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Българско математическо списание

EXPANSION OF COMPLEX FUNCTIONS ANALYTIC IN A STRIPE IN SERIES OF HERMITE FUNCTIONS OF SECOND KIND

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Sufficient conditions are given for a complex function analytic in a stripe to be represented by series in Hermite functions of second kind.

It is well known that the system of Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$ [1, 5.5] is a solution of the difference equation

(1) $$y_{n+1}-2zy_n+2ny_{n-1}=0.$$

The system of Hermite functions of second kind $\{K_n(z)\}_{n=0}^{\infty}$ is defined by the equalities

(2) $$K_n(z)=-\int_{-\infty}^{\infty}\frac{\exp(-t^2)H_n(t)}{t-z}\,dt,\quad n=0,1,2,\ldots,$$

provided that $z\in \mathbb{C}\setminus(-\infty,+\infty)$. It is not difficult to prove that the system $\{K_n(z)\}_{n=0}^{\infty}$ is also a solution of the equation (1). Further, from (2) and the property $H_n(-z)=(-1)^nH_n(z)$ of Hermite polynomials follows that $K_{2n}(z)$ is an odd function and that $K_{2n+1}(z)$ is an even function.

Let us note that the equality (2) defines for every $n=0,1,2,\ldots$ two analytic functions respectively in the half-planes $H^+=\{z\in \mathbb{C}\setminus \operatorname{Im} z>0\}$ and $H^-=\{z\in \mathbb{C}\setminus \operatorname{Im} z<0\}$. In order to make difference between these two functions we accept the following notations, namely $K_n^+(z)=K_n(z)$ for $z\in H^+$ and $K_n^-(z)=K_n(z)$ for $z\in H^-$.

If $0<\tau_0<+\infty$, we define further for $n=0,1,2,\ldots$ and $z\in S(\tau_0)=\{z\in \mathbb{C}\setminus |\operatorname{Im} z|<\tau_0\}$ the functions

(3) $$K_{2n}^{(1)}(z;\,\tau_0)=K_{2n}^+(z+i\tau_0)-K_{2n}^-(z-i\tau_0)$$

resp.

(4) $$K_{2n+1}^{(1)}(z;\tau_0)=K_{2n+1}^+(z+i\tau_0)+K_{2n+1}^-(z-i\tau_0).$$

In this way we introduce a system of complex functions $\{K_n^{(1)}(z;\tau_0)\}_{n=0}^{\infty}$ analytic in the stripe $S(\tau_0)$ and such that $K_n^{(1)}(z;\tau_0)$ is an even function in this stripe for every $n=0,1,2.\ldots$

Further we define

(5) $$K_{2n}^{(2)}(z;\tau_0)=K_{2n}^+(z+i\tau_0)+K_{2n}^-(z-i\tau_0)$$

SERDICA Bulgaricae mathematicae publicationes. Vol. 2, 1976, p. 277—282.

resp.

$$K_{2n+1}^{(2)}(z;\tau_0)=K_{2n+1}^{+}(z+i\tau_0)-K_{2n+1}^{-}(z-i\tau_0) \tag{6}$$

and get a system of complex functions $\{K_n^{(2)}(z;\tau_0)\}_{n=0}^{\infty}$ analytic in the stripe $S(\tau_0)$ and such that $K_n^{(2)}(z;\tau_0)$ is an odd function in this stripe for every $n=0, 1, 2, \ldots$

The main result of the paper is the following

Theorem 1. *Let* $0<\tau_0<+\infty$ *and* $f(z)$ *be a complex function satisfying the following conditions:*

(a) $f(z)$ *is analytic in the stripe* $S(\tau_0)$;

(b) *for every* τ, $0\leq\tau<\tau_0$, *there exists a* $\mu(\tau)>0$ *such that*

$$f(z)=O\{|z|^{-\mu(\tau)}\exp(-z^2)\} \text{ if } z\to\infty \text{ and } z\in \bar{S}(\tau)=\{z\in \mathrm{C}: |\operatorname{Im} z|\leq\tau\}.$$

Then $f(z)$ *can be represented in the stripe* $S(\tau_0)$ *by a series of the kind*

$$f(z)=\sum_{n=0}^{\infty}\{a_n^{(1)}(f)K_n^{(1)}(z;\tau_0)+a_n^{(2)}(f)K_n^{(2)}(z;\tau_0)\} \tag{7}$$

with coefficients

$$a_n^{(1)}(f)=-\frac{1}{4\pi i\, I_n}\int_{-\infty}^{\infty}\{H_n(t+i\tau_0)+(-1)^nH_n(t-i\tau_0)\}f(t)dt \tag{8}$$

and

$$a_n^{(2)}(f)=-\frac{1}{4\pi i\, I_n}\int_{-\infty}^{\infty}\{H_n(t+i\tau_0)-(-1)^nH_n(t-i\tau_0)\}f(t)dt, \tag{9}$$

where $I_n=\sqrt{\pi}2^n n!$. *Moreover, the series on the right side of (9) is absolutely uniformly convergent on every compact subset of* $S(\tau_0)$.

Proof. If $-\tau_0<\sigma<\tau_0$, with $l(\sigma)$ we denote the straight line with parametric equation $\zeta=t+i\sigma$ $(-\infty<t+\infty)$. Let $z\in S(\tau_0)$ and τ be chosen so that $|\operatorname{Im} z|<\tau<\tau_0$. Then we define

$$f^{+}(z)=\frac{1}{2\pi i}\int_{l(-\tau)}\frac{f(\zeta)}{\zeta-z}\,d\zeta=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t-i\tau)}{z-t+i\tau}\,dt \tag{10}$$

and

$$f^{-}(z)=\frac{1}{2\pi i}\int_{l(\tau)}\frac{f(\zeta)}{\zeta-z}\,d\zeta=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t+i\tau)}{z-t-i\tau}\,dt. \tag{11}$$

Let us note that from the condition (b) of the theorem follows that the functions $f^{+}(z)$ and $f^{-}(z)$ are uniquely determined in the stripe $S(\tau_0)$ and that for every $z\in S(\tau_0)$ holds the equality

$$f(z)=f^{+}(z)-f^{-}(z). \tag{12}$$

After dividing the equation (1) by $2I_n$, we can write it in the following canonical form $k_{n+1}y_{n+1}-I_n^{-1}zy_n+k_{n-1}y_{n-1}=0$, where $k_n=(2I_n)^{-1}$. By using this last form of the equation (1) it is not difficult to derive the corresponding formula of Christoffel — Darboux, namely

$$\frac{1}{\zeta-z}=\sum_{n=0}^{\nu}\frac{1}{I_n}H_n(z)K_n(\zeta)+\frac{\Delta_{\nu+1}(z,\zeta)}{\zeta-z}, \tag{13}$$

where $\Delta_{\nu+1}(z,\zeta)=\{H_\nu(z)K_{\nu+1}(\zeta)-H_{\nu+1}(z)K_\nu(\zeta)\}/2I_\nu$.

In the equality (13) we replace ζ with $z+i\tau_0$, z with $t+i(\tau_0-\tau)$, multiply with $-(2\pi i)^{-1}f(t-i\tau)$, integrate over the interval $(-\infty,+\infty)$ and get that

$$f^+(z)=\sum_{n=0}^{\nu}a_n^+(f)K_n^+(z+i\tau_0)+R_\nu^+(z) \tag{14}$$

where

$$a_n^+(f)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}H_n[t+i(\tau_0-\tau)]f(t-i\tau)dt \tag{15}$$

and

$$R_\nu^+(z)=\frac{1}{4\pi i\,I_n}\{K_{\nu+1}^+(z+i\tau_0)\int_{-\infty}^{\infty}\frac{H_\nu[t+i(\tau_0-\tau)]f(t-i\tau)}{z-t+i\tau}dt \tag{16}$$

$$-K_\nu^+(z+i\tau_0)\int_{-\infty}^{\infty}\frac{H_{\nu+1}[t+i(\tau_0-\tau)]f(t-i\tau)}{z-t+i\tau}dt\}.$$

In our paper [2] we have established the following asymptotic formulas for Hermite functions of second kind namely

$$K_n(z)=(-i)^{n+1}\pi\sqrt{2}(2\pi/e)^{n/2}\exp[-z^2/2+iz\sqrt{2n+1}]\{1+k_n^+(z)\} \tag{17}$$

in the half-plane H^+ and respectively

$$K_n(z)=i^{n+1}\pi\sqrt{2}(2n/e)^{n/2}\exp[-z^2/2-iz\sqrt{2n+1}]\{1+k_n^-(z)\} \tag{18}$$

in the half-plane H^-. The functions $\{k_n^+(z)\}_{n=0}^{\infty}$ resp. $\{k_n^-(z)\}_{n=0}^{\infty}$ are analytic in H^+ resp. H^- and $\lim_{n\to\infty}k_n^+(z)=0$ resp. $\lim_{n\to\infty}k_n^-(z)=0$ uniformly on every compact subset of H^+, resp. H^-.

In the same paper we derived from the integral representation of Hermite polynomials [1, (5.6.4)] an inequality of the kind

$$H_n(z)=O\{(2n/e)^{n/2}\exp[z^2+\sigma\sqrt{2n+1}]\} \tag{19}$$

on every closed region $\bar{S}(\sigma)$ $(0\leq\sigma<+\infty)$.

Now, let A be an arbitrary compact subset of the stripe $S(\tau_0)$ and $0<\delta<\tau<\tau_0$ are chosen so that $A\subset\bar{S}(\delta)$. From the asymptotic formula (17) we get that

$$K_{\nu+1}^+(z+i\tau_0)=O\{[(2\nu+2)/e]^{(\nu+1)/2}\exp[-(\tau_0-\delta)\sqrt{2\nu+3}]\} \tag{20}$$

resp.

(21) $$K_\nu^+(z+i\tau_0)=O\{(2\nu/e)^{\nu/2}\exp[-(\tau_0-\delta)\sqrt{2\nu+1}]\}$$

uniformly for $z\in A$.

Since A is compact, there exists $L>0$ such that $|z-t+i\tau|\geqq L|t-i\tau|$ for every $z\in A$ and every $t\in(-\infty,+\infty)$. Then, from the inequality (19) and the condition (b) of the theorem follows that

$$\int_{-\infty}^{\infty}\frac{H_\nu[t+i(\tau_0-\tau)]f(t-i\tau)}{z-t+i\tau}dt=O\{(2\nu/e)^{\nu/2}\exp[(\tau_0-\tau)\sqrt{2\nu+1}]\int_{-\infty}^{\infty}\frac{dt}{|t-i\tau|^{1+\mu(\tau)}}\}$$

resp.

$$\int_{-\infty}^{\infty}\frac{H_{\nu+1}[t+i(\tau_0-\tau)]f(t-i\tau)}{z-t+i\tau}dt=O\{[(2\nu+2)/e]^{(\nu+1)/2}\exp[(\tau_0-\tau)\sqrt{2\nu+3}]\int_{-\infty}^{\infty}\frac{dt}{|t-i\tau|^{1+\mu(\tau)}}\}.$$

Then, from Stirling's formula and the equalities (20) and (21) follows that $R_\nu^+(z)=O\{\exp[-(\tau-\delta)\sqrt{2\nu+1}]\}$ uniformly for $z\in A$. From (16) and (14) we can conclude now that

(22) $$f^+(z)=\sum_{n=0}^{\infty}a_n^+(f)\,K_n^+(z+i\tau_0)$$

uniformly on A. Similarly, by using the asymptotic formula (21) and also the inequality (19), we get the expansion

(23) $$f^-(z)=\sum_{n=0}^{\infty}a_n^-(f)K_n^-(z-i\tau_0)$$

with coefficients $a_n^-(f)=-\frac{1}{2\pi i\,I_n}\int_{-\infty}^{\infty}H_n[t-i(\tau_0-\tau)]f(t+i\tau)dt.$

The Cauchy integral theorem and the condition (b) of the theorem give further that $\int_{l(-\tau)}H_n(\zeta+i\tau_0)f(\zeta)d\zeta=\int_{l(0)}H_n(\zeta+i\tau_0)f(\zeta)d\zeta$, i. e.

$$\int_{-\infty}^{\infty}H_n[t-i\tau+i\tau_0]f(t-i\tau)dt=\int_{-\infty}^{\infty}H_n(t+i\tau_0)f(t)dt.$$

Therefore $a_n^+(f)=-\frac{1}{2\pi i\,I_n}\int_{-\infty}^{\infty}H_n(t+i\tau_0)f(t)dt.$

In the same way we get also $a_n^-(f)=-\frac{1}{2\pi i\,I_n}\int_{-\infty}^{\infty}H_n(t-i\tau_0)f(t)dt.$

1) Let $f(z)$ be even in the stripe $S(\tau_0)$. Then, having in view that $H_n(-z)=(-1)^nH_n(z)\,(n=0,1,2,\ldots)$ we get that $a_{2n}^-(f)=-a_{2n}^+(f)$ resp. $a_{2n+1}^-(f)=a_{2n+1}^+(f)$. Therefore, for $z\in S(\tau_0)$ from (12), (22) and (23) it follows that in this case

$$f(z)=\sum_{n=0}^{\infty}a_n^+(f)K_n^+(z+i\tau_0)-\sum_{n=0}^{\infty}a_n^-(f)K_n^-(z-i\tau_0)$$

$$= \sum_{n=0}^{\infty} a_{2n}^{+}(f)\{K_{2n}^{+}(z+i\tau_0)+K_{2n}^{-}(z-i\tau_0)\}$$

$$+ \sum_{n=0}^{\infty} a_{2n+1}^{+}(f)\{K_{2n+1}^{+}(z+i\tau_0)-K_{2n+1}^{-}(z-i\tau_0)\}$$

$$= \sum_{n=0}^{\infty} a_n^{+}(f)K_n^{(1)}(z;\tau_0).$$

2) If $f(z)$ is odd in the stripe $S(\tau_0)$, in the same way we get the expansion $f(z)=\sum_{n=0}^{\infty} a_n^{+}(f)K_n^{(2)}(z;\tau_0)$.

3) In the general case $f(z)=g(z)+h(z)$ where $g(z)=[f(z)+f(-z)]/2$ and $h(z)=[f(z)-f(-z)]/2$, $g(z)$ is even and $h(z)$ is odd. Moreover, $g(z)$ and $h(z)$ satisfy the same conditions as the function $f(z)$, i. e.

$$g(z)=\sum_{n=0}^{\infty} a_n^{+}(g)K_n^{(1)}(z;\tau_0) \quad \text{and} \quad h(z)=\sum_{n=0}^{\infty} a_n^{+}(h)K_n^{(2)}(z;\tau_0).$$

But $H_n(-z)=(-1)^n H_n(z)$ as it was mentioned above, therefore

$$a_n^{+}(g)=-\frac{1}{4\pi i\, I_n}\sum_{-\infty}^{\infty}\{H_n(t-i\tau_0)+(-1)^n H_n(t-i\tau_0)\}f(t)dt$$

resp.

$$a_n^{+}(h)=-\frac{1}{4\pi i\, I_n}\int_{-\infty}^{\infty}\{H_n(t+i\tau_0)-(-1)^n H_n(t-i\tau_0)\}f(t)dt,$$

i. e. we get the expansion (7) with coefficients given by (8) and (9) and thus Theorem 1 is proved.

Using Rodrigues' formula for Hermite polynomials namely $H_n(z) =(-1)^n \exp z^2\{\exp(-z^2)\}^{(n)}$ [1, (5.5.3)] we can easily derive another integral representation for Hermite functions of second kind namely

$$K_n(z)=(-1)^{n+1}n!\int_{-\infty}^{\infty}\frac{\exp(-t^2)}{(t-z)^{n+1}}dt.$$

From the last representation follows immediately that $K_n^{(k)}(z)=(-1)^k K_{n+k}(z)$ $(k=1, 2, 3, \ldots)$. Then as a conclusion from Theorem 1 we get the following result

Theorem 2. *Let $f(z)$ be a complex function satisfying the conditions of Theorem 1. Then for every $k=1, 2, 3, \ldots$ and every $z\in S(\tau_0)$ holds the equality*

$$f^{(k)}(z)=\sum_{n=0}^{\infty}(-1)^k\{a_n^{(1)}(f)K_{n+k}^{(1)}(z;\tau_0)+a_n^{(2)}(f)K_{n+k}^{(2)}(z;\tau_0)\} \tag{24}$$

$$=\sum_{n=k}^{\infty}\{a_{k,n}^{(1)}(f)K_n^{(1)}(z;\tau_0)+a_{n,k}^{(2)}(f)K_n^{(2)}(z;\tau_0)\}$$

where $a_{n,k}^{(1)}(f)=(-1)^k a_{n-k}^{(1)}(f)$ *resp.* $a_{n,k}^{(2)}(f)=(-1)^k a_{n-k}^{(2)}(f)$.

We shall see now that for the coefficients of the expansion (24) are valid the following integral representations

$$a^{(1)}_{n,k}(f) = -\frac{1}{4\pi i\, I_n}\int_{-\infty}^{\infty}\{H_n(t+i\tau_0)+(-1)^n H_n(t-i\tau_0)\} f^{(k)}(t)\,dt \tag{25}$$

resp.

$$a^{(2)}_{n,k}(f) = -\frac{1}{4\pi i\, I_n}\int_{-\infty}^{\infty}\{H_n(t+i\tau_0)-(-1)^n H_n(t-i\tau_0)\} f^{(k)}(t)\,dt. \tag{26}$$

First of all, from the property $H'_n(z)=2nH_{n-1}(z)$ [1, (5.5.10)] follows that

$$a^{(1)}_{n,k}(f) = -\frac{(-1)^k}{4\pi i\, I_n}\int_{-\infty}^{\infty}\{H^{(k)}_n(t+i\tau_0)+(-1)^n H^{(k)}_n(t-i\tau_0)\} f(t)\,dt. \tag{27}$$

Let $0<\tau<\tau_0$ and $\gamma_\tau(t)$ be the circumference with center at the point t and radius τ. Then, using the integral formulas for the derivatives of an analytic function and also the condition (b) of Theorem 1, we get easily that

$$f^{(k)}(t)=\frac{k!}{2\pi i}\int_{\gamma_\tau(t)}\frac{f(\zeta)}{(\zeta-t)^{k+1}}\,d\zeta=\frac{k!}{2\pi\tau^k}\int_0^{2\pi} e^{-ik\theta} f(t+\tau e^{i\theta})\,dt = O[\exp(-t^2+\tau|t|)]$$

if $|t|\to+\infty$ $(-\infty<t<+\infty)$. Then from (27) after integration by parts we get (25). In a similar way we find the equality (26).

The integral representations (25) and (26) lead to the idea that Theorem 2 can be obtained as an application of Theorem 1 to the function $f^{(k)}(z)$. We shall see now that such a proof of Theorem 2 is impossible. Indeed, if an analytic function $f(z)$ satisfies the conditions of Theorem 1, in general the same is not true for the derivatives $f^{(k)}(z)$ $(k=1, 2, 3\ldots)$. An example of such a function is the following $(z+i\tau_0)^{-1}\exp(-z^2)$.

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Received 12. 6. 1975