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SOME PROPERTIES OF OPERATOR SPACES

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Conditions are obtained under which the Banach spaces $I_p(E, F)$ of p-integral operators and $\Pi_p(E, F)$ of absolutely p-summing operators:
(a) do not contain a subspace isomorphic to c_0,
(b) possess the Radon-Nikodým property.

The present paper is concerned with the study of properties of the Banach spaces $I_p(E, F)$ and $\Pi_p(E, F)$ of p-integral and absolutely p-summing operators, respectively. Gordon, Lewis, Retherford [1] and Saphar [2] dealt with the question of reflexivity of these spaces, and the weak sequential completeness was considered by the author in [3]. In the first part of this note we obtain conditions under which the spaces $I_p(E, F)$ and $\Pi_p(E, F)$ do not contain a subspace isomorphic to c_0. The second part is devoted to the study of the Radon-Nikodým property in these spaces. Kalton [4] and Diestel Morrison [5] studied the space $K(E, F)$ of compact operators from a similar point of view using, however, specific properties of $K(E, F)$.

1. Definitions and notation. Let E and F be Banach spaces. E' denotes the dual space of E and $L(E, F)$ the space of bounded linear operators from E to F with its usual norm. Instead of $L(E, E)$ we write $L(E)$. A Banach space E is said to have the bounded approximation property (b. a. p.) if there exists a net $(\sigma_\gamma)\subset L(E)$ of finite dimensional operators such that $\sup_\gamma \|\sigma_\gamma\|<\infty$, $\lim_\gamma \|\sigma_\gamma x-x\|=0$ for all $x\in E$. A Banach space E has the Radon-Nikodým property if every countably additive E-valued measure of finite total variation has a Bochner derivative with respect to its variation [6, ch. 6]. For given Banach spaces E and F we shall use the notation $E=F$ and $E\subset F$ to indicate the isomorphism and the isomorphic embedding, respectively.

Let $1\leq p<\infty$. An operator $T\in L(E, F)$ is called p-integral if there is a probability measure ν defined on the weak-star compact unit ball U^0 of E' such that jT admits the following factorization:

$$jT: E\xrightarrow{I_1} C(U^0)\xrightarrow{I_2} L_p(U^0, \nu)\xrightarrow{Q} F'',$$

where I_1, I_2 and $j: F\to F''$ are the corresponding canonical embeddings, and Q is a bounded linear operator with $\|Q\|\leq 1$. The p-integral norm of T is defined by $\iota_p(T)=\inf \nu(U^0)^{1/p}$, where the infimum is taken over all possible factorizations.

An operator $T\in L(E, F)$ is called absolutely p-summing whenever there is a constant $\varrho>0$ such that for all finite subsets $\{x_k\}\subset E$ the following inequality holds:

$$(\Sigma\|Tx_k\|^p)^{1/p}\leq\varrho \sup_{x'\in E',\ \|x'\|=1}(\Sigma|\langle x_k, x'\rangle|^p)^{1/p}.$$

The absolutely p-summing norm of T is defined by $\pi_p(T)=\inf\varrho$.

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An operator $T\in L(E,F)$ is said to be p-nuclear if it admits a representation of the form $Tx=\Sigma_{k=1}^{\infty}\langle x, x'_k\rangle y_k$, where $(x'_k)\subset E'$, $(y_k)\subset F$, $(\Sigma_{k=1}^{\infty}\|x'_k\|^p)^{1/p}=C_1<\infty$ and

$$\sup_{y'\in F',\ \|y'\|=1}(\sum_{k=1}^{\infty}|\langle y_k, y'\rangle|^q)^{1/q}=C_2<\infty \quad (1/p+1/q=1).$$

The p-nuclear norm is defined by $\nu_p(T)=\inf C_1C_2$.

Finally an operator $T\in L(E,F)$ is called quasi-p-nuclear, if jT is p-nuclear, where j is an isometric embedding of F into a space $l_\infty(\Gamma)$. The corresponding norm is defined by $\nu_p^Q(T)=\nu_p(jT)$.

The spaces of p-integral, absolutely p-summing, p-nuclear and quasi-p-nuclear operators are Banach spaces, denoted by $I_p(E,F)$, $\Pi_p(E,F)$, $N_p(E,F)$ and $N_p^Q(E,F)$, respectively. If E' or F possesses the bounded approximation property, then $N_p(E,F)\subset I_p(E,F)$ and $N_p^Q(E,F)\subset \Pi_p(E,F)$. For detailed information on the introduced operator spaces see [7].

Let $E\otimes F$ be the algebraic tensor product of E and F, and let α be a crossnorm on $E\otimes F$. The completion of $E\otimes F$ with respect to α is denoted by $E\widehat{\otimes}_\alpha F$. The crossnorm α is called uniform [8] if for all $T_1\in L(E)$ and $T_2\in L(F)$ we have $\|T_1\otimes T_2\|\leq\|T_1\|\,\|T_2\|$. Denote the extension of $T_1\otimes T_2$ to $E\widehat{\otimes}_\alpha F$ by $T_1\otimes_\alpha T_2$.

2. Operator spaces which do not contain c_0. Banach spaces which do not contain a subspace isomorphic to c_0 possess important properties [9]. We shall study here the question under which conditions the property "not containing c_0" is carried across from E' and F to $I_p(E,F)$ and $\Pi_p(E,F)$.

Theorem 1. *Let E and F be Banach spaces such that E' and F have the b. a. p. and let $1\leq p<\infty$. Suppose $E'\urcorner\supset c_0$ and $F\urcorner\supset c_0$.*

(a) If $I_p(E,F)=N_p(E,F)$, then $I_p(E,F)\urcorner\supset c_0$.

(b) If $\Pi_p(E,F)=N_p^Q(E,F)$, then $\Pi_p(E,F)\urcorner\supset c_0$.

The theorem will follow from the two propositions below. First we make a general remark. If α is a uniform crossnorm on $E'\otimes F$ and if F possesses the b. a. p., then the elements of $E'\widehat{\otimes}_\alpha F$ can be identified in the usual way with operators from E to F. It is easily seen that for arbitrary $S_1\in L(E)$, $S_2\in L(F)$ and $T\in E'\widehat{\otimes}_\alpha F$ we have

$$(S_1'\otimes_\alpha S_2)T=S_2TS_1, \tag{1}$$

where on the right-hand side T is regarded as an operator.

Proposition 1. *Let α be a uniform crossnorm on $E'\otimes F$ and let $\sigma\in L(E)$ and $\tau\in L(F)$ be finite dimensional operators. Further suppose $E'\urcorner\supset c_0$, $F\urcorner\supset c_0$ and F has the b. a. p. If a sequence $(T_n)\subset E'\widehat{\otimes}_\alpha F$ is equivalent to the unit vector basis of c_0, then*

$$\lim_{N\to\infty}\sup_{|\xi_k|\leq 1}\alpha(\sum_{k=N}^{\infty}\xi_kT_k\sigma)=0,\ \lim_{N\to\infty}\sup_{|\xi_k|\leq 1}\alpha(\sum_{k=N}^{\infty}\xi_k\tau T_k)=0.$$

Proof. It is sufficient to verify the first equation. By hypothesis, the series ΣT_n is weakly unconditionally convergent. Denote the identity of F by I_F. Since $\sigma'\otimes_\alpha I_F$ is bounded, the series $\Sigma_n(\sigma'\otimes_\alpha I_F)T_n$ is weakly unconditionally convergent, too. We have $\dim \operatorname{Im}\sigma'=m<\infty$, therefore $\operatorname{Im}(\sigma'\otimes_\alpha I_F)=(\operatorname{Im}\sigma')\otimes F$

$\subset E'\hat{\otimes}_\alpha F$. It follows from the properties of crossnorms that $(\mathrm{Im}\,\sigma')\otimes F$ is isomorphic to the direct sum of m copies of F. Since $F\urcorner\supset c_0$, we have $(\mathrm{Im}\,\sigma')\otimes F\urcorner\supset c_0$. Thus, the series $\Sigma_n(\sigma'\otimes_\alpha I_F)\,T_n$ is unconditionally convergent [9]. Now the properties of unconditionally convergent series and (1) together yield the required equation.

Let α be a uniform crossnorm on $E'\otimes F$ and suppose F has the b. a. p. We shall say that α is boundedly complete on $E'\otimes F$ if for each sequence $(T_n)\subset E'\hat{\otimes}_\alpha F$ with $\sup\alpha(T_n)<\infty$ and $\lim_{m,n\to\infty}\|T_m x-T_n x\|=0$ $(x\in E)$ there is a $T_0\in E'\hat{\otimes}_\alpha F$ such that $\lim_{n\to\infty}\|T_n x-T_0 x\|=0$.

Proposition 2. *Suppose E' and F have the b. a. p. Let α be a uniform boundedly complete crossnorm on $E'\otimes F$. If $E'\urcorner\supset c_0$ and $F\urcorner\supset c_0$, then* $[E'\hat{\otimes}_\alpha F\urcorner\subset c_0$.

Proof. Assume that $c_0\subset E'\hat{\otimes}_\alpha F$ and let $(T_n)\subset E'\hat{\otimes}_\alpha F$ be a sequence which is equivalent to the unit vector basis of c_0. Without loss of generality we may assume that $\alpha(T_n)\geq 1$. Since the dual space E' possesses the b. a. p., we can find a net of finite dimensional operators $(\sigma_\gamma)\subset L(E)$ such that $\|\sigma_\gamma\|\leq C_1$ and for every $x'\in E'$, $\lim_\gamma\|\sigma'_\gamma x'-x'\|=0$. We shall define now successively a subsequence (T_{n_k}) and a sequence of finite dimensional operators $(\sigma_k)\subset L(E)$ such that $\|\sigma_k\|\leq C_1$ and for a given $\varepsilon>0$ the following inequalities hold:

$$\sum_{j=1}^{k}\alpha(T_{n_j}\sigma_k-T_{n_j})<\varepsilon, \tag{2}$$

and

$$\alpha\Big(\sum_{j=k+1}^{\infty}T_{n_j}\sigma_k\Big)<\varepsilon. \tag{3}$$

Put $n_1=1$ and choose by the b. a. p. of E' a $\sigma_1\in L(E)$ with $\alpha(T_1\sigma_1-T_1)<\varepsilon$. It follows now from Proposition 1 that

$$\lim_{N\to\infty}\sup_{|\xi_k|\leq 1}\alpha\Big(\sum_{k\geq N}\xi_k T_k\sigma_1\Big)=0.$$

Hence there is an n_2 such that for every increasing sequence $n_2=m_2<m_3<m_4<\cdots$ we have $\alpha(\Sigma_{j=2}^\infty T_{m_j}\sigma_1)<\varepsilon$. Next choose σ_2 such that $\alpha(T_{n_1}\sigma_2-T_{n_1})<\varepsilon/2$ and $\alpha(T_{n_2}\sigma_2-T_{n_2})<\varepsilon/2$. Again by Proposition 1 there is an n_3 with $\alpha(\Sigma_{j=3}^\infty T_{m_j}\sigma_2)<\varepsilon$ for all sequences $n_3=m_3<m_4<m_5<\cdots$. Continuing this selection process, we get the desired sequences.

Since (T_{n_k}) is also equivalent to the unit vector basis of c_0, the series $\Sigma_{k=1}^\infty T_{n_k}x$ is weakly unconditionally convergent in F. By hypothesis, we have $F\urcorner\supset c_0$. Thus, the series is unconditionally convergent [9] and in particular norm convergent. Again by hypothesis there is an element $T_0\in E'\hat{\otimes}_\alpha F$ such that

$$T_0x=\sum_{k=1}^{\infty}T_{n_k}x \tag{4}$$

for each $x\in E$.

Since F possesses the b. a. p., too, there is a finite dimensional operator $\tau \in L(F)$ such that $\|\tau\| \leq C_2$ (the b. a. p. constant of F) and $\alpha(T_0-\tau T_0)<\varepsilon$. It follows from Proposition 1 that there exists an index l with

$$\alpha(\sum_{k=l}^{\infty} \tau T_{n_k})<\varepsilon. \tag{5}$$

Now

$$\alpha[(I_F-\tau) T_0(\sigma_l-\sigma_{l-1})] \leq \|\sigma_l-\sigma_{l-1}\| \alpha[(I_F-\tau) T_0] \leq 2C_1 \varepsilon. \tag{6}$$

Elementary calculations and (4) yield

$$(I_F-\tau) T_0(\sigma_l-\sigma_{l-1})=(I_F-\tau)\sum_{k=1}^{\infty} T_{n_k}(\sigma_l-\sigma_{l-1})$$

$$=(I_F-\tau)\sum_{k=1}^{l-1} T_{n_k}(\sigma_l-\sigma_{l-1})+(I_F-\tau)\sum_{k=l}^{\infty} T_{n_k}(\sigma_l-\sigma_{l-1})$$

$$=(I_F-\tau)\sum_{k=1}^{l-1} T_{n_k}(\sigma_l-\sigma_{l-1})-(I_F-\tau)\sum_{k=l}^{\infty} T_{n_k}\sigma_{l-1}+\sum_{k=l+1}^{\infty} T_{n_k}\sigma_l-\sum_{k=l}^{\infty}\tau T_{n_k}\sigma_l+T_{n_l}\sigma_l.$$

It follows from (2) and (3) that

$$\alpha[(I_F-\tau)\sum_{k=1}^{l-1} T_{n_k}(\sigma_l-\sigma_{l-1})] \leq \|I_F-\tau\| \sum_{k=1}^{l-1}\alpha[T_{n_k}(\sigma_l-\sigma_{l-1})]$$

$$\leq 2C_2\sum_{k=1}^{l-1}[\alpha(T_{n_k}\sigma_l-T_{n_k})+\alpha(T_{n_k}\sigma_{l-1}-T_{n_k})]<4C_2\varepsilon.$$

Likewise

$$\alpha[(I_F-\tau)\sum_{k=l}^{\infty} T_{n_k}\sigma_{l-1}]<2C_2\varepsilon, \quad \alpha(\sum_{k=l+1}^{\infty} T_{n_k}\sigma_l)<\varepsilon, \quad \alpha(T_{n_l}-T_{n_l}\sigma_l)<\varepsilon.$$

Finally it follows from (5) that $\alpha(\Sigma_{k=l}^{\infty}\tau T_{n_k}\sigma_l)<\varepsilon C_1$. Consequently,

$$\alpha[(I_F-\tau) T_0(\sigma_l-\sigma_{l-1})]>\alpha(T_{n_l})-(6C_2+C_1+2)\varepsilon \geqq 1-(6C_2+C_1+2)\varepsilon$$

since we assumed that $\alpha(T_n) \geqq 1$. Choose now ε such that $1-(6C_2+C_1+2)\varepsilon>2C_1\varepsilon$ Then the above inequality together with (6) yield a contradiction, which concludes the proof.

P r o o f of Theorem 1. By hypothesis we have $I_p(E, F)=N_p(E, F)$ and, since F has the b. a. p., $N_p(E, F)=E'\widehat{\otimes}_{g_p} F$, where g_p is a uniform crossnorm [2, 10]. We shall verify that $I_p(E, F)$ satisfies the completeness condition. Let $(T_n)\subset I_p(E, F)$ be a Cauchy sequence in the strong operator topology and suppose $\sup_n \iota_p(T_n)<\infty$. Since

$$I_p(E, F'')=(F'\widehat{\otimes}_{g_q} E)', \quad (1/p+1/q=1)$$

[10] and since $I_p(E, F)$ can be embedded isometrically into $I_p(E, F'')$, the sequence (T_n) is weak-star Cauchy in $I_p(E, F'')$. Consequently (T_n) converges in the weak-star topology to an element $T_0 \in I_p(E, F'')$. Now it follows that $T_0=\lim_{n\to\infty} T_n$ in the strong operator topology and thus $\operatorname{Im} T_0 \subset F$. This implies $T_0 \subset I_p(E, F)$.

The completeness of $\Pi_p(E, F)=E'\widehat{\otimes}_{\varepsilon_p} F$ follows from the relation $\Pi_p(E, F) \subset \Pi_p(E, F'')=(F'\widehat{\otimes}_{g_q} E)'$, $1/p+1/q=1$.

The following proposition is due to A. Persson [11]. It was established in the case when E' is separable or reflexive, but the proof is actually valid for E' possessing the Radon-Nikodým property.

Proposition 3. *If E' has the Radon-Nikodým property, then for $1\leq p<\infty$*

(a) each strongly p-integral operator from E to F is p-nuclear and

(b) each absolutely p-summing operator from E to F is quasi-p-nuclear.

Remark. An operator $T\in L(E, F)$ is called strongly p-integral if it admits a factorization of the form

$$T: E \xrightarrow{I_1} C(U^0) \xrightarrow{I_2} L_p(U^0, \nu) \xrightarrow{Q} F,$$

where I_1, I_2, Q and ν are the same as in definition of p-integral operators.

Theorem 2. *Let E and F be Banach spaces such that E' and F have the b. a. p. and let $1\leq p<\infty$. If E' has the Radon-Nikodým property and $F \neg\supset c_0$, then $I_p(E, F) \neg\supset c_0$ and $\Pi_p(E, F) \neg\supset c_0$.*

Proof. If E' has the Radon-Nikodým property, then $E' \neg\supset c_0$ [6]. Thus, we need to show only $I_p(E, F)=N_p(E, F)$. Let $T\in I_p(E, F)$. Then $jT: E\to F''$ is strongly p-integral, thus p-nuclear. Since E' has the b. a. p., we conclude that $T\in N_p(E, F)$ [12, ch. I, prop. 15].

The following example shows that the assumptions $I_p(E, F)=N_p(E, F)$ and $\Pi_p(E, F)=N_p^Q(E, F)$ of Theorem 1 are essential at least for $1<p<\infty$.

Example. If $1<p<\infty$, then $I_p(C[0, 1], L_p[0, 1])\supset c_0$ and $\Pi_p(C[0, 1], L_p[0, 1])\supset c_0$.

Indeed, let I denote the identity operator from $C[0, 1]$ into $L_p[0, 1]$. I is p-integral but not compact hence not p-nuclear [7]. $L_p[0, 1]$ has an unconditional basis. Denote the associated sequence of one-dimensional projections by (P_n). The series $\Sigma_n P_n I$ is not norm convergent in $I_p(C, L_p)$, since I is not p-nuclear. Thus, there is a block sequence $Q_n=\Sigma_{k=m_n+1}^{m_{n+1}} P_k$ such that $\inf_n \iota_p(Q_n I) \geq C_1>0$. It follows immediately from the properties of an unconditional basis that there is a constant C_2 such that

$$C_1 C_2^{-1} \sup_n |\xi_n| \leq \iota_p(\Sigma_n \xi_n Q_n I)\leq C_2\, \iota_p(T) \sup_n |\xi_n|$$

for each $(\xi_n)\in c_0$. Hence $(Q_n I)$ is equivalent to the unit vector basis of c_0.

The second assertion follows from the relation $\Pi_p(C[0, 1], L_p[0, 1]) = I_p(C[0, 1], L_p[0, 1])$, which is established in [7].

3. Operator spaces with the Radon-Nikodým property. Motivated by Theorem 1 and Proposition 3 we shall show here that the Radon-Nikodým property is carried across from E' and F to $I_p(E, F)$ and $\Pi_p(E, F)$. The additional assumption on E of Theorem 3 (b) seems to be a technical one. However, it is not known if this assumption can be omitted.

Theorem 3. *Let E and F be Banach spaces such that E' and F possess the b. a. p. and let $1\leq p<\infty$. Suppose E' and F have the Radon-Nikodým property. Then*

(a) $\Pi_p(E, F)$ has the Radon-Nikodým property and

(b) $I_p(E, F)$ has the Radon-Nikodým property provided E is WCG.

The essential part of the proof is contained in

Proposition 4. *Suppose E and F are separable and F has the b. a. p. Let α be a uniform boundedly complete crossnorm on $E' \otimes F$. If E' and F have the Radon-Nikodým property, then $E' \widehat{\otimes}_\alpha F$ has this property, too.*

Proof. Let (Ω, Σ) be a measurable space and let $\mu : \Sigma \rightarrow E' \widehat{\otimes}_\alpha F$ be a countably additive measure with finite variation $|\mu|$. Since F is separable and possesses the b. a. p., there is a sequence of finite dimensional operators τ_n such that $\|\tau_n\| \leq C$ and

$$\lim_{n\rightarrow\infty} \|\tau_n y - y\| = 0 \tag{7}$$

for each $y \in F$. $\tau_n \mu$ is a measure of finite variation taking values in the closed subspace $E' \otimes \tau_n(F)$ of $E' \widehat{\otimes}_\alpha F$. This subspace is isomorphic to the direct sum of a finite number of copies of E'. Hence $E' \otimes \tau_n(F)$ has the Radon-Nikodým property. It follows that there is a Bochner integrable function $T_n(\omega)$ with

$$\tau_n \mu(A) = \int_A T_n(\omega) d|\mu| \tag{8}$$

for each $A \in \Sigma$. Moreover $|\tau_n \mu|(A) \leq \|\tau_n\| \, |\mu|(A) \leq C |\mu|(A)$ and $|\tau_n \mu|(A) = \int_A \alpha(T_n(\omega))\, d|\mu|$, $A \in \Sigma$ [13, ch. III]. By the scalar Radon-Nikodým theorem we get

$$\alpha(T_n(\omega)) \leq C \tag{9}$$

for $|\mu|$ — almost all $\omega \in \Omega$.

Let now $\{x_m\}$ be a dense countable subset of E. The measure $\mu(\cdot)x_m$ is F-valued and of finite variation. Thus, there exists a Bochner integrable function $y_m(\omega)$ such that, for $A \in \Sigma$

$$\mu(A)\, x_m = \int_A y_m(\omega)\, d|\mu|. \tag{10}$$

Hence $\tau_n \mu(A)\, x_m = \int_A \tau_n y_m(\omega)\, d|\mu|$ and by (8) $\tau_n y_m(\omega) = T_n(\omega)\, x_m$, $|\mu|$ - a. e. It follows from (7) that

$$\lim_{n\rightarrow\infty} \|T_n(\omega)\, x_m - y_m(\omega)\| = 0, \quad |\mu|\text{-a. e.} \tag{11}$$

By (9) and the density of $\{x_m\}$ we have for $|\mu|$ — almost all $\omega \in \Omega$,

$$\lim_{n_1, n_2 \rightarrow\infty} \|T_{n_1}(\omega)x - T_{n_2}(\omega)x\| = 0 \quad (x \in E).$$

Since α is boundedly complete, there is a function $T : \Omega \rightarrow E \widehat{\otimes}_\alpha F$ with $\lim_{n\rightarrow\infty} \|T_n(\omega)x - T(\omega)x\| = 0$, $(x \in E)$, $|\mu|$-almost everywhere. By (11)

$$T(\omega)x_m = y_m(\omega) \quad |\mu|\text{-a. e.} \tag{12}$$

Thus $\tau_n T(\omega)x_m = \tau_n y_m(\omega) = T_n(\omega)x_m$ and therefore $\tau_n T(\omega) = \tau_n(\omega)$ for almost all $\omega \in \Omega$. Since $T(\omega) \in E' \widehat{\otimes}_\alpha F$, it follows by (7) that $\lim_{n\rightarrow\infty} \alpha[\tau_n T(\omega) - T(\omega)] = 0$, $|\mu|$-a. e. Now the Dominated convergence theorem [13, Ch. III] yields

$$\lim_{n\rightarrow\infty} \int_\Omega \alpha[T_n(\omega) - T(\omega)]\, d|\mu| = 0.$$

Hence $T(\omega)$ is Bochner integrable and it follows from (10) and (12) that $\mu(A)x_m = \int_A T(\omega)x_m\, d|\mu|$. Thus $\mu(A) = \int_A T(\omega)\, d|\mu|$ $(A \in \Sigma)$. This concludes the proof.

Proof of Theorem 3. It will be sufficient to show that separable subspaces of $\Pi_p(E, F)$ and $I_p(E, F)$ possess the Radon-Nikodým property [6].

(a): Let $B \subset \Pi_p(E, F)$ be a separable subspace with a dense sequence (T_n). By the definition of the π_p-norm there is a separable subspace $E_1 \subset E$ such that $\pi_p(T_n|_{E_1}) = \pi_p(T_n)$. Since the norm of the restriction of any $T \in \Pi_p(E, F)$ is not greater than $\pi_p(T)$, we can embed B isometrically into $\Pi(E_1, F)$. The images of $T_n|_{E_1}$ are contained in a separable subspace $F_0 \subset F$. It is easily seen that given a separable subspace F_0 of a space with b. a. p. F there is a separable subspace F_1 such that $F_0 \subset F_1 \subset F$ and F_1 has the b. a. p., too. Now we have $B \subset \Pi_p(E_1, F_0) \subset \Pi_p(E_1, F_1)$. Since E_1' has the Radon-Nikodým property [6], we get $\Pi_p(E_1, F_1) = N_p^Q(E_1, F_1)$ and since F_1 has the b. a. p., $N_p^Q(E_1, F_1) = E_1' \widehat{\otimes}_{\varepsilon_p} F_1$, where ε_p is uniform. It follows in the same way as in the proof of Theorem 2 that ε_p is boundedly complete on $E_1' \otimes F_1$.

(b): Let $B \subset I_p(E, F)$ be separable and let (T_n) be a dense sequence in B. We have $I_p(E, F) \subset I_p(E, F'') = (F' \widehat{\otimes}_{\varepsilon_q} E)'$, $(1/p + 1/q = 1)$. Using the right-injectivity of the crossnorm ε_q [2, 10], we can find a separable subspace $E_1 \subset E$ such that the restriction of each T_n to $F' \widehat{\otimes}_{\varepsilon_q} E$ preserves the norm of T_n. Since E is WCG, we may assume that E_1 is complemented in E [6, ch. 5]. Now

$$(F' \widehat{\otimes}_{\varepsilon_q} E_1)' = I_p(E_1, F'')$$

and we conclude that the restriction $T|_{E_1}$ defines an isometric embedding of B into $I_p(E_1, F)$. Since E_1' has the b. a. p. and the Radon-Nikodým property, we have $I_p(E_1, F) = N_p(E_1, F)$. Using now the definition of the p-nuclear norm, we can find a separable subspace $F_1 \subset F$ such that F_1 has the b. a. p. and B can be embedded isomorphically into $N_p(E_1, F_1)$. Finally,

$$I_p(E_1, F_1) = N_p(E_1, F_1) = E_1' \widehat{\otimes}_{g_p} F_1,$$

and the proof of Theorem 2 yields again that g_p is boundedly complete on $E_1' \otimes F_1$. This concludes the proof.

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