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# SIMPLE LIE ALGEBRAS SATISFYING A NONTRIVIAL IDENTITY

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The main result of the paper is: any locally finite simple Lie algebra over some field of zero characteristic satisfies a nontrivial identity iff it is finite-dimensional over its centroid.

**1. Introduction.** The following problem is probably well-known in the theory of Lie algebras: whether or not there exists an infinite-dimensional over its centroid simple Lie algebra which satisfies a nontrivial identical relation?* The definition of centroid see in the book [1]. Ibidem (chapter X, theorems 2, 3) it is proved that if $L$ is a simple algebra over some commutative field $k$, $P$ is a centroid of $L$, then $L$ is a central simple Lie algebra over $P$; if further $K$ is some extension of $P$ then $L_K = L \otimes_P K$ is a central simple algebra over $K$. These remarks enable us to reduce the problem to the case of an algebraically closed field.

Let now $L$ be a finite-dimensional simple Lie algebra over some field $K$, this latter being infinite-dimensional over some subfield $k$. Then $L$ is an infinite-dimensional simple Lie algebra over $k$ satisfying a nontrivial identity

$$\sum_{\delta \in S_{n+1}} |\sigma| [x, y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n+1)}] = 0, \tag{1}$$

where $n = \dim_K L$, $S_{n+1}$ is a full symmetrical group on the symbols $1, 2, \ldots, n+1$ and $|\sigma| = 1$ $(-1)$ if $\sigma$ is even (odd), so that the coefficients of (1) lie in $k$ [2]. This shows that the condition on the dimension of a Lie algebra over its centroid is necessary.

In this paper following the group-theoretical lectures of O. Kegel [3] (see also [9]) we show that in the case of locally finite Lie algebras the above-mentioned problem can be reduced to the problem of identical relations in some aggregates of finite-dimensional simple Lie algebras. As the main corollary one gets

Theorem 2. *Let $L$ be a locally finite simple Lie algebra over some field $k$ of characteristic zero satisfying a nontrivial identity; then $L$ is finite-dimensional over its centroid.*

For the group-theoretical parallels of the results of this paper see [7] and [8].

**2. Some notations and preliminaries.** The reader is recalled that a Lie algebra $L$ over some field $k$ is called a locally finite Lie al-

* I have been recently informed that an example of such an algebra was constructed in Novosibirsk by Sumenkov, who proved that the generalized Witt algebra $\{W_{-1}, W_0, W_1, W_2, \ldots \mid [W_i, W_j] = (i-j) W_{i+j}\}$ over any field of characteristic zero is infinite dimensional over its centroid and satisfies a nontrivial identity. (*Remark made when reading proofs.*)

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gebra if any finite subset of $L$ lies in some finite-dimensional subalgebra of $L$. A system $\Pi$ of subalgebras of a Lie algebra $L$ is called a local system if (i) $L=\cup_{P\in\Pi}P$ and (ii) for any $P_1, P_2\in\Pi$ there exists $P\in\Pi$ such that $P\subseteq P_1\cup P_2$.

From this definition one readily sees that a Lie algebra $L$ is locally finite iff $L$ possesses a local system whose members are all finite-dimensional.

The theorem 1 below gives a criterion for the simplicity of a Lie algebra in terms of an arbitrary of its local systems. This theorem imitates theorem 3.1 from the already mentioned lecture course [3].

Theorem 1. *A Lie algebra $L$ is simple iff $L$ possesses a local system $\Pi$ such that for any subalgebra $P_1\in\Pi$ and any of its nontrivial ideals $Q_1$ there exists a $P_2\in\Pi$ such that $P_1\subseteq P_2$ and if $Q_2$ is a nontrivial ideal of $P_2$, then $Q_2\cap P_1\neq Q_1$.*

*Conversely, given any simple Lie algebra $L$ with some local system $\Pi$, this latter satisfies the above mentioned conditions.*

Proof. Let the conditions of the theorems be satisfied but $L$ be nonsimple, that is, it possesses a nontrivial proper ideal $T$. Then there exists in $\Pi$ a subalgebra $P_1$ such that $0\neq P_1\cap T=Q_1\neq M_1$. But in this case, given any subalgebra $P_2\supseteq P_1$, one easily sees that $T\cap P_2$ is an ideal of $P_2$ and that $(T\cap P_2)\cap P_1=T\cap P_1=Q_1$ contradicting the assumptions about $\Pi$.

Conversely, let $L$ be a simple Lie algebra and $\Pi$ be a local system of $L$. Let further $P_1\in\Pi$ and $Q_1$ be nontrivial ideal of $P_1$. Either there exists $P_2\in\Pi$ such that $P_1\subseteq P_2$, and for any ideal $Q_2$ of $P_2$ one has $Q_2\cap P_1\neq Q_1$ (what was to be proved) or for any $P\in\Pi$ with $P_1\subseteq P$ there exists some ideal of $P$ whose intersection with $P_1$ gives $Q_1$. In the latter case consider a local system $\Pi'$ whose members are all members of $\Pi$ containing $P_1$ as a subalgebra. For any $P\in\Pi'$ put

$$I(P)=\cap\{Q: Q \text{ an ideal of } P \text{ with } P_1\cap Q=Q_1\}.$$

Then $I(P)$ is the least ideal of $P$ containing $Q_1$ and if $P^*\in\Pi$ and $P\subseteq P^*$, then $I(P)\subseteq I(P^*)$. Put then $R=\cup_{P\in\Pi'}I(P)$. Then $R$ is a subalgebra of $L$ whose local system is the set $\Pi''=\{I(P): P\in\Pi'\}$. But since $I(P)$ is an ideal in $P$ for each $P\in\Pi'$, the same is true for $R$ in $L$, that is $R$ is an ideal in $L$. Now $R\supseteq Q_1\neq\{0\}$ so that $R$ is nontrivial. On the other hand, one readily sees that $R\cap P_1\neq P_1$, and thus $R$ is a proper ideal of $L$. This contradiction completes the proof of the Theorem.

Two simple consequences of the theorem are as follows.

Corollary 1. *Let $L$ be a simple Lie algebra with some local system $\Pi$. Then if some subalgebra $P\in\Pi$ has only finite number of ideals, there exists in $\Pi$ such a subalgebra $P^*$ that $P^*\supseteq P$ and any ideal of $P^*$ intersects $P$ trivially, that is either in $P$ or in $\{0\}$.*

Proof. For any nontrivial ideal $Q$ of $P$ by the theorem there exists a subalgebra $P_Q$ in $\Pi$ with $T_Q\cap P\neq Q$ for any ideal $T_Q$ of $P_Q$. Let $P$ be a member of $\Pi$ such that $P\supseteq P_Q$ for all ideals $Q$ of $P$. Given any ideal $T$ of $P$ and any nontrivial ideal $Q$ of $P$, one has $T\cap P=T\cap(P_Q\cap P)=(T\cap P_Q)\cap P\neq Q$, since $T\cap P_Q$ is an ideal of $P_Q$. Since $T\cap P$ is still an ideal of $P$, it follows that either $T\cap P=\{0\}$, or that $T\cap P=P$.

Corollary 2. *Given any simple Lie algebra $L$ with some local system $\Pi$ whose members are algebras with a finite numbers of ideals, $L$*

*possesses a local system whose members are simple Lie algebras equal to countable unions of subalgebras from* $\Pi$.

*In particular, if $L$ is a simple locally finite Lie algebra over some finite ring $k$, then $L$ has a local system of countable simple algebras.*

Proof. $P=P_0$ being any member of $\Pi$, choose an increasing chain of subalgebras $P_i$, $i=0, 1, 2, \ldots$, determined by the rule: if $P_0 \subset P_1 \subset \ldots \subset P_i$ are already chosen then $P_{i+1}$ is any element of $\Pi$ such that $P_i \subset P_{i+1}$ and for any ideal $Q \in P_{i+1}$ the intersection $Q \cap P_i$ is trivial (see corollary 1 above). Put then $C_P = \cup_{i=0}^{\infty} P_i$; $P_i$, $i=0,1,2,\ldots$ being a local system of $C_P$ satisfying the conditions of the theorem 1. Thus $C_P$ is a simple Lie subalgebra of $L$ equal to a union of a countable chain of subalgebras from $\Pi$, and to complete the proof of the corollary one must only show that the system $\Pi' = \{C_P : P \in \Pi \text{ and } C_P = \cup_{i=0}^{\infty} P_i \text{ for any chain } P_i,\ i=0, 1, 2, \ldots\}$ is local in $L$.

Indeed, the fact that $L = \cup_{C_P \in \Pi'} C_P$ is trivial. Let now $C_P, C_T \in \Pi'$. Find $C_K$ such that $C_K \supseteq C_P \cup C_T$. The chain $K_i$, $i=0, 1, 2, \ldots$ can be constructed in the following manner. Let $K=K_0$ be any subalgebra from $\Pi$ containing $P_0 \cup T_0$. The chain $K_0 \subset K_1 \subset \ldots \subset K_i$ being already constructed, take $U_{i+1} \in \Pi$ with $U_{i+1} \supseteq K_i$, $P_{i+1}$ and $T_{i+1}$ and let $K_{i+1} \in \Pi$ be such a subalgebra that $U_{i+1} \subseteq K_{i+1}$ and for any ideal $I$ of $K_{i+1}$ the intersection $I \cap U_{i+1}$ is trivial. Obviously $C_K$ constructed from $K_i$, $i=0, 1, 2, \ldots$ is such that $C_K \supseteq C_P \cup C_T$, thus proving our corollary 2.

This corollary enables us to reduce in some sense the study of locally finite simple Lie algebras over a finite ring to the countable case.

3. **Zero characteristic case**. Let now $L$ be a locally finite simple Lie algebra over some algebraically closed field $k$ of zero characteristic. Denote by $\Pi$ the system of all finite-dimensional subalgebras of $L$.

Proposition 1. *Let $S$ be semisimple subalgebra from* $\Pi$. *Then there exists an algebra $M$ in* $\Pi$ *with maximal ideal $N$ such that $M$ contains $S$ but $N \cap S = 0$.*

Proof. Indeed, denote by $S^X$ the least ideal of the algebra $X$ containing $S$. Here $X$ is some algebra from $\Pi$. Let $\Pi_1$ be the system of all $S^X$, $X \in \Pi$. Then $\Pi_1$ is a local system of $L$. For $\bigcup_{S^X \in \Pi_1} S^X$ is an ideal of $L$ containing $S$ and hence equal to $L$. Besides, if $z \in \Pi$ is such that $z \supseteq X \cup Y$ then $S^Z \supseteq S^X \cup S^Y$. Apply to $\Pi_1$ the corollary 1 from theorem 1, $S$ having only a finite number of (simple) ideals.

Further let $W \in \Pi$ be such that any ideal of the algebra $S^W$ either contains $S$ or intersects $S$ in $\{0\}$. Let further $T$ be the ideal of $W$, which is maximal among those ideals of $W$ which lie in $S^W$ and for which $T \cap S = \{0\}$. By the definition of $S^W$ and $T$ the factor $S^W/T$ is a main factor of $W$.

Since a solvable radical of a finite-dimensional algebra is invariant under all derivations, either $S^W/T$ is semisimple or $S^W/T$ is solvable. In the latter case $S^W/T$ is even abelian since the members of the derived series are also invariant under all derivations. In both cases thus $S^W/T$ is a direct product of its simple ideals.

Let now $M$ be a minimal ideal of $S^W$ containing $S$ and $T$ and $R$ be a maximal ideal of $S^W$ containing $T$ and such that $R \cap S = \{0\}$. By the choice of $W$ the factor $M+R/R$ is chief and isomorphic to a chief factor of $S^W/T$. Therefore $M+R/R \cong M/R \cap M$ is a simple algebra, and $N = M \cap R$ is the desired maximal ideal of $M$. The proposition is thus proved.

Corollary. *Let $L$ be a locally finite simple Lie algebra over some algebraically closed field $k$ of zero characteristic satisfying a nontrivial identity. Then the dimensions of the finite-dimensional semisimple subalgebras of $L$ are bounded by a finite number.*

Proof. Were this not the case, the dimensions of simple finite-dimensional factors of $L$ would not be bounded. But a simple calculation shows (this can be also seen in [4]) that no infinite aggregate of nonisomorphic finite-dimensional Lie algebras over some algebraically closed field of zero characteristic can satisfy a nontrivial identity. The obtained contradiction proves the corollary.

We now begin the proof of theorem 2.

Lemma 1. *Under the same conditions as in the previous corollary $L$ possesses a finite-dimensional semisimple subalgebra $S$ such that the set of all finite-dimensional subalgebras whose Levy factor equals $S$ forms a local system of $L$.*

Proof. By the corollary the dimensions of all Levy factors (that is maximal semisimple subalgebras) are bounded. Fix some semisimple subalgebra $S$ of maximal dimension. Let $x$ be an arbitrary element of $L$. The subalgebra $M$ generated by $S$ and $x$ is finite-dimensional and one of its Levy factors must contain $S$, therefore coincide with $S$. Thus we checked up the first condition from the definition of a local system. Similarly one can verify the second one which proves the lemma.

Lemma 2. *Let $M_1$, $M_2$ be finite-dimensional subalgebras of $L$ whose Levy factors equal $S$ and $R_1$, $R_2$ be solvable radicals of $M_1$, $M_2$. If $M_1 \subseteq M_2$ then $R_1 \subseteq R_2$.*

Proof. Let the contrary be true. Consider the natural projection $\pi : M_2 \to \bar{S}$ whose kernel coincides with $R_2$. By the conditions of the lemma $\pi$ is an epimorphism of $M_1$ onto $S$. But then $\pi(R_1)$ is a nontrivial solvable ideal of $\bar{S}$ which is impossible. The lemma is proved.

The proof of the following corollary is left to the reader.

Corollary. *Denote by $R(M)$ the radical of a finite-dimensional algebra $M$, and let $\Pi$ be a local system of subalgebras of $L$ containing some fixed semisimple subalgebra $S$ of maximal dimension, $\Pi'$ be a system of $R(M)$, $M \in \Pi$. Then $\Pi'$ is a local system of the subalgebra $R = \cup_{M \in \Pi} R(M)$.*

Now we continue the proof of the theorem 2 for the case when $k$ is algebraically closed. The subalgebra $R$ from the preceding corollary is in fact an ideal of $L$, so that either $R = \{0\}$, or $R = L$. In the first case $L = S$ is finite-dimensional. In the second case $R = L$ and $L$ is locally solvable. Consider then a system $\Pi''$ of subalgebras $R^2$, $R \in \Pi'$. Clearly, this is a local system for $N = \cup_{R \in \Pi'} R^2$. Since $R^2$ is locally nilpotent for a finite-dimensional solvable $R$, we see that $N$ is a locally nilpotent ideal of $L$. If $N = \{0\}$ then $L$ is abelian and thus dim $L \leq 1$.

Proposition 2. *A simple locally nilpotent algebra is abelian.*

Proof. Let $x, y \in L$ be such that $[x, y] \neq 0$. Let $P$ be an ideal of $L$ containing $x, y$; clearly $P = L$. Hence there exist $u_{ij} \in L$ and $\alpha, \beta_i \in k$ such that

$$x = \alpha[x, y] + \sum_{i=1}^{n} \beta_i [x, y, u_{i1}, u_{i2}, \ldots, u_{is_i}]. \tag{2}$$

Let $Q$ be finite-dimensional nilpotent subalgebra containing all elements involved in (2). Repeated application of (2) easily gives $x=0$ since all enough long commutators in $Q$ equal zero. The obtained contradiction proves the proposition.

This (probably well-known) proposition proves the theorem 2 in case of algebraically closed $k$.

Let now $k$ be an arbitrary field of zero characteristic, $K$ be the centroid of $L$, $K^*$ be its algebraic closure. As it is mentioned in the introduction, $L$ is a central simple algebra over $K$ and $L_{K^*}=L\otimes_K K^*$ is a simple locally finite algebra over $K^*$, $\dim_{K^*}L_{K^*}=\dim_K L$. If $L$ possesses a nontrivial identity over $k$, so does $L_{K^*}$ over $K^*$, and our argument above is applicable. Now the proof of the theorem 2 is completed.

**4. Further remarks.** (i) In the case when $k$ has a positive characteristic the main obstacle for the proof of the theorem similar to theorem 2 is, in particular, the lack of knowledge of all simple finite-dimensional algebras over $k$. However, when $k$ is finite the following argument works.

According to corollary 2 of theorem 1 the algebra $L$ possesses a local system of countable simple subalgebras, each of them being equal to a union of a countable chain of finite subalgebras. One readily checks that in the case under consideration proposition 1 is valid for any finite subalgebra $S$. Therefore one may consider only the case when $L$ is the union of the chain $M_1\subset M_2\subset \dots \subset M_n\subset M_{n+1}\subset \dots$ of its finite subalgebras, $M_{n+1}$ possessing a maximal ideal $R_{n+1}$ such that $M_n\cap R_{n+1}=\{0\}$. The family $M_n/R_n$, $n\geqq 1$ is then a set of simple algebras of strictly increasing dimensions. If all these algebras are classical simple algebras (char $k\geqq 5$) then using the results of [4] one can obtain that this family and hence $L$ cannot satisfy a nontrivial identity. We sum up this modest result as

Proposition 3. *Let $k$ be finite field of characteristic $\geqq 5$, $L$ be locally finite simple Lie algebra over $k$ all (but a finite number, up to isomorphism) of whose finite simple factors are classical simple algebras, satisfying a nontrivial identity. Then* $\dim L<\infty$.

(ii) Note also that if $A$ is an associative central simple $PI$-algebra over some field $k$, then its antiisomorphic image $A^*$ is central, simple and $PI$. By Regev's theorem [5] $B=A\otimes_k A^*$ is a $PI$-algebra. On the other hand $B$ is dense in the algebra of all linear transformations of $A$ over $k$ [6, theorem V. 9.2]. Therefore $A$ must be finite-dimensional over $k$ and we have proved the following

Proposition 4. *A simple associative algebra over some field is a PI-algebra iff it is finite-dimensional over its centroid.*

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