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RIEMANN MANIFOLDS WITH CONSTANT GENERALIZED CURVATURE OF RICCI

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A generalized Ricci curvature is introduced and a characteristic of Riemann manifolds with constant sectional curvature and of Einstein-Riemann manifolds is given.

Let M be an n-dimensional Riemann manifold. For every $p \in M$ denote by M_p the tangent space at p to M and E^λ and F^μ be two linear orthogonal subspaces of M_p with dimensions resp. λ and μ. We assume that $E^\lambda \cap F^\mu = \varnothing$, $\lambda, \mu = 2, 3, \dots, n-2$ and $\lambda + \mu \leqq n$. Let further $u_1, \dots, u_\lambda$ be an orthonormal basis of E^λ and $u_{\lambda+1}, \dots, u_{\lambda+\mu}$ of F^μ respectively. Define

$$\varrho(E^\lambda; F^\mu) := \sum_{i=\lambda+1}^{\lambda+\mu} K_{12\dots\lambda i} \tag{1}$$

called curvature of Ricci for the direction E^λ with respect to the μ-dimensional plane F^μ, where $K(E^\lambda) = K_{12\dots\lambda} = \Sigma_{1 \leqq i < j \leqq \lambda} K_{ij}$ is the curvature of the λ-dimensional linear subspace E^λ, defined in [1]. This curvature does not depend on the orthonormal basis of E^λ and F^μ. Indeed

$$\varrho(E^\lambda; F^\mu) := \sum_{i=\lambda+1}^{\lambda+\mu} K_{12\dots\lambda i} = \mu K(E^\lambda) + K(E^\lambda; F^\mu).$$

Theorem 1. *The following formula holds*

$$\varrho(E^\lambda; F^\mu) = (\mu-2) K_{12\dots\lambda} + \sum_{i=1}^{\lambda} \varrho(u_i; G^{\lambda+\mu}), \tag{2}$$

where $\varrho(u_i; G^{\lambda+\mu})$ is the Ricci curvature with respect to the subspace $G^{\lambda+\mu} = E^\lambda \cup F^\mu$ and for the direction u_i [1].

The proof consists of the following equalities:

$$\varrho(E^\lambda; F^\mu) := \sum_{i=\lambda+1}^{\lambda+\mu} K_{12\dots\lambda i} = K_{12\dots\lambda\lambda+1} + \dots + K_{12\dots\lambda\lambda+\mu}$$

$$= \mu K_{12\dots\lambda} + \sum_{i=\lambda+1}^{\lambda+\mu} (K_{1i} + \dots + K_{\lambda i}) \pm \sum_{i=1}^{\lambda} (K_{1i} + \dots + K_{\lambda i})$$

$$= \mu K_{12\dots\lambda} + \sum_{i=1}^{\lambda} \varrho(u_i; G^{\lambda+\mu}) - \sum_{i,j=1}^{\lambda} K_{ij} = (\mu-2) K_{12\dots\lambda} + \sum_{i=1}^{\lambda} \varrho(u_i; G^{\lambda+\mu}).$$

Now consider Riemann manifolds with the property

(3) $$\varrho(E^{\lambda};\, F^{\mu}) = C^{0}_{\lambda,\mu},$$

i. e. the Ricci curvature for every λ-dimensional tangent subspace with respect to any orthogonal to it μ-dimensional tangent subspace at every point of the manifold is constant. Further on one has $\lambda+\mu \leq n$, where λ and μ take the values $2, 3, \ldots, n-2$.

Taking into account (2), the condition (3) becomes

(4) $$(\mu-2)K_{i_1 i_2 \ldots i_\lambda} + \sum_{k=1}^{\lambda} \varrho(u_{i_k};\, G^{\lambda+\mu}) = C^{0}_{\lambda,\mu}.$$

We add these $C^{\lambda}_{\lambda+\mu}$ equations for $1 \leq i_1 < \cdots < i_\lambda \leq \lambda+\mu$. The curvature $\varrho(u_1;\, G^{\lambda+\mu})$ comes out just when $i_1 = 1$ and the remaining $\lambda-1$ indexes become equal to $2, 3, \ldots, \mu+\lambda$ — their number is equal to $\lambda+\mu-1$. Therefore $\varrho(u_1,\, G^{\lambda+\mu})$ appears $C^{\lambda-1}_{\lambda+\mu-1}$ times. The same is true for the other curvatures as well. Using the formula

$$\binom{\lambda+\mu-2}{\lambda-2} S(p;\, G^{\lambda+\mu}) = \sum_{1 \leq i_1 < \ldots < i_\lambda < \lambda+\mu} K_{i_1 \ldots i_\lambda}$$

of [2], we obtain

$$(\mu-2)\binom{\lambda+\mu-2}{\lambda-2} S(p;\, G^{\lambda+\mu}) + \binom{\lambda+\mu-1}{\lambda-1} \sum_{i=1}^{\lambda+\mu} \varrho(u_i;\, G^{\lambda+\mu}) = \binom{\lambda+\mu}{\lambda} C^{0}_{\lambda,\mu}.$$

Hence

(5) $$C^{0}_{\lambda,\mu} = \frac{\lambda\mu(\lambda+1)}{(\lambda+\mu)(\lambda+\mu-1)} S(p;\, G^{\lambda+\mu}).$$

We obtain from (2) $(\mu-2)K_{12\ldots\lambda-1i} + \varrho(u_1;\, G^{\lambda+\mu}) + \cdots + \varrho(u_i;\, G^{\lambda+\mu}) = C^{0}_{\lambda,\mu}$. Considering theorem 1 and summing for every $i=\lambda, \ldots, \lambda+\mu$, one gets

$$(\mu-1)\Big[(\mu-2)K_{12\ldots\lambda-1} + 2\sum_{i=1}^{\lambda-1} \varrho(u_i;\, G^{\lambda+\mu})\Big] = (\mu+1)C^{0}_{\lambda,\mu} - 2S(p;\, G^{\lambda+\mu})$$

or

$$(\mu-2)K_{12\ldots\lambda-1} + 2\sum_{i=1}^{\lambda-1} \varrho(u_i;\, G^{\lambda+\mu}) = \frac{(\mu+1)C^{0}_{\lambda,\mu} - 2S(p;\, G^{\lambda+\mu})}{\mu-1} =: C^{1}_{\lambda,\mu}.$$

We may continue this procedure when $\lambda > 3$ and $\lambda+1 \leq \lambda+\mu \leq n$. If we repeat it k times, where $k+1 < \lambda$, the result will be

$$(\mu+k-2)\Big[(\mu-2)K_{12\ldots\lambda-k} + (k+1)\sum_{i=1}^{\lambda-k} \varrho(u_i;\, G^{\lambda+\mu})\Big] = (\mu+k)C^{k-1}_{\lambda,\mu} - 2kS(p;\, G^{\lambda+\mu}).$$

When $k+1=\lambda$ it becomes $\lambda(\lambda+\mu-3)\varrho(u_1;\, G^{\lambda+\mu}) = (\lambda+\mu-1)C^{\lambda-2}_{\lambda,\mu} - 2(\lambda-1)S(p;\, G^{\lambda+\mu})$ or $\varrho(u_1;\, G^{\lambda+\mu}) = [(\lambda+\mu-1)C^{\lambda-2}_{\lambda,\mu} - 2(\lambda-1)\, S(p;\, G^{\lambda+\mu})]/\lambda(\lambda+\mu-3)$.

If we express $C^{\lambda-2}_{\lambda,\mu}$ by $C^{0}_{\lambda,\mu}$ and use (5), we obtain

$$\varrho(u_1;\, G^{\lambda+\mu}) = \lambda^{-1}[(\lambda+\mu-1)(\lambda+\mu-2)C^{0}_{\lambda,\mu}/\mu(\mu-1)$$

$$(6) \qquad -2(\lambda+\mu-1)(\lambda+\mu-2)\,S(p\,;\,G^{\;+\mu})\sum_{i=1}^{\lambda-1}(\lambda-i)/(\lambda+\mu)(\lambda+\mu-i-1)(\lambda+\mu-i-2)]$$

$$=\frac{2}{\lambda+\mu}\,S(p\,;\,G^{\lambda+\varphi}).$$

It is known [2] that $S(p\,;\,G^{\lambda+\mu})=2^{-1}\Sigma_{i=1}^{\lambda+\mu}\varrho(u_i\,;\,G^{\lambda+\mu})$.

From (6) we obtain

$$(7) \qquad \varrho(u_1\,;\,G^{\lambda+\mu})=\cdots=\varrho(u_{\lambda+\mu}\,;\,G^{\lambda+\mu}),$$

i. e. the curvatures $\varrho(u_i\,;\,G^{\lambda+\mu})$ do not depend on the direction u.

Now let consider the two main cases:

1. $\lambda+\mu=n$; it follows from (7) that M is Einsteinian. Now we may consider the following subcases:

a) $\mu\neq 2$: from (4) and (7) one has $K_{12\ldots\lambda}=\text{const}$ $(2\leqq\lambda\leqq n-2)$ and therefore M is a space with constant sectional curvature.

b) $\mu=2$; using the formula $\varrho(E^\lambda\,;\,F^2)=2K(E^\lambda)+K(F^2\cup E^\lambda)$, considering (3) and expressing $K(F^2\cup E^\lambda)$ by $K(F^2)$ and $K(E^\lambda)$ we obtain $K(F^2)-K(\perp F^2)=\text{const}$. So it follows that M is Einsteinian [1].

2. $\lambda+\mu\neq n$, i. e. $\lambda+\mu<n$.

a) If $\mu\neq 2$ we obtain from (4) that M is with constant sectional curvature.

b) If $\mu=2$, we obtain from (7) that the Ricci curvature of order $\lambda+\mu-1$, where $\lambda+\mu-1\leqq n-2$ is constant for directions with respect to $\lambda+\mu-1$-dimensional orthogonal linear subspaces. It follows from [1] that M is with constant sectional curvature. So we proved the following

Theorem 2. *Let $2\leqq\lambda$, $\mu\leqq n-2$, $\lambda+\mu\leqq n-2$. If the curvature of Ricci for any λ-dimensional subspace E^λ with respect to every μ-dimensional subspace $F^\mu\perp E^\lambda$ is constant, then:*

1. *If $\lambda=n-2$ and $\mu=2$, M is an Einstein-Riemann manifold.*

2. *In all other cases M is a Riemann manifold with constant sectional curvature.*

Note. The cases $\lambda=1$ and $\mu=2$ are treated in [1].

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