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## RIEMANN MANIFOLDS WITH CONSTANT GENERALIZED CURVATURE OF RICCI

#### PENKA P. IVANOVA

A generalized Ricci curvature is introduced and a characteristic of Riemann manifolds with constant sectional curvature and of Einstein-Riemann manifolds is given.

Let M be an n-dimensional Riemann manifold. For every  $p \in M$  denote by  $M_p$  the tangent space at p to M and  $E^{\lambda}$  and  $F^{\mu}$  be two linear orthogonal subspaces of  $M_p$  with dimensions resp.  $\lambda$  and  $\mu$ . We assume that  $E^{\lambda} \cap F^{\mu} = \emptyset$ ,  $\lambda, \mu = 2, 3, \ldots, n-2$  and  $\lambda + \mu \le n$ . Let further  $u_1, \ldots, u_{\lambda}$  be an orthonormal basis of  $E^{\lambda}$  and  $u_{\lambda+1}, \ldots, u_{\lambda+\mu}$  of  $F^{\mu}$  respectively. Define

(1) 
$$\varrho(E^{\lambda}; F^{\mu}) := \sum_{i=\lambda+1}^{\lambda+\mu} K_{12...\lambda i}$$

called curvature of Ricci for the direction  $E^{\lambda}$  with respect to the  $\mu$ -dimensional plane  $F^{\mu}$ , where  $K(E^{\lambda}) = K_{12 \ldots \lambda} = \sum_{1 \leq i < j \leq \lambda} K_{ij}$  is the curvature of the  $\lambda$ -dimensional linear subspace  $E^{\lambda}$ , defined in [1]. This curvature does not depend on the orthonormal basis of  $E^{\lambda}$  and  $F^{\mu}$ . Indeed

$$\varrho(E^{\lambda}; F^{\mu}) := \sum_{i=\lambda+1}^{\lambda+\mu} K_{12\ldots\lambda i} = \mu K(E^{\lambda}) + K(E^{\lambda}; F^{\mu}).$$

Theorem 1. The following formula holds

(2) 
$$\varrho(E^{\lambda}; F^{\mu}) = (\mu - 2) \mathring{K}_{12 \dots \lambda} + \sum_{i=1}^{\lambda} \varrho(u_i; G^{\lambda + \mu}),$$

where  $\varrho(u_i; G^{\lambda+\mu})$  is the Ricci curvature with respect to the subspace  $G^{\lambda+\mu} = E^{\lambda} \cup F^{\mu}$  and for the direction  $u_i$  [1].

The proof consists of the following equalities:

$$\varrho(E^{\lambda}; F^{\mu}) := \sum_{i=\lambda+1}^{\lambda+\mu} K_{12...\lambda i} = K_{12...\lambda \lambda+1} + \cdots + K_{12...\lambda \lambda+\mu}$$

$$= \mu K_{12...\lambda} + \sum_{i=\lambda+1}^{\lambda+\mu} (K_{1i} + \cdots + K_{\lambda i}) \pm \sum_{i=1}^{\lambda} (K_{1i} + \cdots + K_{\lambda i})$$

$$= \mu K_{12...\lambda} + \sum_{i=1}^{\lambda} \varrho(u_i; G^{\lambda+\mu}) - \sum_{i,j=1}^{\lambda} K_{ij} = (\mu-2)K_{12...\lambda} + \sum_{i=1}^{\lambda} \varrho(u_i; G^{\lambda+\mu}).$$

Now consider Riemann manifolds with the property

SERDICA Bulgaricae mathematicae publicationes. Vol. 3, 1977, p. 17-19.

$$\varrho(E^{\lambda}; F^{\mu}) = C_{1...}^{0},$$

i. e. the Ricci curvature for every λ-dimensional tangent subspace with respect to any orthogonal to it μ-dimensional tangent subspace at every point of the manifold is constant. Further on one has  $\lambda + \mu \le n$ , where  $\lambda$  and  $\mu$  take the values  $2, 3, \ldots, n-2$ .

Taking into account (2), the condition (3) becomes

(4) 
$$(\mu-2)K_{i_1i_2...i_{\lambda}} + \sum_{k=1}^{\lambda} \varrho(u_{i_k}; G^{\lambda+\mu}) = C^0_{\lambda,\mu}.$$

We add these  $C^{\lambda}_{\lambda+\mu}$  equations for  $1 \le i_1 < \cdots < i_{\lambda} \le \lambda+\mu$ . The curvature  $\varrho(u_1; G^{\lambda+\mu})$  comes out just when  $i_1=1$  and the remaining  $\lambda-1$  indexes become equal to  $2, 3, \ldots, \mu+\lambda$ — their number is equal to  $\lambda+\mu-1$ . Therefore  $\varrho(u_1, G^{\lambda+\mu})$  appears  $C^{\lambda-1}_{\lambda+\mu-1}$  times. The same is true for the other curvatures as well. Using the formula

$$\binom{\lambda+\mu-2}{\lambda-2}S(\mathfrak{o}; G^{\lambda+\mu})=\sum_{1\leq i_1<\ldots< i_{\lambda}<\lambda+\mu}K_{i_1\ldots i_{\lambda}}$$

of [2], we obtain

$$(\mu-2)\binom{\lambda+\mu-2}{\lambda-2}S(p; G^{\lambda+\mu})+\binom{\lambda+\mu-1}{\lambda-1}\sum_{i=1}^{\lambda+\mu}\varrho(u_i; G^{\lambda+\mu})=\binom{\lambda+\mu}{\lambda}C^0_{\lambda,\mu}.$$

Hence

(5) 
$$C_{\lambda,\mu}^0 = \frac{\lambda\mu(\lambda+1)}{(\lambda+\mu)(\lambda+\mu-1)} S(p; G^{\lambda+\mu}).$$

We obtain from (2)  $(\mu-2)K_{12...\lambda-1i}+\varrho(u_1; G^{\lambda+\mu})+\cdots+\varrho(u_i; G^{\lambda+\mu})=C_{i,\mu}^0$ . Considering theorem 1 and summing for every  $i=\lambda,\ldots,\lambda+\mu$ , one gets

$$(\mu-1)[(\mu-2)K_{12\ldots\lambda-1}+2\sum_{i=1}^{\lambda-1}\varrho(u_i;G^{\lambda+\mu})]=(\mu+1)C_{\lambda,\mu}^0-2S(p;G^{\lambda+\mu})$$

or

$$(\mu-2)K_{12\ldots\lambda-1}+2\sum_{i=1}^{\lambda-1}\varrho(u_i; G^{\lambda+\mu})=\frac{(\mu+1)C_{\lambda,\mu}^0-2S(p; G^{\lambda+\mu})}{\mu-1}=:C_{\lambda,\mu}^1.$$

We may continue this procedure when  $\lambda > 3$  and  $\lambda + 1 \le \lambda + \mu \le n$ . If we repeat it k times, where  $k+1 < \lambda$ , the result will be

$$(\mu + k - 2) [(\mu - 2)K_{12...\lambda-k} + (k+1) \sum_{i=1}^{\lambda-k} \varrho(u_i; G^{\lambda+\mu})] = (\mu + k)C_{\lambda,\mu}^{k-1} - 2kS(p; G^{\lambda+\mu}).$$

When  $k+1=\lambda$  it becomes  $\lambda(\lambda+\mu-3)\varrho(u_1;\ G^{\lambda+\mu})=(\lambda+\mu-1)C_{\lambda,\mu}^{\lambda-2}-2(\lambda-1)S(p;\ G^{\lambda+\mu})$  or  $\varrho(u_1;\ G^{\lambda+\mu})=[(\lambda+\mu-1)C_{\lambda,\mu}^{\lambda-2}-2(\lambda-1)S(p;\ G^{\lambda+\mu})]/\lambda(\lambda+\mu-3)$ . If we express  $C_{\lambda,\mu}^{2-2}$  by  $C_{\lambda,\mu}^0$  and use (5), we obtain

$$\varrho(u_1; G^{\lambda+\mu}) = \lambda^{-1}[(\lambda+\mu-1)(\lambda+\mu-2)C^0_{\lambda,\mu}/\mu(\mu-1)]$$

(6) 
$$-2(\lambda+\mu-1)(\lambda+\mu-2)S(p; G^{+\mu})\sum_{i=1}^{\lambda-1}(\lambda-i)/(\lambda+\mu)(\lambda+\mu-i-1)(\lambda+\mu-i-2)]$$
$$=\frac{2}{\lambda+\mu}S(p; G^{\lambda+\varphi}).$$

It is known [2] that  $S(p; G^{\lambda+\mu}) = 2^{-1} \sum_{i=1}^{\lambda+\mu} \varrho(u_i; G^{\lambda+\mu})$ . From (6) we obtain

(7) 
$$\rho(u_1; G^{\lambda+\mu}) = \cdots = \varrho(u_{\lambda+\mu}; G^{\lambda+\mu}),$$

i. e. the curvatures  $\varrho(u_i; G^{1+\mu})$  do not depend on the direction u.

Now let consider the two main cases:

1.  $\lambda + \mu = n$ ; it follows from (7) that M is Einsteinian. Now we may consider the following subcases:

a)  $\mu \pm 2$ : from (4) and (7) one has  $K_{12...\lambda} = \text{const} \ (2 \le \lambda \le n-2)$  and therefore M is a space with constant sectional curvature.

b)  $\mu=2$ ; using the formula  $\varrho(E^{\lambda}; F^2)=2K(E^{\lambda})+K(F^2\cup E^{\lambda})$ , considering (3) and expressing  $K(F^2\cup E^{\lambda})$  by  $K(F^2)$  and  $K(E^{\lambda})$  we obtain  $K(F^2)-K(\perp F^2)$  = const. So it follows that M is Einsteinian [1].

2.  $\lambda + \mu = n$ , i. e.  $\lambda + \mu < n$ .

- a) If  $\mu \pm 2$  we obtain from (4) that M is with constant sectional curvature.
- b) If  $\mu=2$ , we obtain from (7) that the Ricci curvature of order  $\lambda+\mu-1$ , where  $\lambda+\mu-1 \le n-2$  is constant for directions with respect to  $\lambda+\mu-1$ -dimensional orthogonal linear subspaces. It follows from [1] that M is with constant sectional curvature. So we proved the following

Theorem 2. Let  $2 \le \lambda$ ,  $\mu \le n-2$ ,  $\lambda + \mu \le n-2$ . If the curvature of Ricci for any  $\lambda$ -dimensional subspace  $E^{\lambda}$  with respect to every  $\mu$ -dimensional sub-

space  $F^{\mu} \perp E^{\lambda}$  is constant, then:

1. If  $\lambda = n-2$  and  $\mu = 2$ , M is an Einstein-Riemann manifold.

2. In all other cases M is a Riemann manifold with constant sectional curvature.

Note. The cases  $\lambda = 1$  and  $\mu = 2$  are treated in [1].

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