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ON THE INTEGRABILITY OF ENTIRE FUNCTIONS ON A LINE

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The following theorem is proved: If $F(z)$ is an entire function of exponential type $s\sigma$, where $0<\sigma<\pi$ and $s>0$ is an integer, then the convergence of the series

$$\sum_{n=-\infty}^{\infty} |F^{(\nu)}(n)|^p, \quad \nu=0, 1, 2, \ldots, s-1, \quad p>0$$

mplies

$$\int_{-\infty}^{+\infty} |F(x)|^p\, dx<\infty.$$

The results obtained have already been mentioned in [4], but no proof was given there.

Considering entire functions of exponential type under some conditions about the growth of the functions M. Plancherel and G. Polya [1] proved theorems, which show the equivalence of the inequalities

$$\sum_{n=-\infty}^{\infty} |F(n)|^p<\infty \text{ and } \int_{-\infty}^{\infty} |F(x)|^p dx<\infty, \quad p>0.$$

One of these theorems is

Theorem 1. *Let $f(z)$ be an entire function of exponential type. If $p>0$ and*

$$c=\limsup_{r\to\infty} r^{-1}\log(|F(-ir)|+|F(ir)|)<\pi,$$

then there exists a constant B, depending only on p and c, such that

$$\int_{-\infty}^{\infty} |F(x)|^p dx<B\sum_{n=-\infty}^{\infty} |F(n)|^p.$$

We extend this theorem in a way similar to that in which Korevaar [2] generalised the classic theorem of M. Cartwright [3], i. e. we will increase the type of the function, but shall demand covergence not only of the series $\Sigma|F(n)|^p$, but also of $\Sigma|F^{(\nu)}(n)|^p$, $\nu=1, 2, \ldots, s-1$, where $s\geq 2$ is some integer.

Theorem 2. *Let $F(z)$ be an entire function satisfying the condition*

$$|F(z)|\leq Ce^{s\sigma|z|}, \tag{1}$$

where $C=$const, $s\geq 2$ is an integer and $0<\sigma<\pi$. Let $p>0$ and the series

$$\sum_{n=-\infty}^{\infty} |F^{(\nu)}(n)|^p, \quad \nu=0, 1, 2, \ldots, s-1 \tag{2}$$

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be convergent. Then

$$\int_{-\infty}^{\infty} |F(x)|^p dx < A\Big(\sum_{n=-\infty}^{\infty} |F(n)|^p + \sum_{n=-\infty}^{\infty} |F'(n)|^p + \cdots + \sum_{n=-\infty}^{\infty} |F^{(s-1)}(n)|^p\Big),$$

where A is a constant, which depends only on p, σ and s.

The proof of this theorem is similar to that of theorem 1. Beforehand, we will formulate a lemma [1].

Lemma 1. *Let $p>1$ and the series $\Sigma|x_n|^p$ and $B=\Sigma b_\mu$, $b_\mu>0$ be convergent. If $|V_m|\leqq\Sigma_n b_{m-n}|x_n|$, then $\Sigma_m|V_m|^p\leqq B^p\Sigma_n|x_n|^p$.*

Now we are in a position to prove theorem 1. First of all we note that from the convergence of the series (2) it follows, that the sequences $\{F^{(\nu)}(n)\}_{n=-\infty}^{\infty}$, $\nu=0,1,\ldots,s-1$, are bounded. But the latter, together with condition (1), according to a result of J. Korevaar [2], implies the inequality

$$(1') \qquad |F(z)|\leqq Ae^{s\sigma|y|}, \quad A=\text{const}, \quad z=x+iy,$$

which we shall use later.

Now, let m be an integer and $F_m=\max\{|F(x)|: m-1/2\leqq x\leqq m+1/2\}$. Obviously one has

$$(3) \qquad \int_{-\infty}^{\infty}|F(x)|^p\,dx\leqq\sum_{m=-\infty}^{\infty}F_m^p,$$

so we deal further with F_m.

For simplicity we prove our theorem in the case $s=2$ first and then discuss the general case. When $s=2$, the conditions (1) and (2) have the form

$$(1'') \qquad |F(z)|\leqq Ce^{2\sigma|y|},$$

$$(2') \qquad \sum_n|F(n)|^p<\infty, \quad \sum_n|F'(n)|^p<\infty.$$

Let $\varphi(z)$ be an entire function, that also satisfies an inequality of the form (1″) with some constant $C>0$ and some $\sigma<\pi$. Then, considering the integral

$$J_n=\int_{|\zeta|=n+1/2}\frac{\varphi(\zeta)d\zeta}{(\zeta-z)\sin^2\pi\zeta},$$

which, due to (1″), tends to zero when $n\to\infty$, we obtain the following interpolation formula for the function $\varphi(z)$:

$$(4) \qquad \varphi(z)=\frac{1}{\pi^2}\sin^2\pi z\left[\sum_{n=-\infty}^{\infty}\frac{\varphi(n)}{(n-z)^2}+\sum_{n=-\infty}^{\infty}\frac{\varphi'(n)}{z-n}\right].$$

We shall treat separately the two cases $p\geqq1$ and $0<p<1$.

Let first $p\geqq1$. Consider the function $\varphi(z)=F(z+m)\sin^3\delta z/z$, where m is an integer and $\delta>0$ is such that $2\sigma+3\delta<2\pi$. This function satisfies the condition $|\varphi(z)|<C_1\exp(2\sigma_1|y|)$, $C_1=\text{const}$, $\sigma_1=\sigma+3\delta/2<\pi$. Therefore we may apply (4) for it. Noting also that $\varphi(0)=0$, $\varphi'(0)=0$ we obtain the equality

$$F(z+m)=\frac{z\sin^2\pi z}{\pi^2\sin^3\delta z}\left\{\sum_{n\neq0}\frac{F(n+m)\sin^3\delta n}{n(z-n)^2}+\sum_{n\neq0}\frac{F'(n+m)\sin^3\delta n}{n(z-n)}\right.$$

(5)

$$\left.+\sum_{n\neq0}\frac{3\delta F(n+m)\sin^2\delta n\cos n\delta}{n(z-n)}-\sum_{n\neq0}\frac{F(n+m)\sin^3\delta n}{n^2(z-n)}\right\}.$$

Now let us have $F_n=|F(m+\xi_m)|$, $1/2\leq\xi_m\leq1/2$. Since the expression $x\sin^2\pi x/\pi^2\sin^3\delta x$ is bounded on the interval $-1/2\leq x\leq1/2$, we get from (5) the inequality

(6) $$F_m=|F(m+\xi_m)|\leq K\Big(\sum_{n\neq0}\frac{|F(n+m)|}{|n|(|n|-1/2)^2}+\sum_{n\neq0}\frac{|F'(n+m)|}{|n|(|n|-1/2)}$$

$$+\sum_{n\neq0}\frac{|F(n+m)|}{|n|(|n|-1/2)}+\sum_{n\neq0}\frac{|F(n+m)|}{n^2(|n|-1/2)}\Big),$$

where K is a constant, depending only on δ. Put further

$$a_0=0,\ a_n=\frac{K}{|n|(|n|-1/2)^2}+\frac{K}{|n|(|n|-1/2)}+\frac{K}{|n|^2(|n|-1/2)},\quad n=\pm1,\pm2,\dots$$

$$b_0=0,\ b_n=\frac{K}{|n|(|n|-1/2)},\quad n=\pm1,\pm2,\dots$$

With these notations we write (6) in the form

$$F_m\leq\sum_{n=-\infty}^{\infty}a_n|F(n+m)|+\sum_{n=-\infty}^{\infty}b_n|F'(n+m)|$$

or

(7) $$F_m\leq\sum_{\nu=-\infty}^{\infty}a_{\nu-m}|F(\nu)|+\sum_{\nu=-\infty}^{\infty}b_{\nu-m}|F'(\nu)|.$$

In the special case when $p=1$ from this inequality we get

(8) $$\sum_{m=-\infty}^{\infty}F_m\leq A\sum_{n=-\infty}^{\infty}|F(n)|+B\sum_{n=-\infty}^{\infty}|F'(n)|,$$

where $A=\Sigma a_n$, $B=\Sigma b_n$.

Finally, (3) and (8) imply

$$\int_{-\infty}^{\infty}|F(x)\,dx\leq A\sum_{n=-\infty}^{\infty}|F(n)|+B\sum_{n=-\infty}^{\infty}|F'(n)|.$$

Here the constants A and B depend only on σ.

Let now $p>1$. From (7) again we get

$$F_m^p\leq2^p[(\sum_\nu a_{\nu-m}|F(\nu)|)^p+(\sum_\nu b_{\nu-m}|F'(\nu)|)^p]$$

and summing up along m and applying lemma 1 we obtain

$$\sum_{m=-\infty}^{\infty}F_m^p\leq2^p(A^p\sum_n|F(n)|^p+B^p\sum_n|F'(n)|^p),$$

hence

$$\int_{-\infty}^{\infty} |F(x)|^p dx \leqq 2^p (A^p \sum_{n=-\infty}^{\infty} |F(n)|^p + B^p \sum_{n=-\infty}^{\infty} |F'(n)|^p),$$

where A and B are the same constants as above.

Now let $0<p<1$. In this case we apply (4) to the function $\varphi(z)=F(z+m)\sin^{q+2}\delta z/z^q$, where the integer $q>0$ and $\delta>0$ are chosen so that $1<p(q+1)$ and $2\sigma+(q+2)\delta<2\pi$. Since $\varphi(0)=0$, $\varphi'(0)=0$, we obtain from (4)

$$F(z+m)=\frac{z^q \sin^2 \pi z}{\pi^2 \sin^{q+2}\delta z}\Big(\sum_{n\neq 0}\frac{F(n+m)\sin^{q+2}\delta n}{n^q(n-z)^2}+\sum_{n\neq 0}\frac{F'(n+m)\sin^{q+2}\delta n}{n^q(z-n)}$$

$$+\sum_{n\neq 0}\frac{(q+2)\delta F(n+m)\sin^{q+1}\delta n\cos\delta n}{n^q(z-n)}-\sum_{h\neq 0}\frac{qF(n+m)\sin^{q+2}\delta n}{n^{q+1}(z-n)}\Big).$$

As before, noting that $x^q \sin^2 \pi x/\pi^2 \sin^{q+2}\delta x$ is bounded on the interval $-1/2\leqq x\leqq 1/2$, we get for F_m

$$F_m \leqq L\Big(\sum_{n\neq 0}\frac{|F(n+m)|}{|n|^q(|n|-1/2)^2}+\sum_{n\neq 0}\frac{|F'(n+m)|}{|n|^q(|n|-1/2)}$$

(9)

$$+\sum_{n\neq 0}\frac{|F(n+m)|}{|n|^q(|n|-1/2)}+\sum_{n\neq 0}\frac{|F(n+m)|}{|n|^{q+1}(|n|-1/2)}\Big).$$

Putting

$$a_0=0,\quad a_n=\frac{L}{|n|^q(|n|-1/2)^2}+\frac{L}{n^q(|n|-1/2)}+\frac{L}{|n|^{q+1}(|n|-1/2)},\quad n=\pm 1, \pm 2, \ldots,$$

$$b_0=0,\quad b_n=\frac{L}{|n|^q(|n|-1/2)},\quad n=\pm 1, \pm 2, \ldots,$$

in (9), we get

$$F_m \leqq \sum_{n=-\infty}^{\infty} a_n |F(n+m)| + \sum_{n=-\infty}^{\infty} b_n |F'(n+m)|,$$

or

$$F_m \leqq \sum_{\nu} a_{\nu-m} |F(\nu)| + \sum_{\nu} b_{\nu-m} |F'(\nu)|.$$

Since $0<p<1$, in view of Jensen's inequality, the last result implies

(10) $$F_m^p \leqq \sum_{\nu} a_{\nu-m}^p |F(\nu)|^p + \sum_{\nu} b_{\nu-m}^p |F'(\nu)|^p.$$

Since $1<p(q+1)$, the series $C=\sum_n a_n^p$, $D=\sum_n b_n^p$ are convergent and summing up in (10) along m we get

$$\sum_{-\infty}^{\infty} F_m^p \leqq C \sum_{-\infty}^{\infty} |F(n)|^p + D \sum_{-\infty}^{\infty} |F'(n)|^p$$

and finally

$$\int_{-\infty}^{\infty} |F(x)|^p dx \leqq C \sum_{-\infty}^{\infty} |F(n)|^p + D \sum_{-\infty}^{\infty} |F'(n)|^p,$$

where C and D are constants, which depend only on σ and p.

Thus in the case $s=2$ the theorem 2 is proved. Consider now the case $s>2$. Let the function $F(z)$ satisfy conditions (1) and (2) of theorem 2. If $\varphi(z)$ is an entire function for which (1′) holds with some constant A and some $\sigma<\pi$, then by means of the integral

$$J_n=\int_{|\zeta|=n+1/2}\frac{\varphi(\zeta)\,d\zeta}{(\zeta-z)\sin^s \pi\zeta}, \quad n=1, 2, 3, \ldots$$

which in view of (1′) tends to zero when $n\to\infty$, we obtain an interpolation formula for $\varphi(z)$, which represents $\varphi(z)$ as a finite sum of functions, each being of the form

$$K\sin^s \pi z\sum_{n=-\infty}^{\infty}\frac{\varphi^{(k)}(n)}{(n-z)^r},$$

where $K=\text{const}$ depends only on s and $k>0$, $r>0$ are integers, such that $0\leqq k\leqq s-1$; $1\leqq r\leqq s$. (Of course K, k and r are different for separate summands.) When $p\gtrless 1$, we apply this formula to the function

$$\varphi(z)=F(z+m)\sin^{s+1}\delta z/z,\ \delta>0,\ s\sigma+(s+1)\,\delta<s\pi$$

and in the case $0<p<1$ to the function $\varphi(z)=F(z+m)\sin^{q+s}\delta z/z^q$, $q>0$, $\delta>0$, $1<p(q+1)$, $(s+q)\,\delta+s\sigma<\sigma\pi$.

Then proceeding as in the case $s=2$ we complete the proof.

REFERENCES

1. M. Plancherel, G. Polya. Fonctions entieres et integrales de Fourier multiples. *Comment. Math. Helv.*, **10**, 1938, 110—163.
2. J. Korevaar. Functions of exponential type bounded on sequences of points. *Ann. Soc. Polon. Math.*, **22**, 1949, 207—234.
3. M. L. Cartwright. On certain integral functions of order 1. *Quart. J. Math. Oxford Ser.* (1), 1936, 46—55.
4. T. Argirova. Some remarks on entire functions of exponential type. *C. R. Bulg. Acad. Sci.*, **29**, 1976, No. 1, 11—12.

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