu}\int_{-\infty}^{\infty}\{H_\nu(t)K_{\nu+1}(z)-H_{\nu+1}(t)\,K_\nu(z)\}\frac{F(t)}{z-t}\,dt$$

$$=O\{(2^\nu\nu!)^{-1}[\nu^{1/6}(2\nu/e)^{\nu/2}((2\nu+2)/e)^{(\nu+1)/2}\exp(-|\operatorname{Im} z|\sqrt{2\nu+3})$$

$$+(\nu+1)^{1/6}((2\nu+2)/e^{(\nu+1/2}(2\nu/e)^{\nu/2}\exp(-|\operatorname{Im} z|\sqrt{2\nu+1})]\}$$

$$=O[\nu^{1/6}\exp(-|\operatorname{Im} z|\sqrt{2\nu+1})].$$

Then from (8.4) follows that the series on the right side of (6.1) with coefficients given by the equalities (8.2), converges for every $z\in\mathbb{C}-(-\infty,\infty)$ and its sum is the function defined by (8.1).

Theorem 6. *Let $0<\tau_0<+\infty$ and $F(w)$ be a complex function defined and measurable on the strip $S(\tau_0)$ and satisfy the conditions:*

(a) $\iint_{S(T,\,\tau_0)}|F(w)|\,du\,dv<+\infty$

for every $T>0$, where $S(T,\tau_0)=\{z\in\mathbb{C}:\operatorname{Re} z<T,\ |\operatorname{Im} z|<\tau_0\}$.

(b) $F(w)=O[|w|^{-\alpha}\exp(-w^2)]$ if $|w|\to+\infty$, for some $\alpha>1$.

Then the function

$$f(z)=\iint_{S(\tau_0)}\frac{F(w)}{w-z}\,du\,dv \tag{8.5}$$

can be represented in the "region" $\mathbb{C}\setminus\overline{S(\tau_0)}$ by a series of the kind (6.1) with coefficients

$$b_n=\frac{1}{I_n}\iint_{S(\tau_0)}H_n(w)F(w)\,du\,dv,\quad n=0,1,2,\ldots \tag{8,6}$$

Proof. From condition (a) and the asymptotic formula (1.4) it follows that for every $T>0$

$$\iint_{S(T,T_0)}|H_n(w)\,F(w)|\,du\,dv=O[(2n/e)^{n/2}\exp(\tau_0\sqrt{2n+1})].$$

Using the inequality (7.2) and the condition (b) we get that for every $T>0$

$$\iint_{R(T,\tau_0)}|H_n(w)\,F(w)|\,du\,dv=O\{(2n/e)^{n/2}\exp(\tau_0\sqrt{2n+1})\iint_{R(T,\tau_0)}|w|^{-\alpha}\,du\,dv\}$$

$$=O\{(2n/e^{n/2}\exp(\tau_0\sqrt{2n+1})\},$$

where $R(T,\tau_0)=S(\tau_0)-S(T,\tau_0)$. Therefore

$$\iint_{S(\tau_0)}|H_n(w)\,F(w)|\,du\,dv=O\{(2n/e)^{n/2}\exp(\tau_0\sqrt{2n+1})\} \tag{8.7}$$

for $n\to+\infty$.

In the formula (5.7) of Christoffel — Darboux we replace ζ with z, z with w, multiply with $F(w)$, integrate over the strip $S(\tau_0)$ and get that for every $z\in\mathbb{C}\setminus\overline{S(\tau_0)}$ holds the equality

$$f(z)=\sum_{n=0}^{\nu} b_n K_n(z)+\iint_{S(\tau_0)}\frac{\Lambda_{\nu+1}(w,z)F(w)}{z-w}\,du\,dv \tag{8,8}$$

where the coefficients b_n, $n=0, 1, 2, \ldots$ are given by (8.6). From (5.8), (5,5), (8.7), the asymptotic formulas (3.16), (3.17) and Stirling's formula we get that for every $z\in\mathbb{C}\setminus\overline{S(\tau_0)}$

$$\iint_{S(\tau_0)}\frac{\Lambda_{\nu+1}(w,z)F(w)}{z-w}\,du\,dv=O\{\exp[-(|\operatorname{Im} z|-\tau_0)\sqrt{2\nu+1})]\}$$

and using (8.8) we finish the proof of the Theorem.

9. A general theorem for representation of an analytic function by a series in Hermite functions of second kind. From the formula (5.7) of Christoffel — Darboux also arises the problem that under some conditions it is possible to express the coefficients of the series (6.1) in terms of Hermite polynomials and the analytic function $f(z)$. Bevore giving a result of this kind, we shall prove the following simple

Lemma. *Let $0\leq\tau_0<+\infty$ and $f(z)$ be a complex function satisfying the following conditions:*

(a) *$f(z)$ is analytic in the half-plane $H^+(\tau_0)$;*

(b) *for every τ, $\tau_0<\tau<+\infty$, there exists a $\delta(\tau)>0$ such that $f(z)=O(|z|^{-\delta(\tau)})$ if $z\to\infty$ in the half-plane $\overline{H^+(\tau)}$.*

Then, for every compact set $K\subset H^+(\tau_0)$ and every τ such that $\tau_0<\tau<+\infty$ and $K\subset H^+(\tau)$, uniformly on $z\in K$ holds the equality

$$f(z)=\frac{1}{2\pi i}\int_{l(\tau)}\frac{f(\zeta)}{\zeta-z}\,d\zeta \tag{9.1}$$

where $l(\tau)=\partial H^+(\tau)=\{z\in\mathbb{C}: z=t+i\tau,\ -\infty<t<+\infty\}$.

Proof. Let $R>0$ and denote with $l(R,\tau)$ the segment $[-R+i\tau, R+i\tau]$ and $\gamma(R,\tau)$ be the half-circle $z=i\tau+R\exp i\theta(0\leq\theta\leq\pi)$. If τ, $\tau_0<\tau<+\infty$, is chosen so that $K\subset H^+(\tau)$, for every sufficiently large R holds the equality $(z\in K)$

$$f(z)=\frac{1}{2\pi i}\int_{l(R,\tau)}\frac{f(\zeta)}{\zeta-z}\,d\zeta+\frac{1}{2\pi i}\int_{\gamma(R,\tau)}\frac{f(\zeta)}{\zeta-z}\,d\zeta. \tag{9.2}$$

But from the condition (b) follows that for $R\to+\infty$

$$\int_{\gamma(R,\tau)}\frac{f(\zeta)}{\zeta-z}\,d\zeta=O(R^{-\delta(\tau)})$$

uniformly on $z\in K$ and from (9.2) we get the representation (9.1).

Theorem 7. *Let $0\leq\tau_0<+\infty$ and $f(z)$ be a complex function satisfying the following conditions:*

(a) $f(z)$ is analytic in the half-plane $H^+(\tau_0)$;

(b) for every τ, $\tau_0<\tau<+\infty$, there exists $\delta(\tau)$ such that $f(z)=O(|z|^{-\delta(\tau)})$ if $z\to\infty$ in the half-plane $H^+(\tau)$;

(c) for every τ, $\tau_0<\tau<+\infty$, there exists $\mu(\tau)>1$ such that $f(z=O[|z|^{-\mu(\tau)}\exp(-z^2)]$ if $z\to\infty$ and $z\in l(\tau)=\partial H^+(\tau)$.

Then, the function $f(z)$ can be represented in $H^+(\tau_0)$ by a series of the kind (6.1) with coefficients

$$b_n = -\frac{1}{2\pi i I_n}\int_{l(\tau)} H_n(\zeta)f(\zeta)\,d\zeta \quad (n=0, 1, 2, \dots). \tag{9.3}$$

P r o o f. Using the condition (c) and the inequality (7.2) we can easily prove that integral on the right side of (9.3) is absolutely convergent for every $n=0, 1, 2, \dots$. Indeed, $H_n(\zeta)f(\zeta)=O(|\zeta|^{-\mu(\tau)})$ if $\zeta\to\infty$ and $\zeta\in l(\tau)$.

Let $z\in H^+(\tau_0)$ and $\tau_0<\tau<+\infty$ is chosen so that $z\in H^+(\tau)$. Then the Lemma and the formula (5.7) of Christoffel — Darboux (after changing ζ and z) give us that

$$f(z)-\sum_{n=0}^{\nu} b_nK_n(z)=\frac{1}{2\pi i}\int_{l(\tau)}\frac{f(\zeta)}{\zeta-z}d\xi+\sum_{h=0}^{\nu}\Big\{\frac{1}{2\pi i I_n}\int_{l(\tau)}H_n(\zeta)f(\zeta)d\zeta\Big\}K_n(z) \tag{9.4}$$

$$=\frac{1}{2\pi i}\int_{l(\tau)}\Big\{\frac{1}{\zeta-z}+\sum_{n=0}^{\nu}\frac{1}{I_n}H_n(\zeta)K_n(\zeta)\Big\}f(\zeta)d\zeta=\frac{1}{2\pi i}\int_{l(\tau)}\frac{\Delta_{\nu+1}(\zeta,z)}{\zeta-z}d\zeta$$

$$=\frac{1}{4\pi i I_\nu}\int_{l(\tau)}\{H_\nu(\zeta)K_{\nu+1}(z)-H_{\nu+1}(\zeta)\,K_\nu(z)\}\frac{f(\zeta)}{\zeta-z}d\zeta.$$

Using the condition (c), the inequality (7.2), the asymptotic formula (3.16) for Hermite functions of second kind and Stirling formula, we get that

$$\frac{1}{4\pi i I_\nu}\int_{l(\tau)}\{H_\nu(\zeta)K_{\nu+1}(z)-H_{\nu+1}(\zeta)K_\nu(z)\}\frac{f(\zeta)}{\zeta-z}d\zeta$$

$$=O\{(2^\nu\nu!)^{-1}[(2\nu/l)^{\nu/2}((2\nu+2)/e)^{(\nu+1)/2}\exp(\tau\sqrt{2\nu+1}-\operatorname{Im} z\sqrt{2\nu+3})$$

$$+(2\nu+2)/e)^{(\nu+1)/2}(2\nu/e)^{\nu/2}\exp(\tau\sqrt{2\nu+3}-\operatorname{Im} z\sqrt{2\nu+1})\int_{-\infty}^{\infty}|t+i\tau|^{-1-\mu(\tau)}dt\}$$

$$=O\{\exp[-(\operatorname{Im} z-\tau)\sqrt{2\nu+1}]\}.$$

Then from (9.4) follow the statements of the Theorem.

10. Completeness of the system of Hermite functions of second kind. Let $0\le\tau_0<+\infty$ and $A(\tau_0)$ be the space of all complex functions analytic in the half-plane $H^+(\tau_0)$. From Theorem 3 it follows that if we consider the space $A(\tau_0)$ as a topological vector space with respect to the topology of the uniform convergence on every compact subset of $H^+(\tau_0)$, the system $\{K_n(z)\}_{n=0}^{\infty}$ of Hermite functions of second kind is not a base in the space $A(\tau_0)$. Therefore, it is interesting to solve the problem of completeness of the system $\{K_n(z)\}_{n=0}^{\infty}$ in the space $A(\tau_0)$. The answer is given by the following

T h e o r e m 8. *If $0\le\tau_0<+\infty$, the system $\{R_n(z)\}_{n=0}^{\infty}$ of Hermite functions of second kind is complete in the space $A(\tau_0)$ of all complex functions analytic in the half-piane $H^+(\tau_0)$.*

P r o o f. It is sufficient to prove that every rational function with poles outside the region $H^+(\tau_0)$ and satisfying the condition $R(\infty)=0$, can be represented in $H^+(\tau_0)$ by a series in Hermite functions of second kind. Indeed, if this is the fact, we san use further the classical theorem of Runge for approximation (in the sence of the topology of $A(\tau_0)$) of analytic functions $f(z)\in A(\tau_0)$ by means of rational functions [4, p. 174, (1.5)].

Since every rational function $R(z)$, such that $R(\infty)=0$ has the form

$$R(z)=\sum_{s=1}^{m}\sum_{k=1}^{m_s}\frac{a_{sk}}{(z-z_s)^k}$$

it remains to show that every rational function of the kind $(z-z_0)^{-k}$ $(k=1, 2, 3, \ldots)$ can be expanded in a series of Hermite functions of second kind. We shall prove namely that for every positive integer holds the equality

$$\frac{(k-1)!2^{-k+1}}{(\zeta-z_0)^k}=\sum_{n=k-1}^{\infty} I_n^{-1}n(n-1)\ldots(n-k+2)H_{n-k+1}(z_0)K_n(\zeta) \tag{10.1}$$

uniformly with respect fo ζ on every compact subset of the half-plane $H^+(\tau_0)$ where $\tau_0=|\operatorname{Im} z_0|$.

Using the relation $H_n'(z)=2nH_{n-1}(z)$ [1, (5.5.10)], we get from the Christoffel — Darboux formula (5.7), after derivation with respect to z, the equality $(\nu\geq k)$

$$\frac{(k-1)!\,2^{-k+1}}{(\zeta-z)^k}=\sum_{n=k-1}^{\nu} I_n^{-1}n(n-1)\ldots(n-k+2)H_{n-k+1}(z)K_n(\zeta)$$
$$+\sum_{s=0}^{k-1}\binom{k-1}{s}\frac{(k-s-1)!\,2^{s-k}}{(\zeta-z)^{k-s}}\{I_\nu^{-1}\nu(\nu-1)\ldots(\nu-s+1)H_{\nu-s}(z)K_{\nu+1}(\zeta)$$
$$-I_\nu^{-1}(\nu+1)\nu(\nu-1)\ldots(\nu-s+2)H_{\nu-s+1}(z)K_\nu(\zeta)\}. \tag{10.2}$$

Let $z=z_0$ and $\zeta\in H^+(\tau_0)$ i. e. $\operatorname{Im}\zeta>|\operatorname{Im} z_0|$. Then, using the asymptotic formulas (1.5), (3.16) and Stirling's formula, we get that

$$I_\nu^{-1}\nu(\nu-1)\ldots(\nu-s+2)H_{\nu-s}(z_0)K_{\nu+1}(\zeta)$$
$$=O\{\nu^{s/2}\exp[|\operatorname{Im} z_0|\sqrt{2\nu-2s+1}-\operatorname{Im}\zeta.\sqrt{2\nu+3}]\}$$

and respectively

$$I_\nu^{-1}(\nu+1)\nu(\nu-1)\ldots(\nu-s+2)H_{\nu-s+1}(z_0)K_\nu(\zeta)$$
$$=O\{\nu^{s/2}\exp[|\operatorname{Im} z_0|.\sqrt{2\nu-2s+3}-\operatorname{Im}\zeta.\sqrt{2\nu+1}]\}$$

and then from (10.2) the representation (10.1).

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