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REPRESENTATION OF ANALYTIC FUNCTIONS BY SERIES IN HERMITE POLYNOMIALS

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We consider the problem of representation of the coefficients of a series in Hermite polynomials in terms of Hermite functions of second kind and the given analytic function.

The system of Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$ is a system of polynomials orthogonal on the whole real axis with respect to the weight function $\exp(-x^2)$, i. e.

$$(1) \qquad \int_{-\infty}^{\infty} \exp(-x^2)H_m(x)H_n(x)dx = \sqrt{\pi}2^n n!\ \delta_{mn} \quad (m, n = 0, 1, 2, \ldots).$$

The system $\{H_n(z)\}_{n=0}^{\infty}$ is uniquely determined by the conditions (1) provided that the coefficient of z^n in $H_n(z)$ is positive [1, (5.5.1)].

Hermite polynomials can be defined also by the formula of Rodrigues, namely [1, (5.5.3)]

$$(2) \qquad \exp(-z^2)H_n(z) = (-1)^n \{\exp(-z^2)\}^{(n)} \quad (n = 0, 1, 2, \ldots).$$

It is well known that the system of Hermite polynomials is a solution of the difference equation $y_{n+1} - 2zy_n + 2ny_{n-1} = 0$. The system $\{K_n(z)\}_{n=0}^{\infty}$ of Hermite functions of second kind is defined as a second solution of this equation. More precisely

$$K_n(z) = -\int_{-\infty}^{\infty} \frac{\exp(-t^2)\ H_n(t)}{t-z}\, dt \quad (n = 0, 1, 2, \ldots)$$

provided that $z \in \mathbb{C} - (-\infty, \infty)$ ($\mathbb{C}$ denotes the complex plane).

By using (2) one can easily get another integral representation of Hermite functions of second kind, namely

$$(3) \qquad K_n(z) = (-1)^{n+1}\, n! \int_{-\infty}^{\infty} \frac{\exp(-t^2)}{(t-z)^{n+1}}\, dt, \quad n = 0, 1, 2, \ldots$$

Basing on the asymptotic formula for Hermite polynomials [1, Theorem 8.22.7] it is not difficult to describe the region and the mode of convergence of a series in this polynomials. There is also a formula of Cauchy—Hadamard type [1]. We shall formulate only the corresponding statement without proof.

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Theorem 1 (Cauchy — Hadamard). *Let* $\{a_n\}_{n=0}^{\infty}$ *be an arbitrary sequence of complex numbers and define*

$$\tau_0 = -\varlimsup_{n\to+\infty} (2n+1)^{-1/2} \ln|(2n/e)^{n/2}a_n|. \tag{4}$$

Then: a) if $\tau_0 \leqq 0$, *the series*

$$f(z) = \sum_{n=0}^{\infty} a_n H_n(z) \tag{5}$$

is divergent at every nonreal point: b) if $0<\tau_0\leqq+\infty$, *the series (5) is absolutely uniformly convergent on every compact set* $K\subset S(\tau_0):=\{z\in\mathbb{C}: |\operatorname{Im} z|<\tau_0\}$ *and diverges at every point* $z\in\mathbb{C}-\overline{S(\tau_0)}$.

The solution of the problem of expansion of analytic functions in series of Hermite polynomials was first given by E. Hille [2]. The corresponding result is the following

Theorem 2 (E. Hille). *Let* $f(z)$ *be a complex function defined in the region* $S(\tau_0)$ $(0<\tau_0\leqq+\infty)$. *In order to represent* $f(z)$ *in* $S(\tau_0)$ *by a series of the kind (5) is necessary and sufficient that* $f(z)$ *is analytic in* $S(\tau_0)$ *and to every given* τ, $0\leqq\tau<\tau_0$, *there exists a* $B(\tau)$ *such that*

$$|f(z)|=|f(x+iy)|\leqq B(\tau)\exp\{(x^2-y^2)/2-|x|(\tau^2-y^2)^{1/2}\} \tag{6}$$

for $-\infty<x<+\infty$ *and* $|y|\leqq\tau$.

As it was mentioned above, in this paper our purpose is to discuss the problem of the representation of the coefficients of the series (5) in terms of Hermite functions of second kind and the analytic function $f(z)$. First of all we shall prove some auxiliar statements.

Lemma 1. *Let* $f(z)$ *be a complex function analytic in the strip* $S(\tau_0)$ $(0<\tau_0\leqq+\infty)$ *and suppose that* $f(z)$ *can be represented in* $S(\tau_0)$ *by a series of the kind (5). Then for every* $n=0, 1, 2,\ldots$ *holds the equality*

$$a_n=(I_n)^{-1}\int_{-\infty}^{\infty}\exp(-t^2)H_n(t)f(t)dt, \tag{7}$$

where $I_n=\sqrt{\pi}2^n n!$.

Proof. From Theorem 1 follows that for every $0<\tau<\tau_0$ there exists $A=A(\tau)>0$ such that $|a_n|\leqq A(e/2n)^{n/2}\exp(-\tau\sqrt{2n+1})$ for every $n=0, 1, 2,\ldots$ Using this last inequality and also the inequality [1, (8.91.10)] for Hermite polynomials, we get easily that

$$a_nH_n(t)H_k(t)\exp(-t^2)=O(n^{1/6}\exp(-\tau\sqrt{2n+1})H_k(t)\exp(-t^2/2)\},$$

if $k=0, 1, 2,\ldots$ is fixed. Since

$$\int_{-\infty}^{\infty}|H_k(t)|\exp(-t^2/2)dt<+\infty$$

and the series $\Sigma_{n=0}^{\infty}n^{1/6}\exp(-\tau\sqrt{2n+1})$ is convergent, we conclude that the series $\Sigma_{n=0}^{\infty}a_nH_n(t)H_k(t)\exp(-t^2)$ can be integrated term by term on the interval $(-\infty,\infty)$. But its sum is the function $H_k(t)f(t)\exp(-t^2)$ and having in view the orthogonality relation (1), we get the equalities (7).

L e m m a 2. *Let* $f(z)$ *be a complex function analytic in the region* $S(\tau_0)$ $(0<\tau_0\leq+\infty)$ *and for every* τ, $0<\tau<\tau_0$, *there exists* $\omega(\tau)<1$ *such that* $f(z)=O(|z|^{\omega(\tau)})$ *for* $z\to\infty$ *and* $z\in\overline{S(\tau)}$. *Then for every compact set* $K\subset S(\tau_0)$ *and every* τ *such that* $0<\tau<\tau_0$ *and* $K\subset S(\tau)$, *holds the equality*

$$f^{(n)}(z)=\frac{(-1)^{n+1}n!}{2\pi i}\int_{-\infty}^{\infty}\left\{\frac{f(t-i\tau)}{(z-t+i\tau)^{n+1}}-\frac{f(-t+i\tau)}{(z+t-i\tau)^{n+1}}\right\}dt\quad (n=0,1,2,\ldots) \tag{8}$$

uniformly on $z\in K$.

P r o o f. We denote with $\Delta(R,\tau)$ the rectangle with vertices at the points $A=-R-i\tau$, $B=R-i\tau$, $C=R+i\tau$, $D=-R+i\tau$, i. e. the closed region defined by the inequalities $|x|\leq R$ and $|y|\leq\tau$. From the integral formulas for the derivatives of an analytic function we get that for every $z\in K$ and every $n=0, 1, 2,\ldots$

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_{\partial\Delta(R,\tau)}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta,$$

where $\partial\Delta(R,\tau)$ is the boundary of $\Delta(R,\tau)$.

Having in view that

$$\int_{AB}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta=(-1)^{n+1}\int_{-R}^{R}\frac{f(t-i\tau)}{(z-t+i\tau)^{n+1}}dt$$

$$\int_{CD}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta=(-1)^{n}\int_{-R}^{R}\frac{f(-t+i\tau)}{(z+t-i\tau)^{n+1}}dt$$

and

$$\int_{BC}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta=O(R^{-n-1+\omega(\tau)})\quad (R\to+\infty)$$

$$\int_{DA}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta=O(R^{-n-1+\omega(\tau)})\quad (R\to+\infty)$$

uniformly on $z\in K$, we get the equality (8).

The main result of the paper is the following

T h e o r e m 3. *Let* $0<\tau_0\leq+\infty$ *and* $f(z)$ *be a complex function satisfying the following conditions*:

a) $f(z)$ *is analytic in the region* $S(\tau_0)$;

b) *for every* τ, $0<\tau<\tau_0$, *there exists* $\delta(\tau)>0$ *such that* $f(z)=O(|z|^{-\delta(\tau)})$, *if* $z\to\infty$ *and* $z\in\overline{S(\tau)}$.

Then $f(z)$ *can be expanded in the region* $S(\tau_0)$ *in a series of the kind (5) and for every* τ $(0<\tau<\tau_0)$ *and every* $n=0, 1, 2,\ldots$ *holds the equality*

$$a_n=\frac{1}{2\pi i I_n}\int_{-\infty}^{\infty}\{K_n(t-i\tau)f(t-i\tau)-K_n(-t+i\tau)f(-t+i\tau)\}dt. \tag{9}$$

P r o o f. From the condition b) follows that the function $f(z)$ satisfies the condition (6) of Theorem 2. Therefore $f(z)$ can be represented in the region $S(\tau_0)$ by a series of the kind. Having in view Lemma 1 and formula (2), we get that

(10) $$a_n=(-1)^n (I_n)^{-1} \int_{-\infty}^{\infty} f(x)\{\exp(-x^2)\}^{(n)}dx, \quad (n=0, 1, 2, \dots).$$

We prove now that for every τ $(0<\tau<\tau_0)$ and every $n=0, 1, 2, \dots$

(11) $$f^{(n)}(x)=O(|x|^{\delta(\tau)}) \quad (|x| \to +\infty).$$

From the condition b) of the theorem follows that $f(z)$ satisfies the condition of Lemma 2. Then from (8) we get that for $|x| \to +\infty$

(12) $$f^{(n)}(x)=O\left\{\int_{-\infty}^{\infty}\frac{|t-i\tau|^{-\delta(\tau)}}{|x-t+i\tau|^{n+1}}dt+\int_{-\infty}^{\infty}\frac{|t-i\tau|^{-\delta(\tau)}}{|x+t-i\tau|^{n+1}}dt\right\}$$

$$=O\left\{\int_{-\infty}^{\infty}\frac{(t^2+\tau^2)^{-\delta(\tau)/2}}{[(t-x)^2+\tau^2]^{(n+1)/2}}dt\right\}=O\left\{\int_{-\infty}^{\infty}\frac{[\varphi(t, x)]^{\delta(\tau)/2}}{[(t-x)^2+\tau^2]^{(n+1+\delta(\tau))/2}}dt\right\},$$

where $\varphi(t, x)=[(t-x)^2+\tau^2](t^2+\tau^2)^{-1}$ $(-\infty<t<+\infty;\ -\infty<x<+\infty)$. The function $\varphi(t, x)$ considered as a function of t is bounded on the interval $(-\infty, +\infty)$ for every real x. We shall see that in this interval $\varphi(t, x)\leq\varphi[t_1(x), x]$, if $x>0$, and $\varphi(t, x)\leq\varphi[t_2(x), x]$, if $x<0$, where $2t_1(x)=x-\sqrt{x^2+4\tau^2}$ and $2t_2(x)=x+\sqrt{x^2+4\tau^2}$. Indeed $(\partial/\partial t)\varphi(t, x)=2x(t^2+\tau^2)^{-2}[t-t_1(x)]\times[t-t_2(x)]$ and if $x>0$, $\varphi(t, x)$ increases on the interval $(-\infty, t_1(x)]$, decreases on $[t_1(x), t_2(x)]$ and increases on $[t_2(x), +\infty)$. Therefore, at the point $t=t_1(x)$ this function has a local maximum and its value is $\varphi[t_1(x), x]>1$. Since $\lim_{t\to-\infty}\varphi(t, x)=\lim_{t\to+\infty}\varphi(t, x)\leq 1$, $\varphi[t_1(x), x]$ is the greatest value of $\varphi(t, x)$ on the interval $-\infty<t<+\infty$. In the same way we get that $\varphi[t_2(x), x]$ is the greatest value of $\varphi(t, x)$ on the interval $-\infty<t<+\infty$, if $x<0$. Since $\lim_{x\to+\infty}t_1(x)=\lim_{x\to-\infty}t_2(x)=0$, we get easily that $\varphi[t_1(x), x]=O(x^2)$, if $x\to+\infty$, and respectively $\varphi[t_2(x), x]=O(x^2)$, if $x\to-\infty$. Then from (12) follows that

$$f^{(n)}(x)=O\{|x|^{\delta(\tau)}\int_{-\infty}^{\infty}[(t-x)^2+\tau^2]^{-(n+1+\delta(\tau))/2}dt\}$$

$$=O\{|x|^{\delta(\tau)}\int_{-\infty}^{\infty}(t^2+\tau^2)^{-(n+1+\delta(\tau))/2}dt\}=O(|x|^{\delta(\tau)})$$

and the equality (11) is proved.

From (10) after integration by parts follows that

$$a_n=I_n^{-1}\int_{-\infty}^{\infty}f^{(n)}(x)\exp(-x^2)dx \qquad n=0, 1, 2, \dots$$

and according to Lemma 2 we get further that

(13) $$a_n=\frac{(-1)^{n+1}n!}{2\pi i I_n}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\frac{f(t-i\tau)dt}{(x-t+i\tau)^{n+1}}\right\}\exp(-x^2)dx$$

$$-\frac{(-1)^{n+1}n!}{2\pi i I_n}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\frac{f(-t+i\tau)dt}{(x+t-i\tau)^{n+1}}\right\}\exp(-x^2)dx.$$

Having in view the condition b) of the theorem, it is not difficult to assure that in the last equality we can change the order of integrations. Therefore

$$a_n = \frac{(-1)^{n+1} n!}{2\pi i I_n} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{\exp(-x^2)dx}{(x-t+i\tau)^{n+1}} \right\} f(t-i\tau)dt$$

$$-\frac{(-1)^{n+1} n!}{2\pi i I_n} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{\exp(-x^2)dx}{(x+t-i\tau)^{n+1}} \right\} f(-t+i\tau)dt$$

$$= \frac{1}{2\pi i I_n} \int_{-\infty}^{\infty} \{K_n(t-i\tau) f(t-i\tau) - K_n(-t+i\tau) f(-t+i\tau)\} dt$$

and thus Theorem 3 is proved.

REFERENCES

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