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ON THE NUMERICAL SOLUTION OF CAUCHY-TYPE SINGULAR INTEGRAL EQUATIONS

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A Cauchy-type singular integral equation may be numerically solved by reduction to a system of linear equations in the same way as a Fredholm integral equation. For this reduction, numerical integration methods are used for both regular and singular integrals of the singular integral equation and the points of application of the singular integral equation are selected in such a way that maximum accuracy in the approximations of integrals is obtained.

1. Introduction. Several methods for the numerical solution of Cauchy-type singular integral equations have been up to now developed [1-11]. These methods generally reduce a singular integral equation to a system of linear equations, which can be easily solved to give an approximate expression of the unknown function of the singular integral equation (or the unknown functions in the case of a system of singular integral equations).

These methods present considerable disadvantages, as it will be seen later, due either to their complexity or to the small degree of accuracy obtained by their use. Also most of them are subject to limitations as regards the classes of singular integral equations to which they are applicable.

On the other hand, the numerical solution of Fredholm integral equations by reduction to a system of linear equations is quite easy to be done by using a proper method of numerical expression of the integrals involved in it.

It is the aim of this paper to extend by a direct manner this method of numerical solution of Fredholm integral equations to the case of Cauchy-type singular integral equations, that is to the case when the kernel of a Fredholm integral equation presents, except weak singularities at the ends of the integration interval, Cauchy-type singularities, too.

This is achieved by an extension of the methods of numerical quadrature used up to now for ordinary integrals, so that they can be applied to singular integrals as well. It was Hunter [12] who, first, extended the Gauss-Legendre numerical quadrature formula to the evaluation of singular integrals. Further, Chawla and Ramakrishnan [13] extended the Gauss-Chebyshev and the Gauss-Jacobi numerical quadrature formulae for singular integrals. Here, a general way of extending any known numerical quadrature formula for singular integrals will be presented. In this way, the numerical treatment of a singular integral equation can be achieved in a way similar to that used for Fredholm integral equations.

It must be also remarked that some of the already existing methods for the numerical solution of singular integral equations may result as special cases of the method presented here, which is a general method covering all the cases which could be met in practical applications.

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Moreover, it should be noted that, when speaking about singular integrals and singular integral equations, we mean Cauchy-type singular integrals and Cauchy-type singular integral equations and not singular integrals presenting weak power singularities or logarithmic singularities.

Finally we will say that a function defined in an interval is generalized Hölder-continuous in this interval, if it is Hölder-continuous in every subinterval of this interval not containing the neighbourhoods of the end-points. Such a function may present singularities of power or logarithmic type near the end-points of the interval.

2. The Existing Methods of Numerical Solution of Singular Integral Equations. The problem of numerical solution of a singular integral equation of the first or the second kind, which can be also considered as a Fredholm integral equation, but having a kernel with a Cauchy-type singularity, has been an old subject of investigations by several authors, both applied mathematicians and engineers.

L. V. Kantorovich and V. I. Krylov [1] propose for the numerical solution of a singular integral equation of the form:

$$\varphi(x)-\lambda\int_{\alpha}^{\beta}K(t,x)\varphi(t)dt=f(x), \tag{2.1}$$

where the kernel $K(t, x)$ presents a Cauchy-type singularity, the transformation of it to the following equivalent form:

$$\varphi(x)[1-\lambda\int_{\alpha}^{\beta}K(t,x)dt]-\lambda\int_{\alpha}^{\beta}K(t,x)[\varphi(t)-\varphi(x)]dt=f(x). \tag{2.2}$$

In this way, the first integral of the leftside can be calculated independently, and usually in a closed form, while the second integral does not present any more a singularity and can be approximated by some method of numerical integration.

Thus, by using a formula of approximate calculation of integrals:

$$\int_{a}^{\beta}\psi(t)dt=\sum_{k=1}^{m}A_k\psi(t_k), \tag{2.3}$$

the following system of linear equations approximating the singular integral equation (2.2) is obtained:

$$\varphi(x_i)[1-\lambda\int_{\alpha}^{\beta}K(t,x_i)dt]-\lambda\sum_{k=1}^{n}A_kK(x_k,x_i)[\varphi(x_k)-\varphi(x_i)]=f(x_i), \tag{2.4}$$

$$i=1,\ 2,\ldots,\ n.$$

The disadvantages of this method of numerical solution of singular integral equations are the necessity of calculation of the first integral in (2.2) and (2.4) by a direct method and of computation of the term in the sum of (2.4) for $i=k$, when a form $0:0$ results. Kantorovich and Krylov propose that this term be computed by some interpolation formula like linear interpolation:

$$(2.5) \qquad \{K(x_k, x_i)[\varphi(x_k)-\varphi(x_i)]\}_{i=k} \cong \frac{x_{k+1}-x_k}{x_{k+1}-x_{k-1}} K(x_k, x_{k-1})\{\varphi(x_{k-1})-\varphi(x_k)\}$$

$$+\frac{x_k-x_{k-1}}{x_{k+1}-x_{k-1}} K(x_k, x_{k+1})\{\varphi(x_{k+1})-\varphi(x_k)\}.$$

Nevertheless, it can be noted that the accuracy of interpolation, even of higher order. is not enough and that the system (2.4) takes in this way a rather complicated form.

Although these disadvantages of the above-described method of numerical solution of singular integral equations were known rendering it inferior to the corresponding method for Fredholm integral equations, this method has been widely used and remains a standard method of treating singular integral equations.

On the other hand, during the last twenty years, effective methods for the numerical solution of singular integral equations, based on the properties of systems of orthogonal polynomials over the integration interval, have been developed. In this way, singular integral equations of the form

$$(2.6) \qquad \int_{-1}^{1} w(t)K(t, x)\varphi(t)dt = f(x),$$

where functions $K(t, x)$, $\varphi(t)$ and $f(x)$ are assumed Hölder-continuous in the integration interval, without singularities at the ends ± 1, and the weight function $w(t)$ is of the form :

$$(2.7) \qquad w(t) = (1-t)^{\pm 1/2}(1+t)^{\pm 1/2}$$

have been considered at first by V. V. Ivanov [2] and A. I. Kalandiya [3] and afterwards by F. Erdogan [4], F. Erdogan and G. Gupta [5] and F. Erdogan, G. Gupta and T. Cook [6], who gave two methods of reduction of (2.6) to a system of linear equations. Both these methods are based on the properties of Chebyshev polynomials and on an expansion of the unknown function $\varphi(t)$ in a series of such polynomials. The difference between these methods lies in the fact that the unknown quantities in the system of linear equations approximating (2.6) may be either the coefficients in the expansion of the unknown function in a series of Chebyshev polynomials [4, 6], or the values of the unknown function at the points of the integration interval $[-1, 1]$ used as abscissae in the Gauss-Chebyshev numerical integration method [5, 6].

This second method was thought by Erdogan, Gupta and Cook [6] as a Gaussian integration method for the approximation of singular integrals because of its similarity to the well-known Gauss-Chebyshev method for regular integrals. Although this is true, one may note that Erdogan, Gupta and Cook [5, 6] have developed this method in a quite different way than that normally used for the development of Gaussian integration formulae for regular integrals and that they proved it to be accurate for integrals of the form:

$$(2.8) \qquad I = \int_{-1}^{1} w(t) \frac{\varphi(t)}{t-x} dt$$

only when the function $\varphi(t)$ is of degree up to $(n-1)$ and the points x are selected as roots of Chebyshev polynomials.

In the case when:

$$w(t)=(1-t^2)^{-1/2}, \tag{2.9}$$

P. Theocaris and N. Ioakimidis [7] have proved, using the same method as Erdogan, Gupta and Cook [5, 6], that the above-mentioned method of approximating the integral (2.8) by a sum and coinciding with the Gauss-Chebyshev method for ordinary integrals is accurate for functions $\varphi(t)$ up to the degree $2n$, that is, in some way, more accurate than the corresponding method for ordinary integrals, which is accurate for integrands $\varphi(t)$ polynomials up to the degree $2n-1$. In the same paper [7] a second method for the numerical evaluation of singular integrals of the form (2.8) was also developed, based on the properties of the Chebyshev polynomials. This method is equivalent to the Lobatto-Chebyshev method for ordinary integrals and accurate for functions $\varphi(t)$ polynomials up to $(2n-2)$ degree, while the corresponding method for regular integrals is accurate for functions $\varphi(t)$ polynomials up to $(2n-3)$ degree.

Analogous methods for the numerical solution of singular integral equations of the form:

$$A\varphi(x)+\frac{B}{\pi}\int_{-1}^{1} w(t)\varphi(t)\frac{dt}{t-x}+\int_{-1}^{1} w(t)K(t, x)\varphi(t)dt=f(x), \tag{2.10}$$

where the weight function $w(t)$ is of the form

$$w(t)=(1-t)^a(1+t)^b, \tag{2.11}$$

with constants a and b determined as

$$a=\frac{1}{2\pi i}\log\frac{A-iB}{A+iB}+N,\quad b=-\frac{1}{2\pi i}\log\frac{A-iB}{A+iB}+M,\ -1<a,\ b<1, \tag{2.12}$$

where N and M are appropriate integer numbers, have also been developed by using the properties of the Jacobi polynomials instead of the Chebyshev polynomials considered previously. L. N. Kaprenko [8], Erdogan [4] and Erdogan, Gupta and Cook [6] reduced the singular integral equation (2.10) to a system of linear equations for the determination of the coefficients in the expansion of the unknown function $\varphi(t)$ in a series of Jacobi polynomials corresponding to the weight function (2.11), while S. Krenk [9] reduced (2.10) to a system of linear equations for the determination of the values of the unknown function at the points of the integration interval $[-1, 1]$ used as abscissae in the Gauss-Jacobi numerical integration method after a selection of the points of application of the singular integral equation as roots of an appropriate Jacobi polynomial.

As in the case of the Gauss-Chebyshev method, the method used by Krenk [9] for the evaluation of singular integrals is equivalent to the Gauss-Jacobi method for regular integrals, but it was proved to be accurate for functions $\varphi(t)$ polynomials only up to the degree $n-1$, while, as it can be realized, it is accurate for functions $\varphi(t)$ polynomials up to the degree $2n$.

The case of a singular integral equation of the form (2.10), but of the first kind (that is with $A=0$) and without the first two of limitations (2.12), was considered by Erdogan, Gupta and Cook [6], who reduced it to a system of linear equations by using directly the Gauss-Jacobi numerical integration method for regular integrals and choosing the points x of application of (2.10) in a rather unjustified way.

Finally S. Majumdar [10, 11] considered a singular integral equation of the form

$$\int_{-1}^{1} K(t-x)\varphi(t)dt = f(x) \tag{2.13}$$

with kernel $K(t-x)$ presenting a Cauchy-type singularity without weak singularities at the points ± 1. Majumdar reduced this singular integral equation to a system of linear equations with unknowns the coefficients in the expansion of the unknown function $\varphi(t)$ in a series of Chebyshev polynomials after an arbitrary selection of the points x of application of (2.13). The method of Majumbar is evidently subject to limitations as it is applicable to singular integral equations of the first kind only with kernels of special type and without weight functions.

3. Quadrature Formulae for Cauchy-Type Integrals. Several quadrature formulae exist for the numerical evaluation of integrals of the general form

$$I = \int_{\alpha}^{\beta} w(t) f(t)dt, \tag{3.1}$$

where $[\alpha, \beta]$ is the integration interval lying on the real axis and being finite or infinite, $w(t)$ is the weight function and $f(z)$ is the integrated function, which is defined in the interior D of a curve C surrounding the integration interval $[\alpha, \beta]$. Function $f(z)$ may have simple poles outside the integration interval but not on the integration interval, as in the case of Cauchy-type integrals. Except for these poles, $f(z)$ is analytic in the domain D.

General approaches to quadrature formulae for integrals of the form (3.1) and estimation of their remainders are given by H. Takahasi and M. Mori [14, 15] and J. Donaldson and D. Elliott [16]. The case of an integrated function $f(z)$ having simple poles in the integration interval $[\alpha, \beta]$ was treated by D. Hunter [12], who modified the Gauss-Legendre numerical integration method for the evaluation of such integrals, and by M. Chawla and T. Ramakrishnan [13], who modified the Gauss-Chebyshev and the Gauss-Jacobi methods for the evaluation of such integrals.

The developments of Hunter and Chawla and Ramakrishnan can be extended to give a general method of approach to guadrature rules for integrals of the form (3.1) with an integrated function $f(z)$ having m simple poles at points $z_k(k=1, 2, \ldots, m)$ with corresponding residuals $\varrho_k(k=1, 2, \ldots, m)$. Points z_k are inside curve C and poles of the function $f(z)$ outside this curve are not taken into account. Of course, none of the poles z_k is permitted to coincide with the end-points α or β of the integration interval.

We define now the function

$$\sigma_n(z) = k \prod_{k=1}^{n} (z-t_k), \tag{3.2}$$

where t_k are the abscissae used in the quadrature rule for ordinary integrals which we will modify as to apply to singular integrals, too, and k is an arbitrary constant.

Now we consider the contour integral

$$I_0 = \frac{1}{2\pi i} \int \frac{f(\tau)}{(\tau - z)\sigma_n(\tau)} d\tau, \tag{3.3}$$

on curve C surrounding the integration interval $[\alpha, \beta]$. If this interval is of infinite length, curve C is also of infinite length having one branch above interval $[\alpha, \beta]$ and one branch below interval $[\alpha, \beta]$. By applying the Cauchy residue theorem to integral (3.3) and taking into account the simple poles z_k of the function $f(z)$ inside curve C as well as the roots t_k of polynomial $\sigma_n(z)$ in the interval $[\alpha, \beta]$, we find that

$$I_0 = \frac{f(z)}{\sigma_n(z)} + \sum_{k=1}^{n} \frac{f(t_k)}{(t_k - z)\sigma'_n(t_k)} + \sum_{k=1}^{m} \frac{\varrho_k}{(z_k - z)\sigma_n(z_k)}. \tag{3.4}$$

To derive expression (3.4), we have supposed that no pole z_k of the function $f(z)$ coincides with a zero t_k of polynomial $\sigma_n(z)$.

Further, combining Eqs. (3.3) and (3.4) we find that

$$f(z) = \sigma_n(z)\left\{ \sum_{k=1}^{n} \frac{f(t_k)}{(z - t_k)\sigma'_n(t_k)} + \sum_{k=1}^{m} \frac{\varrho_k}{(z - z_k)\sigma_n(z_k)} + \frac{1}{2\pi i} \int_C \frac{f(\tau)d\tau)}{(\tau - z)\sigma_n(\tau)} \right\}. \tag{3.5}$$

This expression for $f(z)$ if applied to the points t of the interval $[\alpha, \beta]$, multiplied by the weight function $w(t)$ of integral (3.1) and integrated along the interval $[\alpha, \beta]$ except of its parts contained in small cycles of radius $\varepsilon \to 0$ with centres coinciding with those simple poles of $f(z)$ lying on the integration interval, which is in accordance with the definition of principal value of Cauchy-type integrals, gives:

$$I = \int_\alpha^\beta w(t) f(t) dt = \sum_{k=1}^{n} A_k f(t_k) - 2 \sum_{k=1}^{m} \frac{\varrho_k q_n(z_k)}{\sigma_n(z_k)} + E_n, \tag{3.6}$$

where

$$q_n(z) = -\frac{1}{2} \int_\alpha^\beta \frac{w(t)\sigma_n(t)}{t - z} dt, \tag{3.7}$$

$$A_k = \frac{1}{\sigma'_n(t_k)} \int_\alpha^\beta \frac{w(t)\sigma_n(t)}{t - t_k} dt = -2\frac{q_n(t_k)}{\sigma'_n(t_k)}, \tag{3.8}$$

$$E_n = \frac{1}{\pi i} \int_C \frac{q_n(\tau)}{\sigma_n(\tau)} f(\tau) d\tau. \tag{3.9}$$

This numerical integration formula (3.6) coincides with the corresponding integration formula for regular integrals if the function $f(z)$ does no have any simple pole in the interval $[\alpha, \beta]$. It may be also noted that the expression (3.9) for the error term E_n was obtained after a change of the order of integration on curve C and interval $[\alpha, \beta]$. Further, there is no distinction as regards the influence of simple poles z_k of $f(z)$ lying in the integration interval $[\alpha, \beta]$ or outside it.

Finally, the accuracy of formula (3.6) is exactly the same with the accuracy of the corresponding formula for regular integrals because there is no difference in the expression (3.9) of the error term. This means that if the numerical integration formula under consideration is exact for polynomials up to p degree for regular integrals, when it is valid for $z \to \infty$:

$$q_n(z)/\sigma_n(z) = O(z^{-p-2}), \tag{3.10}$$

then it will be exact for functions $f(z)$ of the form considered here:

$$f(z) = g(z) / \prod_{k=1}^{m} (z - z_m), \tag{3.11}$$

if $g(z)$ is a polynomial up to $(p+m)$ degree. This fact can be easily confirmed if curve C tends to infinity and expression (3.9) is taken into account. In some way, we can say that a numerical integration formula is more accurate for functions $f(z)$ having simple poles than for functions $f(z)$ not having such poles.

One further remark is that (3.7) should be valid for points z inside the integration interval $[\alpha, \beta]$ as well as outside it. If we take into account the Plemelj formulae, we find from (3.7)

$$q_n^+(t_0) - q_n^-(t_0) = -\pi i \sigma_n(t_0), \tag{3.12a}$$

$$q_n^+(t_0) + q_n^-(t_0) = -\int_\alpha^\beta \frac{w(t)\sigma_n(t)}{t - t_0}\, dt, \tag{3.12b}$$

where t_0 are the points of the real axis and $q_n^\pm(t_0)$ are the boundary values of function $q_n(z)$ as z approaches a point t_0 of the real axis, but lies in the positive or negative half-plane respectively. From formulae (3.7) and (3.12b) for the points t_0 of the real axis, on which the interval $[\alpha, \beta]$ lies, we find

$$q_n(t_0) = \frac{1}{2}\, [q_n^+(t_0) + q_n^-(t_0)]. \tag{3.13}$$

In another wording, (3.13) is a definition of function $q_n(z)$ on the real axis consistent with the previous development.

Finally, we can apply formula (3.6) to a Cauchy-type integral of the form

$$I' = \int_\alpha^\beta w(t)\, \frac{f(t)}{t - x}\, dt, \quad x \in [\alpha, \beta], \quad x \neq t_k \ (k = 1, 2, \ldots, n), \tag{3.14}$$

where the function $f(z)$ has no poles inside curve C and point x does no coincide with some of abscissae t_k or points α or β, when we find

$$I' = \int_\alpha^\beta w(t) \frac{f(t)}{t - x}\, dt = \sum_{k=1}^{n} A_k \frac{f(t_k)}{t_k - x} - 2f(x) \frac{q_n(x)}{\sigma_n(x)} + E_n, \tag{3.15}$$

where

$$E_n = \frac{1}{\pi i} \int_C \frac{q_n(\tau)}{\sigma_n(\tau)} \frac{f(\tau)}{\tau - x} d\tau. \tag{3.16}$$

4. On the Behaviour of Functions the $q_n(z)$. Under the generally satisfied assumptions that:

i) The weight function $w(t)$ is positive along the integration interval $[\alpha, \beta]$, is generalized Hölder-continuous in this interval and at the end-points α and β either it is bounded, or it presents singularities of power or logarithmic or complex type of the form

$$w(t) \sim (t-c)^{-\gamma_c} \ln^{p-1}(t-c), \quad \gamma_c < 1, \quad t \to c, \quad c = \alpha, \beta, \tag{4.1}$$

where γ_c is a constant and p a positive integer, and:

ii) The weights A_k $(k = 1, 2, \ldots, n)$ are positive numbers, and because of the fact that functions $\sigma_n(z)$, defined by (3.2), are analytic inside the curve C, we can conclude that functions $q_n(z)$, defined by (3.7), are sectionally analytic inside the curve C except on the interval $[\alpha, \beta]$, are generalized Hölder-continuous in this interval and at the end-points α and β either they are bounded (if $w(\alpha)\sigma_n(\alpha) = w(\beta)\sigma_n(\beta) = 0$), or they present simple logarithmic singularities (if $w(\alpha)\sigma_n(\alpha)$, $w(\beta)\sigma_n(\beta) \neq 0$, but bounded), or they present singularities of power or logarithmic or complex type of the form (see F. D. Gakhov [17]):

$$\begin{aligned} & q_n(z) \sim S(z, c), && z \notin [\alpha, \beta] \quad z \to c, \\ & && \qquad\qquad c = \alpha, \beta, \\ & q_n(z) \sim \frac{1}{2}[S^+(z, c) + S^-(z, c)], && z \in [\alpha, \beta] \quad z \to c, \end{aligned} \tag{4.2}$$

where the function $S(z, c)$ depends on the behaviour (4.1) of the weigh function $w(t)$ and has near the end-points α and β of the integration interval $[\alpha, \beta]$ the behaviour

$$S(z, c) \sim \frac{e^{\gamma'_c \pi i}}{2i \sin \gamma'_c \pi} \frac{\ln^p(z-c)}{(z-c)^{\gamma'_c}}, \quad z \to c, \quad c = \alpha, \beta, \quad z \notin [\alpha, \beta], \tag{4.3}$$

where the constant γ'_c is equal to γ_c if the end-point c under considerationt does not coincide with any of the abscissae t_k $(k = 1, 2, \ldots, n)$, roots of the polynomial $\sigma_n(z)$, or equal to $(\gamma_c - 1)$ if some of the abscissae t_k coincides with the end-point c.

If only the function $q_0(z)$ is considered, when the corresponding function $\sigma_n(z)$ reduces to a constant, because $n = 0$, the constant γ'_c coincides with γ_c. As it can be seen from expressions (4.2) and (4.3), the function $q_0(z)$ presents singularities of the form

$$q_0(t) \sim (t-c)^{-\gamma_c} \ln^p(t-c), \quad c = \alpha, \beta, \quad t \in [\alpha, \beta], \tag{4.4}$$

if the weight function $w(t)$ does not tend to zero when $t \to c$, except for the case when $\gamma_c = 1/2$ and $p = 0$, when it does not present any singularity inside the integration interval $[\alpha, \beta]$, while it presents a singularity of the form (4.4) outside it.

Because of the assumption that the weight function $w(t)$ is positive along the integration interval $[\alpha, \beta]$, we can conclude from (3.7) that the function $q_0(z)$, if not identical to zero, has one and only one root inside the integration interval $[\alpha, \beta]$.

As regards the roots of the functions $q_n(z)$, we can investigate the number and distribution of them inside the integration interval $[\alpha, \beta]$ as follows. We apply formula (3.15) for $f(t)=-1/2$, when $E_n\equiv 0$, and we obtain, taking also into account the definition (3.7) of functions $q_n(z)$:

$$q_0(t)=\frac{1}{2}\sum_{k=1}^{n}\frac{A_k}{t-t_k}+\frac{q_n(t)}{\sigma_n(t)},\quad t\in(\alpha, \beta). \tag{4.5}$$

Because of the facts that the function $q_0(t)$ is generalized Hölder-continuous inside the interval $[\alpha, \beta]$ and it is an increasing function of t and the assumption that all the weights A_k $(k=1, 2, \ldots, n)$ are positive numbers, it is easy to prove that function $\sigma_n(t)$ has one and only one root inside each subinterval (t_k, t_{k+1}) $(k=1, 2, \ldots, n-1)$ of the integration interval $[\alpha, \beta]$, provided that $t_1<t_2<\ldots<t_{n-1}<t_n$. If the abscissa t_1 does not coincide with the end-point α, one more root of the function $q_n(z)$ lies inside the subinterval (α, t_1), if the function $q_0(t)$ presents a singularity of the form (4.4) near the point α tending to $(-\infty)$, as it can be easily seen. In a similar manner, if the abscissa t_n does not coincide with the end-point β, one more root of the function $q_n(z)$ lies inside the subinterval (t_n, b), if the function $q_0(t)$ presents a singularity of the form (4.4) near the point β tending to $(+\infty)$, as it can be easily seen.

Summarizing, we can say that functions $q_n(z)$ have $(n-1)$ up to $(n+1)$ roots inside the integration interval $[\alpha, \beta]$ under assumptions mentioned at the beginning of this paragraph, the exact number of these roots depending on the possible coincidence of points t_1 and t_n with the end-points α and β and the behaviour of the weight function $w(t)$ near the same end-points. The number of these roots is of great importance for the numerical solution of singular integral equations, as it will be seen below.

5. **On the Numerical Solution of Singular Integral Equations of the First Kind**. Let us consider the following system of m singular integral equations of the first kind:

$$\sum_{j=1}^{m}\int_{\alpha_j}^{\beta_j} w_j(t)K_{ij}(t, x)\varphi_j(t)dt=f_i(x),\quad x\in(\alpha_i, \beta_i),\quad i=1, 2, \ldots, m. \tag{5.1}$$

The functions $f_i(x)$ are defined in the intervals $[\alpha_i, \beta_i]$ respectively and are generalized Hölder-continuous in these intervals. At the end-points α_i and β_i they are permitted to present power or logarithmic or even complex singularities of the form

$$f_i(x)\sim(t-c_i)^{-\gamma_{ic}}\ln^{p_i-1}(t-c_i),\quad \gamma_{ic}<1,\ x\to c_i,\quad c_i=\alpha_i,\beta_i, \tag{5.2}$$

where γ_{ic} are constants and p_i positive integers. In the same way, the weight functions $w_i(t)$ are also generalized Höider-continuous, but they may present singularities at the corresponding end-points α_j, β_j of the same form as $_i(x)$, subject to conditions (i) of the previous paragraph. Finally, the kernels $K_{ij}(t, x)$ are defined in the domains: $\alpha_j \leq t \leq \beta_j$, $\alpha_i < x < \beta_i$ and may be written under the form

$$K_{ij}(t, x) = \frac{K_{ij1}(t, x)}{t-x} + K_{ij2}(t, x), \quad i, j = 1, 2, \ldots, m, \tag{5.3}$$

where $K_{ij1}(t, x) \equiv 0$ for $i \neq j$. The functions $K_{ij1,2}(t, x)$ are supposed to be Hölder-continuous with respect to both their variables in the intervals where they are defined and bounded at the end-points of these intervals except for the functions $K_{ij2}(t, x)$ for $i \neq j$ when both their variables tend at the same time to the end-points of the corresponding intervals $[\alpha_j, \beta_j]$ and $[\alpha_i, \beta_i]$, when they may present singularities as strong as $(t-x)^{-1}$. The expressions (5.3) for the kernels $K_{ij}(t, x)$ are those occurring in most practical problems which may be reduced to singular integral equations.

It may be also noted that the singularities of the weight functions $w_j(t)$ at the end-points of the corresponding intervals $[\alpha_j, \beta_j]$ are not arbitrary, but they are taken in such a way that the corresponding functions $\varphi_j(t)$ be bounded at the points α_j and β_j. Under the above-mentioned assumptions, the unknown functions $\varphi_j(t)$ in the system of singular integral equations (5.1) are expected to be Hölder-continuous in the corresponding intervals $[\alpha_j, \beta_j]$. The correct behaviour of the functions $w_j(t)$ near the points α_j and β_j can be found in a way analogous to that used by Erdogan, Gupta and Cook [6].

The main difficulty in solving a system of singular integral equations of the form (5.1) has been up to now the non-existence of effective methods for the numerical expression of the singular integrals involved in it. Now taking into account the development of paragraph 3, in which a method for extending the application of any numerical integration rule to the case of singular integrals was given, and of paragraph 4, in which it was proved that for a proper choice of point x in a singular integral of the form (3.14), this integral may be numerically expressed in exactly the same way, as if it were an ordinary integral, we can approximate the system of singular integral equations (5.1) by the following system of linear equations:

$$\sum_{j=1}^{m} \sum_{k=1}^{n_j} A_{jk} K_{ij}(t_{jk}, x_{ir}) \varphi_j(t_{jk}) = f_i(x_{ir}), \quad r = 1, 2, \ldots, r_{0i}, \; i = 1, 2, \ldots, m. \tag{5.4}$$

In system (5.4), A_{jk} and t_{jk} are the weights and abscissae respectively of the numerical quadrature rule used in the interval $[\alpha_j, \beta_j]$ and x_{ir} are the roots of the functions $q_{in_i}(t)$, associated to this quadrature rule, in accordance with the developments of paragraphs 3 and 4. We can also note that the expressions (5.3) of the kernels $K_{ij}(t, x)$ were also taken into account for the development of system (5.4), as, due to the fact that only the kernels with $i = j$ have Cauchy-type singularities, the points x_{ir} of application of the i-th singular integral equation were selected as the roots of the functions $q_{in_i}(t)$ associated with the numerical integration rule used for the i-th integral in each one of the equations (5.1).

We can further note that the number r_{0i} of points x_{ir} of application of the i-th singular integral equation is equal to $(n_i - 1)$ or n_i or $(n_i + 1)$, where n_i is the number of points used in the numerical integration of the i-th integral. Thus, the total number of linear equations of the system (5.4) may vary from $\Sigma_{i=1}^{m}(n_i-1)$ up to $\Sigma_{i=1}^{m}(n_i+1)$, while the number of unknowns in this system is exactly $\Sigma_{i=1}^{m} n_i$. It is also possible that the functions $\varphi_j(t)$ satisfy conditions of the form

$$(5.5) \qquad \sum_{j=1}^{m} \int_{\alpha_j}^{\beta_j} w_j(t) K_{0hj}(t) \varphi_j(t) = C_h, \quad h=1, 2, \ldots, h_0,$$

where $K_{0hj}(t)$ are Hölder-continuous functions in the corresponding intervals $[\alpha_j, \beta_j]$ without any singularities at the end-points of these intervals and C_h are given constants. Conditions (5.5) may be written, after a numerical approximation of the integrals, under the form

$$(5.6) \qquad \sum_{j=1}^{m} \sum_{k=1}^{n_j} A_{jk} K_{0hj}(t_{jk}) \varphi_j(t_{jk}) = C_h, \quad h=1, 2, \ldots, h_0,$$

in an analogous way to that used for the reduction of the system of singular integral equations (5.1) to the system of linear equations (5.4).

Moreover, some of the functions $\varphi_j(t)$ are possible to have known values at one or both the end-points of the corresponding intervals $[\alpha_j, \beta_j]$. In such a case, it is recommended that, for the numerical expression of integrals in these intervals, a numerical quadrature rule including the end-points where the values of functions $\varphi_j(t)$ are known be used. In this way there results a number, let l_0, of conditions, which reduce the number of unknows in the system of linear equations (5.4) and (5.6) to $\Sigma_{i=1}^{m} n_i - l_0$. In order that a solution of this system of equations is possible, the number of equations must be greater or equal to the number of unknowns. This can be always achieved, especially when quadrature rules not including the end-points of the corresponding intervals among their abscissae are used.

In the case that the number of linear equations is greater than the number of unknowns, some of them should be ignored. Such a case must be avoided by simply using quadrature formulae containing among their abscissae one or both the end-points of the corresponding integration intervals, instead of quadrature formulae not including them between their abscissae. If this technique is used for some integration intervals $[\alpha_j, \beta_j]$, it will be possible to obtain exactly the same number of linear equations and unknowns. Of course, in no way it is permitted to neglect anyone of conditions (5.5), which generally result from physical considerations and are of the same importance with the singular integral equations (5.1). On the contrary, one may neglect to take into account the values taken by functions $\varphi_j(t)$ at the end-points of the corresponding integration intervals, even if one does know them, but this method of equalization of the number of linear equations to the number of unknowns is not advised, because there is no approximation in these neglected values.

6. The Gauss, Radau and Lobatto Quadrature Formulae Applied to Cauchy-Type Singular Integrals and the Corresponding Integral Equations. For the numerical solution of a system of singular integral equations, it is recommended to express the integrals using quadrature rules of high accuracy, so that the number of abscissae n of the quadrature rule be small but the accuracy in the approximate expressions of the integrals be satisfactory. In this way, the number of linear equations we will have to solve will not be too large. Kantorovich and Krylov [1] proposed that the Gauss quadrature formula be used in general, as it is accurate in the case of ordinary integrals for integrands polynomials up to the degree $2n-1$. We may add that, when we want to have among the abscissae used in some numerical integration formula one of the ends of the corresponding integration interval, either because we know the value of the unknown function in the integral we want to approximate at these points, or because we would like to know it as much accurately as possible at these points, the use of the Radau quadrature formula is the best possibility, as it is accurate for ordinary integrals for integrands polynomials up to the degree $2n-2$. Finally, in the case we want to have among the abscissae used both ends of the integration interval, the use of the Lobatto quadrature formula is the best choice, as it is accurate for ordinary integrals for integrands polynomials up to degree $2n-3$.

Taking into account the development of paragraph 3, we can find out that, for singular integrals of the form (3.14), the above-mentioned Gauss, Radau and Lobatto quadrature rules are accurate for integrands polynomials up to the degree $(2n)$, $(2n-1)$ and $(2n-2)$ respectively.

We can further note that, if $p_n(z)$ is the system of orthogonal polynomials associated with some specific form of the above-mentioned quadrature formulae (i. e. related to a specific integration interval and weight function), then the corresponding polynomials $\sigma_n(z)$ will be given by:

$$\sigma_n(z)=kp_n(z) \tag{6.1}$$

for the Gauss quadrature formula,

$$\sigma_n(z)=k[p_n(z)+c_c p_{n-1}(z)],\quad c_c=-\frac{p_n(c)}{p_{n-1}(c)},\quad c=\alpha \text{ or } \beta, \tag{6.2}$$

for the Radau quadrature formula, the value of the constant c_c depending on the end-point (α or β) of the integration interval included among the points used in this formula, and :

$$\sigma_n(z)=k\{p_n(z)+cp_{n-1}(z)+dp_{n-2}(z)\}, \tag{6.3a}$$

$$c=-\frac{p_n(\alpha)p_{n-2}(\beta)-p_n(\beta)p_{n-2}(\alpha)}{p_{n-1}(\alpha)p_{n-2}(\beta)-p_{n-1}(\beta)p_{n-2}(\alpha)},$$

$$d=+\frac{p_n(\alpha)p_{n-1}(\beta)-p_n(\beta)p_{n-1}(\alpha)}{p_{n-1}(\alpha)p_{n-2}(\beta)-p_{n-1}(\beta)p_{n-2}(\alpha)} \tag{6.3b}$$

for the Lobatto quadrature formula, where k is an arbitrary constant. These formulae were obtained according to the method proposed by M. Bouzitat [18].

Further, it is possible to find the corresponding system of functions $q_n(z)$, defined by (3.7) for the Gauss, Radau and Lobatto quadrature formulae. It may be noted that, once the system of these functions is found for the Gauss quadrature formula (by simply replacing in (3.7) the polynomial

$\sigma_n(z)$ by $kp_n(z)$, according to (6.1)), then the systems of these functions for the Radau and Lobatto quadrature formulae can be found in a direct way, because of the linearity of (6.2) and (6.3a).

The roots of these functions $q_n(z)$ are the points of application of those singular integral equations of the system (5.1) for which the numerical integration formula corresponding to these functions was used for the approximate expression of the Cauchy-type integral occurring in them, independently of which integration formulae were used for the approximate expression of the ordinary integrals occurring in the same integral equations.

In general, after the solution of the system of linear equations approximating a system of singular integral equations of the form (5.1) and the determination of the unknown functions $\varphi_j(t)$ at the abscissae t_{jk} in the corresponding integration intervals $[\alpha_j, \beta_j]$ used for the numerical integration in these intervals, we have to find expressions giving these unknown functions $\varphi_j(t)$ along the intervals $[\alpha_j, \beta_j]$ and not only at the points t_{jk}. This can be generally achieved by using the methods of interpolation. Here only the cases when the Gauss, Radau and Lobatto formulae are used as integration rules in intervals $[\alpha_j, \beta_j]$ will be dealt with.

Let us consider at first the case of the Gauss quadrature formula. We have generally to find an approximate expression for a function $\varphi(t)$, the values of which are known at the points t_k $(k=1, 2, \ldots, n)$, roots of the polynomial $p_n(z)$ of the system of orthogonal polynomials associated with the Gauss quadrature formula used. To express the function $\varphi(t)$ along the corresponding interval $[\alpha, \beta]$, we assume it to have a polynomial form of degree $(n-1)$ as follows:

$$\varphi(t) = \sum_{i=0}^{n-1} c_i p_i(t), \tag{6.4}$$

where c_i are coefficients, which can be determined according to the following formula given by D. Paget and D. Elliott [19]:

$$c_i = \frac{1}{h_i} \sum_{k=1}^{n} A_k p_i(t_k)\varphi(t_k), \quad i=0, 1, \ldots, n-1, \tag{6.5}$$

where A_k are the weight coefficients corresponding to abscissae t_k and h_i are constants given by

$$h_i = \int_\alpha^\beta w(t) p_i^2(t) dt \tag{6.6}$$

with $w(t)$ the weight function.

In the case of use of the Radau quadrature formula, it can be shown that (6.4) and (6.5) remain valid. The same is also true for the case of the Lobatto quadrature formula, except of the fact that the coefficient c_i for $i=n-1$ must be determined by

$$c_i = \sum_{k=1}^{n} A_k p_i(t_k)\varphi(t_k) \Big/ \sum_{k=1}^{n} A_k p_i^2(t_k). \tag{6.7}$$

Formula (6.7) may be used for the computation of all the coefficients c_i $(i=0, 1, \ldots, n-1)$ instead of formula (6.5). This is true not only for the

Lobatto method, but also for the Gauss and the Radau methods. In reality, ormula (6.5) results from (6.7) after an application of the numerical inte- fration method under consideration for the integral of the right side of (6.6).

7. **On the Numerical Solution of Singular Integral Equations of the Second Kind**. The method of numerical solution of systems of singular integral equations of the first kind developed in paragraph 5 can be extended so as to apply to systems of singular integral equations of the second kind too. However, such an extension causes the method to become more complicated as regards the selection of points x_{ir} of application of the singular integral equations.

We will consider in the present development only one singular integral equation of the second kind of the form:

$$A(x)w(x)\varphi(x)+B(x)\int_{\alpha}^{\beta}\frac{w(t)\varphi(t)}{t-x}dt+\int_{\alpha}^{\beta}w(t)k(t,x)\varphi(t)dt=f(x), \tag{7.1}$$

as the extension of the method which will be presented here to systems of such equations is obvious after the development of paragraph 5.

In (7.1), all the given functions are assumed to be generalized Hölder-continuous inside the integration interval $[\alpha, \beta]$. At the ends α, β of this interval the functions $A(x)$ and $B(x)$ are assumed bounded and different from zero, the functions $w(x)$ and $f(x)$ are assumed to present singularities, as in paragraph 5, and the kernel $k(t, x)$ is assumed to have the same behaviour as the kernels $K_{ij2}(t, x)$ in paragraph 5. It may be noted that the singularities of the weight function $w(x)$ should be consistent with that of the function $f(x)$, depending also on the functions $A(x)$ and $B(x)$ and the integration interval $[\alpha, \beta]$, in a way that the unknown function $\varphi(t)$ be Hölder-continuous along the whole integration interval $[\alpha, \beta]$ without any singularity at its end-points.

Now, applying a method of numerical integration to the second integral of the left side of (5.1), which is an ordinary integral, with $x \neq \alpha$, β and the generalized form of the same method to the first integral of the left side of this equation, which is a singular integral, according to formula (3.15), we find the following approximate expression for (7.1):

$$A(x)w(x)\varphi(x)+B(x)\left[\sum_{k=1}^{n}A_k\frac{\varphi(t_k)}{t_k-x}-2\varphi(x)\frac{q_n(x)}{\sigma_n(x)}\right]+\sum_{k=1}^{n}A_kk(t_k,x)\varphi(t_k)=f(x), \tag{7.2}$$

where t_k are the abscissae and A_k the weights relative to the numerical integration method with integration interval $[\alpha, \beta]$ and weight function $w(x)$ in use.

From (7.2), it is obvious that the best selection of points x_r of application of this equation, in order that it is reduced to a system of linear equations, is that these points be the roots of the following transcendental equation:

$$\frac{2q_n(x)}{w(x)\sigma_n(x)}=\frac{A(x)}{B(x)}. \tag{7.3}$$

In the case examined in paragraph 5, when $A(x)\equiv 0$, these points were simply the roots of the function $q_n(x)$. The number and distribution of these roots were investigated in paragraph 4. An analogous investigation for the roots of (7.3) reveals that this equation has at least $(n-1)$ roots and at maximum $(n+1)$ roots. From these roots, $(n-1)$ lie inside the $(n-1)$ intervals (t_k, t_{k+1}) $(k=1, 2, \ldots, n-1)$ and the last two, if they exist, lie in the intervals (α, t_1) and (t_n, β), supposed that these intervals are not of zero length.

In accordance with this selection of points x_r of application of (7.2), this equation is reduced to the following system of linear equations:

$$\sum_{k=1}^{n} A_k\left[\frac{B(x_r)}{t_k-x_r}+k(t_k, x_r)\right]\varphi(t_k)=f(x_r), \quad r=1, 2, \ldots, r_0, \tag{7.4}$$

where r_0 may be equal to $(n-1)$, n or $(n+1)$. This system of equations may be complemented by one or more conditions resulting from physical considerations, as in the case of systems of singular integral equations considered in paragraph 5.

The disadvantage of the method presented here is that the points x_r of application of the singular integral equation depend not only on the numerical integration formula in use, as in the case of singular integral equations of the first kind, but also on the ratio $A(x)/B(x)$ that is on the form of the singular integral equation itself. This disadvantage is partly eliminated by the fact that in most practically occurring problems which are reduced to singular integral equations both functions $A(x)$ and $B(x)$ are constants, when the points $x_{r'}$, once found for some problem, can be used for all cases of this problem when only the kernel $k(t, x)$ and the right side function $f(x)$ change.

Another possibility would be to use as points x_r the roots of functions $q_n(x)$, as if $A(x)$ were identical to zero, and then to express the values $\varphi(x_r)$ of the unknown function at these points by interpolation through the values $\varphi(t_k)$, which will finally be the unknowns in the system of linear equations resulting from (7.2). Nevertheless, such an approach has the serious disadvantage that the interpolation used reduces the accuracy of the quadrature formula especially when this formula is a very accurate one.

8. Application of Usually Used Numerical Integration Rules to the Solution of Singular Integral Equations. In this paragraph we will consider the most generally used numerical integration rules of the Gauss, Radau and Lobatto type and we will determine the corresponding functions $q_n(z)$, the roots of which should be taken as the points of application of singular integral equations of the first kind, as we have seen in paragraph 5, or which are involved, together with the corresponding polynomials $\sigma_n(z)$ in (7.3), the roots of which should be taken as the points of application of singular integral equations of the second kind.

Every numerical integration rule is characterized by the type of it (that is Gauss, Radau or Lobatto for the rules considered in this paragraph), as well as by the name of the corresponding orthogonal polynomials (as Legendre, Chebyshev, Jacobi, Hermite, Laguerre etc.).

I) The Gauss-Legendre numerical integration rule. This rule is applicable to integrals with integration interval $[-1, 1]$ and weight function $w(t)=1$. The system of associated orthogonal polynomials is the system of Legendre polynomials $P_n(z)$, when the functions $\sigma_n(z)$ and $q_n(z)$ are given by formulae

$$\sigma_n(z)=p_n(z), \quad q_n(z)=Q_n(z), \tag{8.1}$$

where $Q_n(z)$ are the Legendre functions of the second kind. These functions have $(n+1)$ roots inside the integration interval $[-1, 1]$ alternating with the roots of Legendre polynomials according to a theorem reported by G. Szegö [20]. As the functions $Q_n(z)$ are either even or odd, their roots appear by pairs of opposite roots, except the root $x=0$, when the point 0 is a root of function $Q_n(z)$.

Table of roots of the Legendre functions $Q_n(z)$ and $Q'_{n-1}(z)$ for $n=2(1)11$

x_k $(Q_n(\pm x_k)=0)$	x_k $(Q'_{n-1}(\pm x_k)=0)$	x_k $(Q_n(\pm x_k)=0)$	x_k $(Q'_{n-1}(\pm x_k)=0)$
$n=2$		$n=8$	
0.000000	0.000000	0.000000	0.000000
0.937612		0.360623	0.409449
$n=3$		0.672548	0.748081
0,429707	0.623175	0.893670	0.957206
0.967802		0.994486	
$n=4$		$n=9$	
0.000000	0.000000	0.164368	0.184704
0.639003	0.805405	0.475294	0.529164
0.980429		0.734720	0.802140
$n=5$		0.914550	0.966663
0.280595	0.348370	0.995584	
0.752756	0.881699	$n=10$	
0.986867		0.000000	0.000000
$n=6$		0.294423	0.326049
0.000000	0.000000	0.562687	0.616763
0.463374	0.547340	0.780957	0.840627
0.820643	0.920609	0.929855	0.973300
0.990584		0.996384	
$n=7$		$n=11$	
0.207455	0.241443	0.136039	0.149549
0.586501	0.668994	0.398027	0.435360
0.864172	0.943076	0.630496	0.682483
0.992921		0.816208	0.868954
		0.941401	0.978137
		0.996985	

In the table we give the roots of the functions $Q_n(z)$ for $n=2(1)11$ and with an accuracy of six decimal digits, because it seems that these roots have not been tabulated up to now.

II) The modified Gauss-Legendre numerical integration rule. This rule is an extension of the above-mentioned Gauss-Legendre numerical integration rule applicable to integrals with integration interval [0, 1] and weight function either $w(t)=1/\sqrt{1-t}$ or $w(t)=1/\sqrt{t}$.

In the first case the functions $\sigma_n(z)$ and $q_n(z)$ are given by

$$\sigma_n(z)=P_{2n}(\sqrt{1-z}), \quad q_n(z)=\frac{1}{\sqrt{1-z}}\, Q_{2n}(\sqrt{1-z}) \tag{8.2a}$$

or by

(8.2b) $$\sigma_n(z)=\sqrt{1-z}P_{2n-1}(\sqrt{1-z}), \quad q_n(z)=Q_{2n-1}(\sqrt{1-z})$$

depending on whether the end-point $t=1$ is not included among the abscissae t_k, or it is included, while in the second case the functions $\sigma_n(z)$ and $q_n(z)$ are given by

(8.3a) $$\sigma_n(z)=P_{2n}(\sqrt{z}), \quad q_n(z)=\frac{1}{\sqrt{z}}\,Q_{2n}(\sqrt{z}),$$

or by

(8.3b) $$\sigma_n(z)=\sqrt{z}P_{2n-1}(\sqrt{z}), \quad q_n(z)=Q_{2n-1}(\sqrt{z})$$

depending on whether the end-point $t=0$ is not included among the abscissae t_k or it is included.

Eqs. (8.2) and (8.3) mean that the roots of functions $q_n(z)$, x'_k are given by

(8.4) $$x'_1=1-x_k^2, \quad x'_k=x_k^2$$

in the first and the second case respectively, where x_k are the roots of the Legendre functions $Q_n(z)$.

It may be also remarked that the modified Gauss-Legendre numerical integration method is either a Radau or a Gauss method depending on whether one of the end-points 0 and 1 of the integration interval is included among the abscissae used, or not.

III) The Lobatto-Legendre numerical integration rule. This rule is applicable to integrals with integration interval $[-1, 1]$ and weight function $w(t)=1$. The system of associated orthogonal polynomials is again the system of Legendre polynomials $P_n(z)$, when functions $\sigma_n(z)$ and $q_n(z)$ are given, because of (6.3), by

(8.5) $$\sigma_n(z)=P_n(z)-P_{n-2}(z)=\frac{2n-1}{n(n-1)}(z^2-1)P'_{n-1}(z),$$
$$q_n(z)=Q_n(z)-Q_{n-2}(z)=\frac{2n-1}{n(n-1)}(z^2-1)Q'_{n-1}(z).$$

The number of useful roots of functions $q_n(z)$ is $(n-1)$, as their roots ± 1 coincide with the ends of the integration interval $[-1,1]$ and cannot be used for the numerical solution of singular integral equations. In the Table we give the roots of functions $q_n(z)$, except ± 1, for $n=2(1)11$ and with an accuracy of six decimal digits.

IV) The modified Lobatto-Legendre numerical integration rule. This rule is an extension of the above-mentioned Lobatto-Legendre numerical integration rule, in the same way as the modified Gauss-Legendre numerical integration rule is an extension of the Gauss-Legendre numerical integration rule. The integration interval is $[0, 1]$ and the weight function either $w(t)=1/\sqrt{1-t}$ or $w(t)=1/\sqrt{t}$.

In the first case the functions $\sigma_n(z)$ and $q_n(z)$ are given by

$$\sigma_n(z) = P_{2n}(\sqrt{1-z}) - P_{2n-2}(\sqrt{1-z}) - \frac{4n-1}{2n(2n-1)} zP'_{2n-1}(\sqrt{1-z}),$$

(8.6a)
$$q_n(z) = \frac{1}{\sqrt{1-z}} \{Q_{2n}(\sqrt{1-z}) - Q_{2n-2}(\sqrt{1-z})\}$$
$$- \frac{1}{\sqrt{1-z}} \frac{4n-1}{2n(2n-1)} zQ'_{2n-1}(\sqrt{1-z}),$$

or by

$$\sigma_n(z) = \frac{4n-1}{2n}\sqrt{1-z}\{P_{2n-1}(\sqrt{1-z}) - P_{2n-3}(\sqrt{1-z})\}$$

(8.6b)
$$- \frac{(4n-1)(4n-3)}{4n(n-1)(2n-1)} \sqrt{1-z}\, zP'_{2n-2}(\sqrt{1-z}),$$

$$q_n(z) = \frac{4n-1}{2n}\{Q_{2n-1}(\sqrt{1-z}) - Q_{2n-3}(\sqrt{1-z})\} - \frac{(4n-1)(4n-3)}{4n(n-1)(2n-3)} zQ'_{2n-2}\sqrt{1-z})$$

depending on whether the end-point $t = 1$ is not included among the abscissae $t_{k'}$ or it is included, while, in the second case, the functions $\sigma_n(z)$ and $q_n(z)$ are given by

(8.7a)
$$\sigma_n(z) = P_{2n}(\sqrt{z}) - P_{2n-2}(\sqrt{z}) - \frac{4n-1}{2n(2n-1)}(1-z)P'_{2n-1}(\sqrt{z}),$$
$$q_n(z) = \frac{1}{\sqrt{z}}\{Q_{2n}(\sqrt{z}) - Q_{2n-2}(\sqrt{z})\} - \frac{1}{\sqrt{z}}\frac{4n-1}{2n(2n-1)}(1-z)Q'_{2n-1}(\sqrt{z})$$

or by

(8.7b)
$$\sigma_n(z) = \frac{4n-1}{2n}\sqrt{z}\{P_{2n-1}(\sqrt{z}) - P_{2n-3}(\sqrt{z})\} - \frac{(4n-1)(4n-3)}{4n(n-1)(2n-1)}\sqrt{z}(1-z)P'_{2n-2}(\sqrt{z}),$$
$$q_n(z) = \frac{4n-1}{2n}\{Q_{2n-1}(\sqrt{z}) - Q_{2n-3}(\sqrt{z})\} - \frac{(4n-1)(4n-3)}{4n(n-1)(2n-3)}(1-z)Q'_{2n-2}(\sqrt{z}),$$

depending on whether the end-point $t = 0$ is not included among the abscissae t_k or it is included.

Equations (8.6) and (8.7) mean that the roots of functions $q_n(z)$, $x'_{k'}$ are given by formulae (8.4) in the first and the second case respectively, where x_k are the roots of the corresponding functions $q_n(z)$ of the second of (8.5).

It may be also remarked that the modified Lobatto-Legendre numerical integration method is either a Lobatto or a Radau method depending on whether both the end-points 0 and 1 of the integration interval are included among the abscissae used or only one of them is.

V) The Gauss-Chebyshev numerical integration rule. This rule is applicable to integrals with integration interval $[-1, 1]$ and weight function $w(t) = 1/\sqrt{1-t^2}$. The system of associated orthogonal polynomials is the system of Chebyshev polynomials of the first kind $T_n(z)$, while the functions $q_n(z)$ are proportional to the Chebyshev functions of the second kind $U_{n-1}(z)$ [13] and take the form of the Chebyshev polynomials of the second kind $U_{n-1}(t)$ on the integration interval $[-1, 1]$ because of Eq. (3.13). That is

(8.8) $$\sigma_n(z) = T_n(z), \quad q_n(z) = -\frac{\pi}{2} U_{n-1}(z), \quad U_{-1}(z) \equiv 0.$$

It may be further noted that the roots x_k of polynomials $U_{n-1}(t)$, that is the roots of functions $q_n(z)$ are given by

(8.9) $$x_k = \cos(k\pi/n), \quad k = 1, 2, \ldots, n-1.$$

VI) The Lobatto-Chebyshev numerical integration rule. This rule is also applicable to integrals with integration interval $[-1, 1]$ and weight function $w(t) = 1/\sqrt{1-t^2}$. Because of Eq. (6.3), the functions $\sigma_n(z)$ and $q_n(z)$ are given by:

(8.10) $$\sigma_n(z) = T_n(z) - T_{n-2}(z) = 2(z^2-1)U_{n-2}(z),$$
$$q_n(z) = -\frac{\pi}{2}\{U_{n-1}(z) - U_{n-3}(z)\} = -\pi T_{n-1}(z),$$

when the roots x_k of the functions $q_n(z)$ are given by

(8.11) $$x_k = \cos\frac{(2k-1)\pi}{2\,n-1)}, \quad k = 1, 2, \ldots, n-1.$$

VII) The Gauss-Jacobi numerical integration rule. This rule is applicable to integrals with integration interval $[-1, 1]$ and weight function $w(t) = (1-t)^a(1+t)^b$ $(a, b > -1)$. The system of associated orthogonal polynomials is the system of Jacobi polynomials $P_n^{(a,b)}(z)$ and the functions $q_n(z)$ are related to the Jacobi functions of the second kind $Q_n^{(a,b)}(z)$ [13]. That is

(8.12) $$\sigma_n(z) = P_n^{(a,b)}(z), \quad q_n(z) = (z-1)^a(z+1)^b Q_n^{(a,b)}(z).$$

The roots x_k of the functions $q_n(z)$ should be taken as the points of application of a corresponding singular integral equations, although such a selection is not in agreement with the developments of Erdogan, Gupta and Cook [6], which are, in some way, unjustified. Erdogan, Gupta and Cook propose that these points x_k be selected as roots of a proper Jacobi polynomial.

VIII) The Gauss-Laguerre numerical integration rule. This rule is applicable to integrals with integration interval $[0, \infty)$ and weight function $w(t) = e^{-t}$. The system of associated orthogonal polynomials is the system of Laguerre polynomials $L_n(z)$, when the functions $\sigma_n(z)$ and $q_n(z)$ are given by

(8.13) $$\sigma_n(z) = L_n(z), \quad q_n(z) = -\frac{1}{2}\int_0^\infty e^{-t}\frac{L_n(t)}{t-z}\,dt.$$

It can be easily shown that the functions $q_n(z)$ satisfy the recurrence relations for the Laguerre polynomials

(8.14) $$q_1(z) = (1-z)q_0(z), \quad n\sigma_n(z) = (2n-1-z)q_{n-1}(z) - (n-1)q_{n-2}(z),$$

while $q_0(z)$ may be expressed as

(8.15) $$q_0(z)=\frac{1}{2}e^{-z}\,\mathrm{Ei}(z),\qquad \mathrm{Ei}(z)=-\int_{-z}^{\infty}\frac{e^{-t}}{t}\,dt=\int_{-\infty}^{z}\frac{e^{t}}{t}\,dt.$$

The integral $\mathrm{Ei}(z)$ is the well-known exponential integral. It may be also shown that the functions $q_n(z)$ are confluent hypergeometric functions for the Gauss-Laguerre rule, while the corresponding functions for the Gauss-Jacobi rule (whose special cases are the Gauss-Legendre and the Gauss-Chebyshev rules) are hypergeometric functions.

IX) The Gauss-Hermite numerical integration rule. This rule as applic able to integrals with integration interval $(-\infty, \infty)$ and weight function $w(t)=e^{-t^2}$. The system of associated orthogonal polynomials is the system of Hermite polynomials $H_n(z)$, when the functions $\sigma_n(z)$ and $q_n(z)$ are given by

(8.16) $$\sigma_n(z)=H_n(z),\quad q_n(z)=-\frac{1}{2}\int_{0}^{\infty}e^{-t^2}\frac{H_n(t)}{t-z}\,dt.$$

It can be easily shown that functions $q_n(z)$ satisfy the recurrence relations for the Hermite polynomials:

(8.17) $$q_1(z)=2zq_0(z),\quad q_n(z)=2zq_{n-1}(z)-2(n-1)q_{n-2}(z),$$

while $q_0(z)$ may be expressed as

(8.18) $$q_0(z)=-\pi i w(z)\ (\mathrm{Im}\,z>0),\ q_0(z)=\pi i w(-z)(\mathrm{Im}\,z<0),$$
$$q_0(z)=2\sqrt{\pi}F(z)\ (\mathrm{Im}\,z=0),$$

where $w(z)$ is the complex error function and $F(z)$ the Dawson integral. These two functions are defined as

(8.19) $$w(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z-t}\,dt,\qquad F(x)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{x-t}\,dt.$$

It may be also shown that the functions $q_n(z)$ are confluent hypergeometric functions, as in the case of the Gauss-Laguerre numerical integration rule.

X) The modified Gauss-Hermite numerical integration rule. This rule is an extension of the above-mentioned Gauss-Hermite numerical integration rule applicable to integrals with integration interval $[0, \infty)$ and weight function $w(t)=1/\sqrt{t}$. The functions $\sigma_n(z)$ and $q_n(z)$ are given by

(8.20a) $$\sigma_n(z)=H_{2n}(\sqrt{z}),\quad q_n(z)=\frac{1}{\sqrt{z}}q^*_{2n}(\sqrt{z})$$

or by

(8.20b) $$\sigma_n(z)=\sqrt{z}H_{2n-1}(\sqrt{z}),\quad q_n(z)=q^*_{2n-1}(\sqrt{z}),$$

where $q^*_n(z)$ denote the functions $q_n(z)$ for the previously studied Gauss-Hermite method. When formulae (8.20a) are used, the end-point $t=0$ of the integration interval is not included among the abscissae t_k and the numerical integration rule considered is in reality a Gauss rule. On the contrary, when formulae (8.20b) are used, the end-point $t=0$ of the integration interval is included among the abscissae t_k and the numerical integration rule considered is in reality a Radau rule.

9. Applications to Crack Problems in Plane Elasticity. The method of numerical solution of singular integral equations presented here may be successfully applied to a lot of problems of Mathematical Physics that could be reduced to singular integral equations. In this paragraph we will consider only crack problems ocurring in the Theory of Plane Elasticity for isotropic media.

Consider a complicated crack L (Fig. 1), possibly branched and with more than two tips, in an infinite isotropic elastic medium. If the loading at infinity as well as on both edges $(+)$ and $(-)$ of the crack (the symbols $(+)$ and $(-)$ assigned to them in an arbitrary way) are known, then this problem can be reduced to a singular integral equation of the form

$$(9.1)\qquad \frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\tau-t}\,d\tau - \frac{1}{\pi i}\int_L \frac{\overline{\varphi(\tau)}}{\overline{\tau}-\overline{t}}\,d\overline{\tau}$$

$$\frac{d\overline{t}}{dt}\left\{\frac{1}{\pi i}\int_L \frac{\omega(\tau)}{\tau-t}\,d\tau + \frac{1}{\pi i}\int_L \varphi(\tau)\frac{\overline{\tau}-\overline{t}}{(\tau-t)^2}\,d\tau\right\} = f(t),$$

where $f(t)$ is a known complex function and $\varphi(t)$ the unknown complex function along the crack [N. Ioakimidis, 21]. This function $\varphi(t)$ may be considered as representing the dislocation density along the composite crack. Moreover, it presents singularities of the order of $(-1/2)$ at the crack tips and, possibly, weaker power or logarithmic singularities at the branch points, corner points and points of loading discontinuities. These singularities may be found by the general method proposed by P. Theocaris [22] for wedges and are supposed to be known.

Under these conditions, the complex singular integral equation (9.1) may be reduced to a system of real singular integral equations of a number equal at maximum to the double of the number of the branches of the composite crack, in which it can be divided, so that no irregular point of geometry or loading exists in each one of them.

Several plane crack problems have been solved by reduction to singular integral equations or systems of such equations. These equations, resulting from (9.1), are of the first kind and can be solved by the numerical method proposed in paragraph 5 and using the numerical integration rules considered in paragraph 8. Among these problems we can quote the problems of periodic collinear cracks, periodic parallel cracks and doubly-periodic arrays of cracks (solved by using the Gauss or Lobatto-Chebyshev numerical integration rule), the problem of a star-shaped crack with branches of equal lengths and symmetrically oriented (solved by using the modified Gauss or Lobatto-Legendre numerical integration rule), the problem of a cruciform crack with equal or unequal arms (solved by using either the Gauss or Lobatto-Chebyshev numerical integration rule or the modified Gauss or Lobatto-Legendre numerical integration rule), the problem of an edge crack in a half-plane (solved by using both the Gauss-Laguerre and modified Gauss or Lobatto-Legendre numerical integration rules), the problem of a simple curvilinear crack in an infinite medium (solved by using the Gauss or Lobatto-Chebyshev numerical integration rule), the problem of a branched crack consisting of a main crack and two or more branches with equal or unequal lengths and arbitrary orientation (solved by using either both the modified Gauss and Lobatto-Legendre numerical integration rules or only the modified Lobatto-Legendre numerical integration rule) and, finally, the problem of parallel semi-infinite periodic cracks (solved by using the modified Gauss-Hermite numerical integration rule).

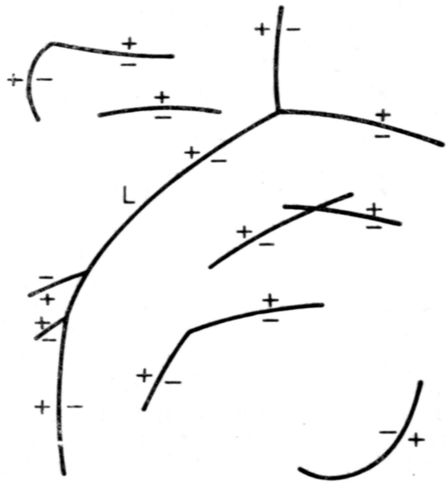

Fig. 1. Crack of a complicated shape in an infinite medium

In some of these problems, the condition of single-valuedness of displacements was taken into account, complementing (9.1). The problems, where it was not taken into account, were those problems where the symmetry assured that this condition was satisfied. Also, in the problem of a branched crack, when solved by using only the modified Lobatto-Legendre numerical integration rule, special conditions resulting from physical considerations at the common point of the branches were also taken into consideration, according to the procedures developed in paragraph 5.

It can be also noted that, when we are interested in the evaluation of the stress intensity factors at the tips of the cracks, then the Lobatto rules including between their abscissae these points should be preferred over the corresponding Gaussian rules, because, in this way, no extrapolation, introducing errors, is needed for the evaluation of the stress intensity factors. The values of the stress intensity factors found by the numerical solution of (9.1) together with the condition of single-valuedness of displacements, were found to be in satisfactory agreement with their values estimated theoretically or by numerical methods or by experimental techniques like the method of caustics, developed by P. Theocaris [23-25]. This fact proves the effectivenes of the method of solution of singular integral equations presented here.

Moreover, except of the first fundamental problem for cracks in plane isotropic elastic media, the cases of the second and mixed fundamental problems can be also reduced to singular integral equations [21]. Although the equation for the second fundamental problem is quite similar to (9.1), valid for the first fundamental problem, the equation for the mixed fundamental problem is more complicated and contains also the unknown function $\varphi(t)$ as a free term. In a similar manner, the case of more complex crack problems in isotropic media, as well as of crack problems in anisotropic media can be reduced to singular integral equations [21], a numerical solution of which can be found by the method developed here.

10. Discussion and Generalizations. The method of numerical solution of singular integral equations developed here can be successfully applied to all problems of Mathematical Physics which can be reduced to singular integral equations. The only difficulty in such an application is that the points used as points of application of the singular integral equations, that is the roots of functions $q_n(z)$, or (7.3), should be computed before the solution of the singular integral equation. Nevertheless, the roots of functions $q_n(z)$, depending only on the numerical integration rule in use and not on the singular integral equation to which this rule is applied, can be computed once, tabulated and then used whenever needed, considered associated with the quadrature rule, like abscissae and weights. The same would be possible for some standard and frequently used forms of (7.3).

The problem in these considerations is that functions $q_n(z)$ are usually transcendental, difficult to compute functions and the computation of their roots is not a very simple matter. Up to now, we have tried the computation of the roots of $q_n(z)$ only in the case of rules associated with the Legendre polynomials with an accuracy of twenty-six digits and we hope to be able to compute the roots of other forms of $q_n(z)$ as well.

Further, extension of the method considered here for singular integral equations along closed curves is a problem which can be studied, as well as the estimation of the error of the solution of a singular integral equation, although the error analysis is quite the same as for Fredholm integral equations and can be considered as known.

Finally, it is believed that the method proposed here will be applied in future to other problems, except crack problems, with, we hope, an equal success.

REFERENCES

1. L. V. Kantorovich, V. I. Krylov. Approximate methods of higher analysis. Groningen, 1958.
2. B. B. Иванов. Приближенное решение сингулярных интегральных уравнений в случае разомкнутых контуров интегрирования. *Доклады АН СССР*, **111**, 1956, 933-936.
3. А. И. Каландия. О приближенном решении одного класса сингулярных интегральных уравнений. *Доклады АН СССР*, **125**, 1959, 715—718.
4. F. Erdogan. Approximate solutions of systems of singular integral equations. *SIAM J. Appl. Math.*, **17**, 1969, 1041-1059.
5. F. Erdogan, G. D Gupta. On the numerical solution of singular integral equations. *Quart. Appl. Math.*, **30**, 1972, 525-534.
6. F. Erdogan, G. D. Gupta, T. S. Cook. Numerical solution of singular integral equations. In: Methods of Analysis and Solutions of Crack Problems. Leyden, 1973, 368-425.
7. P. S. Theocaris, N. I. Ioakimidis. Numerical integration methods for the solution of singular integral equations. *Quart. Appl. Math.* **34**, 1976 (to appear).
8. Л. Н. Карпенко. Приближенное решение одного сингулярного интегрального уравнения при помощи многочленов Якоби. *Прикладная мат. и мех.*, **30**, 1966, 564-569.
9. S. Krenk. On quadrature formulas for singular integral equations of the first and the second kind. *Quart. Appl. Math.*, **33**, 1975, 225-232.
10. S. R. Majumdar. The numerical solution of singular integral equations. *J. Math and Phys. Sci.*, **7**, 1973, 415-439.
11. S. R. Majumdar. Note on the numerical solution of singular Fredholm linear integral equations. *Z. Angew. Math. u. Mach.*, **54**, 1974, 515-517.

12. D. B. Hunter. Some Gauss-type formulae for the evaluation of Cauchy principal values of integrals. *Num. Math.*, **19**, 1972, 419-424.
13. M. M. Chawla, T. R. Ramakrishnan. Modified Gauss-Jacobi quadrature formulas for the numerical evaluation of Cauchy type singular integrals. *BIT*, **14**, 1974, 14-21.
14. H. Takahasi, M. Mori. Error estimation in the numerical integration of analytic functions. *Reports of the Computer Centre, Univ. of Tokyo*, **3**, 1970, 41-108.
15. H. Takahasi, M. Mori. Estimation of errors in the numerical quadrature of analytic functions. *Applicable Analysis*, 1, 1971, 201-229.
16. J. D. Donaldson, D. Elliott. A unified approach to quadrature rules with asymptotic estimates of their remainders. *SIAM J. Numer. Anal.*, **9**, 1972. 573-602.
17. F. D. Gakhov. Boundary Value Problems. Pergamon Press, Oxford, 1966.
18. M. J. Bouzitat. Intégration numérique approchée par la méthode de Gauss généralisée et extension de cette méthode. In: Mineur Techniques de Calcul Numérique. Paris, 1952.
19. D. F. Paget, D. Elliott. An algorithm for the numerical evaluation of certain Cauchy principal value integruls. *Num. Mat.*, **19**, 1972, 373-385.
20. G. Szegö. Orthogonal Polynomials. New York, 1959.
21. N. I. Ioakimidis. General methods for the solution of crack problems in the theory of plane elasticity (Thesis), Athens, 1976.
22. P. S. Theocaris. The order of singularity at a multiwedge corner of a composite plate. *Intern. J. Eng. Sci.*, **12**, 1974, 107-120.
23. P. S. Theocaris. Local yielding around a crack-tip in plexiglas. *J. Appl. Mech.*, **37**, 1970, 409-415.
24. P. S. Theocaris. New methods based on geometric optics for the solution of fracture mechanics problems. *Technika Chronika*, **41**, 1972, 145-153.
25. P. S. Theocaris, C. H. Blonzou. Symmetric branching of cracks in PMMA (plexiglas), *Materialprüfung*, **15**, 1973, 123-130.

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