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CHARACTERIZATION OF B^*-EQUIVALENT BANACH ALGEBRAS BY MEANS OF THEIR POSITIVE CONES

HRISTO N. BOYADŽIEV

The complex Banach star algebra $\mathcal{A}$ is called B^*-algebra if $\| x^*x \| = \| x \|^2$ for every $x \in \mathcal{A}$ and B^*-equqivalent algebra if it is isomorphic and homeomorphic to a B^*-algebra. It is well known, that in a B^*-algebra (and hence in a B^*-equivalent algebra) the set K of all self-adjoint elements with non-negative spectrum is a cone ($K+K\subset K$, $\lambda K\subset K$ for $\lambda\geqq 0$ and $K\cap -K=\{0\}$). Our aim is when given a partially ordered Banach algebra $\mathcal{A}$ with positive cone K to impose on K some conditions, necessary and sufficient for $\mathcal{A}$ to be a B^*-equivalent star algebra. Necessary and sufficient conditions for a Banach star algebra to be symmetric and to be B^*-equivalent are given.

1. Throughout this part $\mathcal{A}$ will be a complex Banach star algebra with norm $\|.\|$, spectral radius $\varrho\,(.)$, continuous involution $x\rightarrow x^*$ and for $x\in\mathcal{A}$ we denote by $Sp(x)$ its spectrum. Let H be the set of self-adjoint elements and K the wedge $K=\{z\,|\,z=\Sigma_{k=1}^n x_k^* x_k,\ x_k\in\mathcal{A},\ 1\leqq k\leqq n\}$ (evidently $K+K\subseteq K$ and $\lambda K\subseteq K$ when $\lambda\geqq 0$). We shall consider only algebras with a unit, so let e be the unit of $\mathcal{A}$. Also let $P=\{f\,|\,f$ a positive linear functional on $\mathcal{A}$ and $f(e)=1\}$ and $\operatorname{Extr} P$ — the extreme points of P.

Let $p(.)$ be a real function defined on the wedge K with the properties:

1) $p(\lambda x)=\lambda p(x)$ when $\lambda\geqq 0$ and $x\in K$.

2) $p(x)\leqq p(x+z)$ when $xz=zx$, x, $z\in K$.

If $p(.)$ is such a function, it follows from 1) that $p(0)=p(0.0)=0p(0)=0$ and from 2) with $x=0$ and $z\in K$ that $0\leqq p(0)\leqq p(0+z)=p(z)$. So $p(.)$ is non-negative on K. Let $x\in K$. We take $t>0$, $t<\varrho(x)^{-1}$ (when $\varrho(x)$ is zero, $\varrho(x)^{-1}$ means ∞). There exists $y\in H$ with $e-tx=y^2$, i. e. $e=tx+y^2$ and we have according to 2) and 1) $p(tx)\leqq p(e)$, $p(x)\leqq t^{-1}p(e)$. Let now $t\rightarrow\varrho(x)^{-1}$. We obtain $p(x)\leqq\varrho(x)p(e)$. So if $p(e)=0$, then $p(x)=0$ for every $x\in K$. For the next lemma we consider $p(.)$ normed $p(e)=1$.

Lemma 1. *Let $x\in K$. There exists $f\in P$, $f(x)=p(x)$ (and if $f(z)\leqq p(z)$ for every $z\in K$ and $f\in P$, then f can be chosen from* $\operatorname{Extr}.P$*).*

Proof. Let $L=\{z\,|\,z=\lambda e+\mu x,\ \lambda,\mu\in R\}$ (R — the set of real numbers). Evidently L is a real linear subspace of the real Banach space H. For $z=\lambda e+\mu x\in L$ we define $f(z)=\lambda+\mu p(x)$. We obtain thus a linear functional f on L. We'll show that for $z\in L\cap K$, $f(z)\geqq 0$. Let $z=\lambda e+\mu x\in L\cap K$. There are four possibilities:

a) $\lambda\geqq 0$, $\mu\geqq 0$. Then $\lambda+\mu p(x)=f(x)\geqq 0$.

b) $\lambda<0$, $\mu\geqq 0$. Then $\mu x=-\lambda e+z$ and from 2) and 1) it follows $p(-\lambda e)=-\lambda\leqq p(\mu x)=\mu p(x)$, $\lambda+\mu p(x)\geqq 0$.

c) $\lambda\geqq 0$, $\mu<0$. Then $\lambda e=-\mu x+z$, $p(-\mu x)=-\mu p(x)\leqq p(\lambda e)=\lambda$, $\lambda+\mu p(x)\geqq 0$.

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d) $\lambda<0$, $\mu<0$, $0=-\lambda e-\mu x+z$, $\lambda=0$ because $0\leq p(-\lambda e)=-\lambda\leq 0$. This case is impossible.

We use now a well known theorem to extend f from L to H as positive linear functional.

Theorem ([1, theorem 2.5.2]). *Let X be a real linear space partially ordered by the wedge N. The linear functional f on the linear subspace $M\subseteq X$ can be extended as linear and positive on X if for every $u\in X$ there exists $v\in M$ with $v-u\in N$ and if $f(u)\geq 0$ for $u\in M\cap N$.*

In our case if $y\in H$ and $t>\|y\|$ then $e-t^{-1}y=z^2$ with $z\in H$, so $te-y\in K$ and $te\in L$.

With $H=X$, $L=M$ and $K=N$ we apply the theorem. (We can also extend f directly with the Zorn's lemma.) As every $z\in\mathcal{A}$ is represented uniquely $z=a+ib$, $a, b\in H$ $(a=(z+z^*)/2, b=(z-z^*)/2i)$, we extend f to $\mathcal{A}$ $(f(z)=f(a)+if(b)$. Evidently $f\in P$ and $f(x)=p(x)$.

If $f(z)\leq p(z)$ for every $f\in P$ and $z\in K$, let $P_x=\{f\mid f\in P,\ \ f(x)=p(x)\}$. This set is convex and compact in the weak topology (pointwise convergence). As P_x is non-void, the set of its extreme points $\operatorname{Extr} P_x$ is also non-void according to the Krein-Milman's theorem. One can easily see that $\operatorname{Extr}. P_x\subseteq\operatorname{Extr} P$, so if $f\in\operatorname{Extr}\ P_x$ and $f\neq 0$ then $f\in\operatorname{Extr} P$. The proof is completed.

With this method a more general lemma can be proved.

Lemma 1a. *Let X be a partially ordered normed real linear space with positive wedge K and let $p(.)$ be a monotone increasing real function on K $(p(x)\leq p(y)$ when $0\leq x\leq y)$ with $p(\lambda x)=\lambda p(x)$, $x\in K$, $\lambda\geq 0$. Let X has an order unit e. Then for every $x\in K$ there exists a positive linear functional f on X with $f(x)=p(x)$ and $f(e)=p(e)$.*

If $\mathcal{A}$ is commutative and B^*, then the norm $\|.\|$ is monotone increasing on K.

Proof. [3] a) If $x, y\in H$, then $\|x^2\|\leq\|x^2+y^2\|$. We have $x=(x+iy)/2+(x-iy)/2$, and so $\|x\|\leq\|x+iy\|/2+\|x-iy\|/2=\|x+iy\|$ $(\|z^*\|=\|z\|)$, $\|x^2\|=\|x\|^2\leq\|x+iy\|^2=\|(x+iy)(x-iy)\|=\|x^2+y^2\|$.

b) If $u, v\in H$, there exists $w\in H$ with $u^2+v^2=w^2$. Let u, $v\in H$ and $0<t<\min(\|u\|^{-1}, \|v\|^{-1})$. Then $\|(tu)^2\|=\|tu\|^2<1$, $\|(tv)^2\|=\|tv\|^2<1$ and there exist x, $y\in H$ such that $e-t^2u^2=x^2$, $e-t^2v^2=y^2$. According to a) $\|x^2\|\leq 1$ and $\|y^2\|\leq 1$ hence $\|e-(t^2u^2+t^2v^2)/2\|\leq\|e-t^2u^2\|/2+\|e-t^2v^2\|/2\leq 1$, and so $(t^2u^2+t^2v^2)/2=z^2$ for $z\in H$. Take $w=\sqrt{2}\,z/t$.

As for every $x\in\mathcal{A}$ $x^*x=u^2+v^2$ $(u=(x+x^*)/2,\ v=(x-x^*)/2i\in H)$, every $z\in K$ can be represented in the form $z=a^2$ with $a\in H$. According to a) for every x, $y\in K$ we obtain $\|x\|\leq\|x+y\|$. (It is shown later that if in the non-commutative case the norm is monotone increasing on K, then $\mathcal{A}$ is B^*-equivalent.)

Now according to Lemma 1 and to the fact that every f from $\operatorname{Extr} P$ is multiplicative, for every $x\in\mathcal{A}$ we obtain a multiplicative linear functional f with $f(x^*x)=\|x^*x\|$. Now $f(x^*x)=f(x^*)f(x)=\overline{f(x)}f(x)=|f(x)|^2=\|x^*x\|=\|x\|^2$. Finally: $|f(x)|=\|x\|$. This gives the isometry in the Gel'fand's representation of $\mathcal{A}$, which is the main point in the proof the Gel'fand-Naimark theorem.

This method can be applied to real commutative Banach algebras to obtain a classical result:

If $\mathcal{A}$ is a real commutative Banach algebra with unit e and $\|x\|^2\leq\|x^2+y^2\|$ (it follows $\|x^2\|=\|x\|^2$ and $\|x^2\|\leq\|x^2+y^2\|$) for every $x, y\in\mathcal{A}$, then for every $x\in\mathcal{A}$ there exists a multiplicative linear functional f with $|f(x)|=\|x\|$. This gives the isometrical isomorphism on $C_R(\Lambda)$ — the continuous real functions on $\Lambda=\{f \mid f$ multiplicative and linear on $\mathcal{A}\}$. In this case positive linear functionals are those linear functionals f for which $f(z^2)\geq 0$, $z\in\mathcal{A}$ and the extreme points of the set $P=\{f \mid f$ positive, linear on $\mathcal{A}$ and $f(e)=1\}$ are multiplicative. (See also [3] where $f\in$ Extr P is obtained otherwise.)

Let us now see when the spectral radius $\varrho(x)=\lim_n \|x^n\|^{1/n}$ (who has the property $|f(x)|\leq\varrho(x)$, $f\in P$, $x\in H$) is monotone increasing on K.

Lemma 2. *In $\mathcal{A}$ the condition*

2a) $\varrho(x^2)\leq\varrho(x^2+y^2)$, $x^2y^2=y^2x^2$, $x, y\in H$

is equivalent to symmetry.

Proof. If $\mathcal{A}$ is symmetric ($(e+a^*a)^{-1}$ exists for every $a\in\mathcal{A}$), then 2a) follows from [2, 4. 7. 12.], [4, V. 37. 6.] which states that if $\mathcal{A}$ is symmetric, then for every self-adjoint $x\in\mathcal{A}$

(*) $$\varrho(x)=\sup\{|f(x)| \mid f\in \operatorname{Extr} P\}.$$

Conversely, let $\varrho(x^2)\leq\varrho(x^2+y^2)$ when $x^2y^2=y^2x^2$, $x, y\in H$ and let $a\in H$, $a^2\neq 0$ and $0<t<\|a^2\|^{-1}$. There exists $y\in H$ with $e-ta^2=y^2$, i. e. $e=y^2+ta^2$. From 2a) $\varrho(e-ta^2)=\varrho(y^2)\leq 1<1+t$ and $\varrho((1+t)^{-1}e-t(1+t)^{-1}a^2)<1$. Then

$$e-[(1+t)^{-1}e-t(1+t)^{-1}a^2]=t(1+t)^{-1}(e+a^2)$$

is invertible, so $e+a^2$ is too. (In fact we need only $\varrho(y^2)\leq 1$ when $e=y^2+x^2$. with $x\in h$, $y\in H$ and $y^2x^2=x^2y^2$.) So every $x\in H$ has real spectrum according to [2, 4. 1. 7., p. 184]. Now $\mathcal{A}$ is symmetric according to the Shirali-Ford's theorem [6].

Remark 1. In fact 2a) and 2) (from the first page, with $\varrho(.)$ instead of $p(.)$) are equivalent (2) $\rightarrow$ 2a) is trivial, and 2a) $\rightarrow$ 2) without $xz=zx$ according to the symmetry of $\mathcal{A}$ and (*)).

We consider now the norm $\|.\|$ ($|f(x)|\leq\|x\|$ for $f\in P$ and $x\in P$ and $x\in H$).

Lemma 3. *If in $\mathcal{A}$*

2b) $\|x^2\|\leq\|x^2+y^2\|$, $x^2y^2=y^2x^2$, $x, y\in H$ *then $\mathcal{A}$ is symmetric and for every $x\in H$ with non-negative spectrum $\varrho(x)=\|x\|$.*

Proof. Let $a\in H$, $a^2\neq 0$ and $0<t<\|a^2\|^{-1}$. There exists $y\in H$, $e-ta^2=y^2$, i. e. $e=y^2+ta^2$. We obtain $\varrho(e-ta^2)\leq\|e-ta^2\|\leq 1<1+t$. The proof continues as in Lemma 2 and we obtain that $e+a^2$ is invertible, so $\mathcal{A}$ is symmetric.

We will see now that the norm is monotone increasing on K (the condition 2) with $\|.\|$ instead of $p(.)$). Let $x, z\in K$ and $xz=zx$. The elements of K have non-negative spectrum [2, 4. 7. 10.], so if $\varepsilon>0$, $Sp(x+\varepsilon e)=Sp(x)+\varepsilon>0$, $Sp(z+\varepsilon e)>0$ ($Sp(y)>0$ means $\lambda>0$ for every $\lambda\in Sp(y)$) and hence there exist $a, b\in H$ with $a^2=x+\varepsilon e$, $b^2=z+\varepsilon e$ [2, 4. 7. 2.]. Evidently $a^2b^2=b^2a^2$, so according to 2b) we have $\|x+\varepsilon e\|\leq\|x+z+2\varepsilon e\|$. Letting $\varepsilon\rightarrow 0$ we obtain $\|x\|\leq\|x+z\|$. As $\|.\|$ is monotone increasing on commuting elements of K and $\|\lambda x\|=|\lambda|\,\|x\|$ then $\|x\|\leq\varrho(x)$ for every $x\in K$ (proved for $p(.)$ at the beginning). The inverse inequality also holds, so $\|x\|=\varrho(x)$ for every $x\in K$. If now $x\in H$ and $Sp(x)\geq 0$, we have $Sp(x+\varepsilon e)>0$ for $\varepsilon>0$ and $x+\varepsilon e=a^2$ with $a\in H$. Now $\|x+\varepsilon e\|^2=\varrho^2(x+\varepsilon e)=\varrho((x+\varepsilon e)^2)=\|(x+\varepsilon e)^2\|=\|x^2+2\varepsilon x+\varepsilon e\|$. Let $\varepsilon\rightarrow 0$. We obtain $\|x\|^2=\|x^2\|$. As this holds for every $x\in H$ with $Sp(x)\geq 0$, then $\|z\|=\varrho(z)$ for every such element ($\|z\|=\|z^2\|^{1/2}=\|z^4\|^{1/4}=\cdots=\|z^{2^n}\|^{1/2^n}=\cdots=\lim_n\|z^{2n}\|^{1/2^n}=\varrho(z)$). Now 2) holds for $\|.\|$ without $xz=zx$.

Remark 2. If in the inital Banach symmetric star algebra $\mathcal{A}$ for every self-adjoint element x with non-negative spectrum we have $\|x\|\leq a\varrho(x)$, then $\|z\|\leq(2a+1)\,\varrho(z)$ for every self-adjoint element z.

Proof. Let z be self-adjoint and e be the unit of $\mathcal{A}$. Then $Sp(\varrho(z)e+z) = \varrho(z)+Sp(z) \geq 0$ $(Sp(z) \subset R)$, $\|z\| \leq \|z+\varrho(z)e\| + \|\varrho(z)e\| \leq \alpha\varrho(z+\varrho(z)e)+\varrho(z) \leq (2\alpha+1)\varrho(z)$. Now we can state the theorem:

Theorem 1. *Let $\mathcal{A}$ be a complex Banach star algebra with continuous involution $x \to x^*$ and unit. Then the following conditions are equivalent:*

a) *The algebra $\mathcal{A}$ is symmetric and $\varrho(x) = \|x\|$ for every self-adjoint element with non-negative spectrum.*

b) *In $\mathcal{A}$ 2b) holds (see lemma 3).*

c) *The algebra $\mathcal{A}$ is B^*-equivalent (the norm $\|.\|$ is equivalent to the B^* norm $|.|$, $|x| = \varrho(x^*x)^{1/2}$ for $x \in \mathcal{A}$) and $\varrho(x) = \|x\|$ for every self-adjoint element with non-negative spectrum.*

Proof. According to lemma 2, a) implies b), and b) implies a) according to lemma 3. Evidently c) implies a).Now a) together with the above Remark 2 implies c) according to the following theorem of Horst Behncke [5]:

If $\mathcal{A}$ is a complex unital Banach star algebra which is symmetric and $\|x\| \leq \beta\varrho(x)$ for every x self-adjoint, then $\mathcal{A}$ is B^*-equivalent (the norm $\|.\|$ is equivalent to the B^*-norm $|x| = \varrho(x^*x)^{1/2}$).

(That in a symmetric algebra $|x| = \varrho(x^*x)^{1/2}$ is a B^*-semi-norm was proved by Ptak [4], and it is easy to see that $\|.\|$ and $|.|$ are equivalent when a) holds.)

2. We consider now partially ordered Banach algebras.

Theorem 2. *Let $\mathcal{A}$ be a complex Banach algebra with unit e which is a partially ordered complex linear space with positive cone K such that if we denote $H = K - K$:*

a) *If $x \in H$, then $x^2 \in K$.*

b) *The cone K is generating, i. e. $\mathring{A} = H + iH$.*

c) *The norm is monotone increasing on commuting elements of K, i. e $\|x\| \leq \|x+z\|$ when $xz = zx$, $x, z \in K$.*

d) *The algebra $\mathcal{A}$ is semi-simple.*

e) *If $a, b \in H$, then $i(ab-ba) \in H$.*

f) *$H \cap iH = \{0\}$.*

Then a continuous involution $x \to x^*$ can be introduced in $\mathcal{A}$, so that it becomes B^*-equivalent and $H = \{x \mid x = x^*\}$, $K = \{x \mid x = x^*, Sp(x) \geq 0\}$ and $\varrho(x) = \|x\|$ for $x \in K$.

Conversely: *If $\mathcal{A}$ is B^*-equivalent and $\varrho(x) = \|x\|$ for every self-adjoint element with non-negative spectrum, then for $H = \{x \mid x = x^*\}$, $K = \{x \mid x = x^*, Sp(x) \geq 0\}$ we have that K is a cone, $H = K - K$ (this is well known) and the above conditions* a) — f) *hold.*

Proof. If $a, b \in H$, then $ab+ba = (a+b)^2 - (a^2+b^2)$ is also in H according to a) and the fact that H is a real linear subspace of $\mathcal{A}$. According to b) and f) every $x \in \mathcal{A}$ has an unique representation $x = a+ib$ with $a, b \in H$. Now $x = a+ib \to x^* = a-ib$, $a, b \in H$ is an algebraic involution [4, 1. 12. 7.], which is continuous, as $\mathcal{A}$ is semi-simple (a result of Johnson [4]). The algebra $\mathcal{A}$ becomes a star algebra with continuous involution and H is obviously the set of self-adjoint elements.

Now if $x, y \in H$ and $x^2y^2 = y^2x^2$ we have according to c) that $\|x^2\| \leq \|x^2+y^2\|$. Applying theorem 1 we obtain that $\mathcal{A}$ is B^*-equivalent with B^*-norm $|x| = \varrho(x^*x)^{1/2}$ and $\|x\| = \varrho(x)$ if $x \in H$ and $Sp(x) \geq 0$.

If $x \in H$ and $Sp(x) \geq 0$, there exists $u \in H$, $u^2 = x$ so $x \in K$. Conversely, if $x \in K$, $e+x$ is invertible (as in lemma 3), so every $x \in K$ has non-negative spectrum.

To prove that $\mathcal{A}$ is B^*-equivalent we needed in fact only:

c1-$\|x^2\| \leq \|x^2+y^2\|$ when $x^2y^2=y^2x^2$, $x, y \in H$

But with c1) only $K \supseteq \{x \mid x \in H, Sp(x) \geq 0\}$.

Conversely: If $\mathcal{A}$ is B^*-equivalent and $H=\{x \mid x=x^*\}$, $K=\{x \mid x=x^*, Sp(x) \geq 0\}$, it is well known that K is a cone, $H=K-K$ and a), b), d), e), f) hold. If $\varrho(x)=\|x\|$ for every $x \in K$ we obtain c) using theorem 1 and lemma 3 in which we proved that the norm is monotone increasing on K. The proof is completed.

In the same way, with the help of lemma 2 we can prove the following theorem:

Theorem 3. *If $\mathcal{A}$ is a complex unital Banach algebra, and a partially ordered complex linear space in the same time, with positive wedge K, such that if we denote $H=K-K$ the conditions a), b), d), e), f) from theorem 2 and also:*

c') The spectral radius is monotone increasing on commuting elements of K ($\varrho(x) \leq \varrho(x+z)$ when $xz=zx$, $x, z \in K$ holds), then a continuous involution $x \to x^*$ can be introduced in $\mathcal{A}$ so that it becomes a symmetric Banach star algebra and $H=\{x \mid x=x^*\}$, $K=\{x \mid x=x^*, Sp(x) \geq 0\}$.

The converse is also true (except d)).

Remark 3. We needed $\mathcal{A}$ to be semi-simple in the above theorems (condition d)) only to obtain continuity of the involution. It can be replaced with any other condition giving that continuity. For example, if H is closed (which is so, if K is closed), the involution is continuous [4, V. 36. 1.]. (In this case the converse is also true — if $\mathcal{A}$ is symmetric, K is closed according to a recent result of Aupetit about continuity of the spectrum in symmetric algebras.)

A theorem similar to theorem 2 can be proved, if c) is replaced with the condition:

c'') If $x, z \in K$ and $xz=zx$, then $\|x\| \leq \alpha \|x+z\|$ (α — a constant). (If K is a normal cone, this follows.) The proof will appear in another paper.

(So we can drop c) and d) if K is a closed normal cone. The converse is also true — if $\mathcal{A}$ is B^*-eguivalent, the set of all self-adjoint elements with non-negative spectrum is a closed normal cone.)

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Centre for Research and Education in Mathematics and Mechanics

1000 Sofia P. O. Box 373

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