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QUASILINEAR DEGENERATE ELLIPTIC-PARABOLIC EQUATIONS OF SECOND ORDER

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The first boundary value problem is studied for a class of quasilinear second order partial differential equations with nonnegative characteristic form. Existence and uniqueness theorems for classical (i. e. at least two times differentiable) solutions are proved, the hypotheses for the equations, as well as the results obtained being quite naturally arising analogues of the linear case.

In the present paper the first boundary value problem is studied for the following class of second-order quasilinear equations:

$$L(u)\equiv a^{ij}(x,u)u_{ij}+a^{i}(x,u)u_{i}-a(x,u)u=f(x) \tag{1}$$

with

$$a^{ij}(x,y)\xi_i\xi_j\geqq 0 \tag{2}$$

(subscripts are used to denote differentiation and summation convention is accepted where $(x,y)\in G\times[-M,M]$, G denotes a region in R^n and M is a positive constant.

The correct formulation of the boundary value problem for the linear equation

$$L(u)\equiv a^{ij}(x)u_{ij}+a^{i}(x)u_{i}-a(x)u=f(x) \tag{3}$$

with

$$a^{ij}(x)\xi_i\xi_j\geqq 0 \tag{4}$$

is given in [1]. Namely let according to [1] $G\subset R^n$ be a region with a piece wise smooth boundary Σ, which is divided into three parts Σ_1, Σ_2 and Σ_3 n the following manner: Σ_3 is the set of points $x\in\Sigma$ with $a^{ij}(x)\nu_i(x)\nu_j(x)>0$ (where $\nu_i(x)$ denotes the i-th component of the unit exterior normal at a point $x\in\Sigma$) and Σ_2 and Σ_1 are the subsets of $\Sigma\setminus\Sigma_3$, for which $b(x)>0$ and $b(x)\leqq 0$ respectively with $b(x)=(a^i(x)-a^{ij}_j(x))\nu_i(x)$. Then the first boundary value problem for (3) is that of finding a solution with given values on $\Sigma_2\cup\Sigma_3$ only.

Taking into account the linear case let us assume that the boundary of the region G in which we study the equation (1) is piecewise smooth and $\partial G=S_3\cup S$, where

$$S_3=\{x\in\partial G: a^{ij}(x,y)\nu_i(x)\nu_j(x)>0,\ y\in[-M,M]\}, \tag{5}$$

$$S=\{x\in\partial G: a^{ij}(x,y)\nu_i(x)\nu_j(x)=0,\ y\in[-M,M]\}; \tag{6}$$

if

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(7) $$b(x, y)=(a^i(x, y)-a^{ij}_j(x, y))\nu_i(x),$$

let $S=S_1\cup S_2$, where

(8) $$S_1=\{x\in S: b(x, y)\leqq 0,\ y\in[-M, M]\},$$

(9) $$S_2=\{x\in S: b(x, y)>0,\ y\in[-M, M]\}.$$

The crucial assumptions made in the present paper are:

(i) $a(x, y)$ in (1) is very large positive compared to $f(x)$, to the other coefficients and their derivatives up to second order, including the derivatives of $a(x, y)$ as well;

(ii) the equation (1) is defined in (or can be extended to) a larger region $G'\supset \bar{G}$, the characteristic form of the extended equation being also nonnegative. The assumption (i) is widely used in the linear case (3), cf. [2—6].

We shall first prove the existence and the uniqueness of the solution of (1) for a region G with $\partial G=S_3$; next, using the technique of extension of the equation (1) to a larger region as in the linear case [2—5], we study the case $\partial G=S_1\cup S_2\cup S_3$ with some additional assumptions concerning the disposition of the sets S_1, S_2 and S_3 on ∂G. In a previous article [7] we have studied the case $\partial G=S_2$.

The proof of the main result (theorem 1) is carried out by a combined version of the methods of elliptic regularisation and the method of succesfive iterations; all the necessary a priori estimations are made by the aid of modifications of the well-known method of Bernstein [8].

The following traditional notations are accepted. If $f(x)$ and $g(x, y)$ are differentiable functions with domains in R^n and R^{n+1} respectively, let by definition

$$D^\alpha f(x)=\partial^{|\alpha|}f(x_1, \ldots, x_n)/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n},$$

$$D^{\alpha+p}g(x, y)=\partial^{|\alpha|+p}g(x_1, \ldots, x_n, y)/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}\partial y^p,$$

where $\alpha=(\alpha_1, \ldots, \alpha_n)$ is an ordered n-tuple of nonnegative intgers and $|\alpha|=\Sigma_{\nu=1}^n\alpha_\nu$. If k is nonnegative integer and $f(x)$ is k times continuously differentiable in $\bar{\Omega}$, let $\|f\|_{C^\circ(\bar{\Omega})}=\max_{\bar{\Omega}}|f(x)|$ and

(10) $$\|f\|^2_{C^k(\bar{\Omega})}=\Sigma_{|\alpha|\leqq k}\|D^\alpha f\|^2_{C^\circ(\bar{\Omega})}.$$

The following inequalities are frequently used:

(11) $$|ab|\leqq|a^p|/p+|b^q|/q\ (p, q>0,\ 1/p+1/q=1),$$

(12) $$2\,ab\leqq\delta a^2+b^2/\delta\quad(\delta>0)$$

and $(\Sigma_{\nu=1}^N|a_\nu|)^2\leqq N\,\Sigma_{\nu=1}^N a_\nu^2$.

The next three propositions concerning the operator (3) are repeatedly used:

1. A variant of the maximum principle: let the operator (3) with (4) be defined in $\Omega\subset R^n$ and $a(x)>0$ in $\bar{\Omega}$; then for every function $u\in C^2(\Omega)\cap C^\circ(\bar{\Omega})$, $L(u)\geqq 0$, resp. $L(u)\leqq 0$, in Ω and $u|_{\partial\Omega}\leqq 0$, resp. $u|_{\partial\Omega}\geqq 0$, imply $u\leqq 0$, resp. $u\geqq 0$, in Ω [9; 11].

2. An estimate due to Fichera: let the operator (3) with (4) be defined in a region $G \subset R^n$ with a piecewise smooth boundary, $a(x) > 0$ in G, $a^{ij} \in C^2(G)$, $a^i \in C^1(G)$, $a \in C^\circ(G)$; then for every function $u \in C^2(G) \cap C^1(\bar{G})$, such that $L(u)$ is bounded in G and $u|_{\Sigma_2 \cup \Sigma_3} = 0$, the inequality $\max_G |u| \leq \sup_G |L(u)/a|$ holds [1, (V)].

3. An inequality concerning the leading part of the operator (3) with (4): let $a^{ij} \in C^2(R^n)$ be bounded in R^n; then for every function $u \in C^2(R^n)$ the inequality

$$(13) \qquad (a_k^{ij}(x) u_{ij}(x))^2 \leq (C(n) \sup_{|\alpha| = 2} |D^\alpha a^{ij}|) \, a^{ij}(x) u_{ik}(x) u_{jk}(x)$$

holds, where $C(n)$ is a constant depending on n only [5].

Throughout this paper, unless the contrary is explicitly mentioned, we shall assume that:

a) Ω is a bounded region in R^n with a boundary $\partial\Omega$ which is locally defined in the form $\varphi(x_1, \ldots, x_n) = 0$ with $\operatorname{grad} \varphi \neq 0$ and is of the class C^6, i. e. φ is of the class C^6.

b) the coefficients of the leading part of the operator (1) with (2) are defined in $\Omega' \times [-M, M]$, where $\Omega' \supset \bar{\Omega}$, $M > 0$ and are at least four times continuously differentiable in their domain, $\partial\Omega = S_3$ and

$$(14) \qquad a^{ij}(x, y) \varphi_i(x) \varphi_j(x) \geq 4a_0 > 0 \quad (a_0 = \text{constant})$$

on $\partial\Omega \times [-M, M]$.

c) the remaining coefficients are defined and at least four times continuously differentiable in $\Omega \times [-M, M]$ and

$$(15) \qquad a(x, y) = a_0 + c(x, y),$$

where a_0 is a positive constant and $c(x, y) \geq 0$ in $\Omega \times [-M, M]$.

If $u \in C^3(\Omega)$, let by definition $|D^k u| = \max_{|\alpha| = k} |D^\alpha u|_{C^\circ(\bar{\Omega})}$ $(k = 1, 2, 3)$, and for

$$(16) \qquad m_k = \max_{\Omega \times [-M, M]} \{ |D^{\alpha+p} a^{ij}|, \ |D^{\alpha+p} a^i|, \ |D^{\alpha+p} c| \}$$

$(|\alpha| + p = k, \ k = 0, 1, 2, 3)$, let

$$(17) \qquad C_1(u) = m_1(1 + |D^1 u|),$$

$$(18) \qquad C_2(u) = m_2(1 + |D^1 u| + |D^1 u|^2) + m_1 |D^2 u|,$$

$$(19) \qquad C_3(u) = m_3(1 + |D^1 u| + |D^1 u|^2 + |D^1 u|^3) + m_2 |D^2 u| (1 + |D' u|) + m_1 |D^3 u|.$$

We shall first study the operators

$$(20) \qquad L_\varepsilon(v, u) \equiv \varepsilon \Delta u + a^{ij}(x, v) u_{ij} + a^i(x, v) u_i - a(x, v) u,$$

where Δ denotes the Laplace operator, $v \in C^4(\Omega)$, $|v|_{C^\circ(\Omega)} < M$, $\varepsilon > 0$ and find a priori estimates (lemmas 1—3) for the solutions and their derivatives up to second order of the boundary value problem

$$(21) \qquad L_\varepsilon(v, u) = f(x)$$

in Ω and

$$(22) \qquad u|_{\partial\Omega} = 0$$

with some additional boundedness assumptions about the functions v. The existence and uniqueness of solutions $u \in C^4(\Omega)$ follow from the well known theorem of Schauder [10, p. 235] taking into account the hypotheses a) — c) and the assumption $v \in C^4(\Omega)$.

Lemma 1. *If u is a solution of the boundary value problem* (21), (22), *then*

$$\|u\|_{C^0(\Omega)} \leq a_0^{-1} \|f\|_{C^0(\Omega)}. \tag{23}$$

Proof. For $\lambda = a_0^{-1}\|f\|_{C^0(\Omega)}$, (21), (15) and (20) imply

$$L_\varepsilon(v, \pm u - \lambda) = \pm f + a\lambda \geq -\|f\|_{C^0(\Omega)} + a_0\lambda = 0.$$

From $\pm u - \lambda|_{\partial\Omega} \leq 0$, the above inequality and the maximum principle it follows $\pm u - \lambda \leq 0$ in Ω, i. e. (23).

Let $\xi \in \partial\Omega$ and V_ξ be a neighbourhood of ξ in $\bar{\Omega}$, such that

$$a^{ij}(x, y)\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j} \geq 2a_0$$

holds for $(x, y) \in V_\xi \times [-M, M]$. The existence of such a neighbourhood follows from (14). Since grad $\varphi \neq 0$ we may suppose (may be with a change of numeration) that $\partial\varphi/\partial x_n \neq 0$. Now without loss of generality we may suppose that the image of V_ξ by the transformation

$$y_k = x_k,\ k = 1, \ldots, n-1,\ t = \varphi(x_1, \ldots, x_n) \tag{24}$$

contains the set $\{(y, t) \in R^n : \Sigma_{k=1}^{n-1} y_k^2 + t^2 < 4,\ t \geq 0\}$.

Let $\psi(y_1, \ldots, y_{n-1})$ be a twice continuously differentiable function, satisfying

$$0 \leq \psi \leq 1 \tag{25}$$

and

$$\psi(y) = \begin{cases} 1 \text{ for } |y| < 1 \\ 0 \text{ for } |y| > 2. \end{cases} \tag{26}$$

Since $\partial\Omega$ is a compact set, there exists a finite number of functions φ such that the inverse images of the sets $\{|y| < 1,\ t = 0\}$ by the transformations (24) cover $\partial\Omega$. In what follows we shall denote by φ whichever of these functions and by K a constant that majorizes the derivatives up to third order of the transformations (24) and their inverse transformations. Let $\varkappa$ denote a constant such that

$$|\partial\psi/\partial y_k| \leq \varkappa,\quad |\partial^2\psi/\partial y_k \partial y_l| \leq \varkappa \quad k, l = 1, \ldots, n-1. \tag{27}$$

By means of the transformations (24) the equations (21) take the form

$$\tilde{L}_\varepsilon(v; u) \equiv Au_{tt} + 2A^k u_{kt} + A^{kl} u_{kl} + B^k u_k + Bu_t - Cu = F(y, t), \tag{28}$$

where

$$
(29)\qquad
\begin{aligned}
A &= a^{ij}\varphi_i\varphi_j + \varepsilon\varphi_i\varphi_i \geqq 2a_0, \\
A^k &= a^{ik}\varphi_i + \varepsilon\varphi_k & k=1,\dots,n-1 \\
A^{kl} &= a^{kl} + \delta_k^l\,\varepsilon & k,l=1,\dots,n-1 \\
B &= a^{ij}\varphi_{ij} + a^i\varphi_i + \varepsilon\Delta\varphi \\
B^k &= a^k & k=1,\dots,n-1 \\
C &= a
\end{aligned}
$$

and δ_k^l denotes the Kronecker symbol. We shall consider also the operator

$$(30)\qquad \widetilde{M}_\varepsilon(v\,;u) \equiv Au_{tt} + 2A^k u_{kt} + A^{kl}u_{kl} + B^k u_k + Bu_t.$$

Obviousy

$$(31)\qquad \widetilde{L}_\varepsilon(v\,;u) = \widetilde{M}_\varepsilon(v\,;u) - Cu.$$

Further we shall need the value of the operator (30) for functions of the form

$$(32)\qquad w(y,t) = \exp\left[-\gamma t + \beta\psi(y)\right],$$

where β and γ are positive constants and ψ is defined by (26). Now (29), (30) and (32) imply

$$
\begin{aligned}
M_\varepsilon(v\,;w) &= w\{A\gamma^2 - (2A^k\beta\psi_k + B)\gamma - (A^{kl}\beta^2\psi_k\psi_l + A^{kl}\beta\psi_{kl} + B^k\beta\psi_k)\} \\
&= w\{\gamma^2(a^{ij}\varphi_i\varphi_j + \varepsilon\varphi_i\varphi_i) - \gamma[(a^{ik} + \varepsilon\delta_i^k)\varphi_i\psi_k\beta + a^{ij}\varphi_{ij} + \varepsilon\Delta\varphi + a^i\varphi_i] \\
&\quad - (a^{kl} + \varepsilon\delta_k^l)(\beta^2\psi_k\psi_l + \beta\psi_{kl}) + a^k\beta\psi_k\},
\end{aligned}
$$

where the summation is carried out from 1 to n with respect to i and j and from 1 to $n-1$ with respect to k and l. The first equality (29) implies the existence of a constant $\gamma_0(\beta)$ such that $\gamma \geqq \gamma_0(\beta)$ implies

$$(33)\qquad \widetilde{M}_\varepsilon(v\,;w) \geqq a_0 w\gamma^2.$$

Elementary calculations show that (33) holds for instance when

$$
\begin{aligned}
\gamma_0 \geqq \max\{&a_0^{-1}\,|(a^{ik} + \varepsilon\delta_i^k)\varphi_i\psi_k\beta + a^{ij}\varphi_{ij} + \varepsilon\Delta\varphi + a^i\varphi_i| \\
&+ a_0^{-1/2}\,|(a^{kl} + \varepsilon\delta_k^l)(\beta^2\psi_k\psi_l + \beta\psi_{kl}) + a^k\beta\psi_k|^{1/2}\}
\end{aligned}
$$

or because of (27), (15) and (16) with ε sufficiently small, when

$$
\begin{aligned}
(34)\qquad \gamma_0(\beta) &= m_0 a_0^{-1/2}(\varkappa\beta + 1)[Ka_0^{-1/2}(n+1)^2 - (n-1)] + \beta \\
&> m_0 K a_0^{-1}(n^2\varkappa\beta + n^2 + n + 1) + m_0 a_0^{-1/2}[(n-1)^2\varkappa^2\beta^2 + (n-1)^2\varkappa\beta + (n-1)\varkappa\beta + 1]^{1/2}.
\end{aligned}
$$

We shall make use of the identities

$$(35)\qquad L_\varepsilon(v\,;uw) = uL_\varepsilon(v\,;w) + wL_\varepsilon(v\,;u) + a^{ij}(u_i w_j + u_j w_i) + auw$$

and

$$(36)\qquad L_\varepsilon(v\,;u^N) = Nu^{N-1}L_\varepsilon(v\,;u) + (N-1)au^N + N(N-1)u^{N-2}S_\varepsilon(u)$$

for every integer $N \geqq 2$, where by definition

$$(37)\qquad S_\varepsilon(u) = a^{ij}u_i u_j + \varepsilon u_i u_i\,.$$

The identity (36) is proved by induction, using (35).

As it is easily seen (for instance by straightening the boundary of Ω and reflexions as in [11, § 4.8] or [12, § 8.1]), a function $v \in C^2(\bar{\Omega})$ can be extended to a function $\tilde{v} \in C^2(R^n)$ in such a manner, that

$$\|\tilde{v}\|_{C^2(R^n)} \leqq \varkappa_1 \|v\|_{C^2(\Omega)}, \tag{38}$$

where the constant $\varkappa_1$ depends on the boundary of Ω only.

Let $\chi \in C^2(R^n)$, $\chi = 1$ in a neighbourhood of Ω and supp $\chi \subset \Omega'$. Now if $\varkappa_1 \|v\|_{C^0(\Omega)} < M$ then $\chi(x) a^{ij}(x, \tilde{v})$ satisfies the hypothesis 3 from above and (13), (18), (37) and (54) imply the inequality

$$\left(\frac{\partial}{\partial x_k} a^{ij}(x, v) u_{ij}\right)^2 \leqq \varkappa_2 C_2(v) a^{ij}(x, v) u_{ik} u_{jk} + \varepsilon u_{ik} u_{ik} \tag{39}$$

$$= \varkappa_2 C_2(v) \sum_{k=1}^{n} S_\varepsilon(u_k)$$

in Ω, where the constant $\varkappa_2$ depends on the boundary of Ω, the distance between $\partial\Omega$ and $\partial\Omega'$, and n only.

Lemma 2. *Let the constant a_0 be such that the set V' of the functions $v \in C^4(\Omega)$ with $\varkappa_1 \|v\|_{C^0(\Omega)} < M$, satisfying*

$$2a_0 > 3 + 2(n+1) C_1(v) \tag{40}$$

be non empty. Then the first derivatives of the solutions $u = u(v, \varepsilon)$ of the boundary value problem (21), (22) *are uniformly bounded in Ω for $v \in V'$ and $\varepsilon > 0$.*

Proof. We shall first estimate the derivatives on the boundary. Let u be a solution of the boundary value problem (21), (22) and let us consider the function $w_1 = \|f\|_{C^0(\bar{\Omega})} \exp[-\gamma_1 t + \beta_1 \psi]$ in the region $0 < t < \beta_1 \gamma_1^{-1} \psi$. Lemma 1 (25), (26) and $e^\beta > 1 + \beta$, $\beta > 0$, at $\beta_1 = a_0^{-1}$ imply

$$\|f\|_{C^0(\Omega)} \pm u \leqq \|f\|_{C_0(\Omega)} + a_0^{-1} \|f\|_{C^0(\Omega)} < \|f\|_{C^0(\Omega)} e^{1/a_0}$$

hence the maximum of the functions $w_1 \pm u$ on the boundary is attained for $t = 0$, $\psi = 1$. Let

$$\gamma_1 = \gamma_0(a_0^{-1}) + a_0^{-1/2}(2 + m_0/a_0)^{1/2}. \tag{41}$$

Now (31), (28), lemma 1 and (16) imply

$$|\tilde{M}_\varepsilon(v; \pm u)| = |F \pm Cu| \tag{42}$$

$$\leqq \|f\|_{C^0(\Omega)} + (m_0 + a_0) a_0^{-1} \|f\|_{C^0(\Omega)} = (2 + m_0/a_0) \|f\|_{C^0(\Omega)}.$$

From (33), (34), (41) and (42) it follows

$$\tilde{M}_\varepsilon(v; w_1 \pm u) \geqq a_0 w_1 \gamma_1^2 \pm F \pm Cu > 0$$

since the minimum of the function w_1 is $\|f\|_{C^0(\Omega)}$. The maximum principle now implies, that the maximum of the functions $w_1 \pm u$ in the region under consideration is attained on its boundary, whence

$$-\frac{\partial}{\partial t}(w_1 \pm u)\Big|_{\substack{t=0\\ \psi=1}} = \|f\|_{C^0(\Omega)}\gamma_1 e^{\beta_1} \mp \frac{\partial u}{\partial t}\Big|_{t=0} \geqq 0.$$

Therefore (40) implies

(43) $$|\partial u/\partial t| \leqq \gamma_1 e^{1/a_0}\|f\|_{C^0(\Omega)} \leqq 2\gamma_1\|f\|_{C^0(\Omega)}$$

on the set $t=0$, $\psi=1$. Since (22) implies $\partial u/\partial y_k|_{t=0}=0$, $k=1,\ldots,n-1$, (43) now implies

(44) $$|\partial u/\partial x_i| \leqq 2\gamma_1 K\|f\|_{C^0(\Omega)}, \qquad i=1,\ldots,n$$

on the inverse image of the set $\{(y,t): t=0,\ \psi(y)=1\}$ by the transformation (24). Since these sets cover $\partial\Omega$, (44) holds on $\partial\Omega$, i. e.

(45) $$|\partial u/\partial x_i|_{\partial\Omega} \leqq 2\gamma_1 K\|f\|_{C^0(\Omega)}, \qquad i=1,\ldots,n.$$

Let for any positive integer

(46) $$P^m = \sum_{k=1}^{n} u_k^{2m}.$$

From (36) it follows

(47) $$L_\varepsilon(v;P^m) = 2mu_k^{2m-1}L_\varepsilon(v;u_k) + (2m-1)aP^m + 2m(2m-1)u_k^{2(m-1)}S_\varepsilon(u_k).$$

By differentiation with respect to $x_k (k=1,\ldots,n)$ (12) implies

(48) $$L_\varepsilon(v;n_k) = f_k - \bar{a}_k^{ij}u_{ij} - \bar{a}_k^{i}u_i + \bar{a}_k u,$$

here and below line denoting total derivatives, for instance

$$\bar{a}_k = \frac{\partial a}{\partial x_k} + \frac{\partial a}{\partial y}\frac{\partial y}{\partial x_k} = \frac{\partial a}{\partial x_k} + \frac{\partial a}{\partial y}v_k.$$

From (11) with $p=2m$, $q=2m/(2m-1)$ and (17) follow

(49) $$2m\, f_k u_k^{2m-1} \leqq \sum_{k=1}^{n} f_k^{2m} + (2m-1)P^m,$$

(50) $$2m|\bar{a}_k^i u_k^{2m-1}u_i| \leqq C_1(v)\sum_{i,k=1}^{n}|u_i^{2m} + (2m-1)u_k^{2m}u_i| = 2mnC_1(v)P^m,$$

(51) $$2m|\bar{a}_k u_k^{2m-1}u| \leqq C_1(v)\sum_{k=1}^{n}|u^{2m} + (2m-1)u_k^{2m}|$$
$$= nC_1(v)u^{2m} + (2m-1)C_1(v)P^m.$$

At last (12), (18) and (39) imply

(52) $$2m|u_k^{2m-1}\bar{a}_k^{ij}u_{ij}| \leqq m\sum_{k=1}^{n} u_k^{2(m-1)}(\delta^{-1}u_k^2 + \delta(\bar{a}_k^{ij}u_{ij})^2)$$
$$\leqq \delta^{-1}mP^m + mn\delta\varkappa_2 C_2(v)u_k^{2(m-2)}S_\varepsilon(u_k).$$

Now (47) — (52) imply

(53) $$L_\varepsilon(v;P^m) \geqq m\{2(2m-1) - \delta n\varkappa_2 C_2(v)\}u_k^{2(m-2)}S_\varepsilon(u_k) + \{(2m-1)a_0$$
$$-[m\delta^{-1} + 2m - 1 + (2mn + 2m - 1)C_1(v)]\}P^m - nC_1(v)u^{2m} - \sum_{|\alpha|=1}\|D^\alpha f\|^{2m}_{C^0(\bar{\Omega})}.$$

Lemma 1 and (40) imply

(54) $$nC_1(v)u^{2m} \leqq \|f\|^{2m}_{C^\circ(\bar{\Omega})}.$$

From (40) follows, that if $\delta=1$, then

(55) $$2(2m-1)-n\varkappa_2 C_2(v)>0$$

and the coefficient of P^m in (53) is nonnegative for all sufficiently large m. Now (53)–(55) imply

(56) $$L_\varepsilon(v\,;P^m) \geqq -\sum_{|\alpha|\leqq 1} \|D^\alpha f\|^{2m}_{C_\circ(\bar{\Omega})}.$$

Let $\lambda_m=\max\{a_0^{-1}\Sigma_{|\alpha|\leqq 1}\|D^\alpha f\|^{2m}_{C^\circ(\bar{\Omega})},\ n(2K\gamma_1)^{2m}\|f\|^{2m}_{C^\circ(\bar{\Omega})}\}$.
From (46), (45) and (56) now follow $L_\varepsilon(v\,;P^m-\lambda_m)\geqq 0$ in Ω and $P^m-\lambda_m|_{\partial\Omega}\leqq 0$. The maximum principle together with the last two inequalities imply $P^m\leqq\lambda_m$ in $\bar{\Omega}$, whence

$$|\partial u/\partial x_k| \leqq \sqrt[2m]{P^m} \leqq \sqrt[2m]{\lambda^m} \leqq \max\{a_0^{-1/2m}\sum_{|\alpha|\leqq 1}\|D^\alpha f\|_{C^\circ(\bar{\Omega})},\ \sqrt[2m]{n}\,2K\gamma_1\|f\|_{C^\circ(\bar{\Omega})}\}.$$

Since this holds for all sufficiently large values of m, taking limits for $m\to\infty$ one obtains

(57) $$|\partial u/\partial x_k| \leqq \max\{\sum_{|\alpha|\leqq 1}\|D^\alpha f\|_{C^\circ(\bar{\Omega})},\ 2K\gamma_1\|f\|_{C^\circ(\bar{\Omega})}\}$$
$$\leqq \sum_{|\alpha|\leqq 1}\|D^\alpha f\|_{C^\circ(\bar{\Omega})}+2K\gamma_1\|f\|_{C^\circ(\bar{\Omega})}, \qquad k=1,\dots,n.$$

Since the right hand side does not depend on v and ε, the proof of lemma 2 is completed.

Lemma 3. *Let the constant a_0 be such that the set V'' of the functions $v\in C^3(\bar{\Omega})$ with $\varkappa_1\|v\|_{C^\circ(\bar{\Omega})}<M$ and $|D^1 v|\leqq\Sigma_{|\alpha|\leqq 1}\|D^\alpha f\|_{C^\circ(\bar{\Omega})}+2K\gamma_1\|f\|_{C^\circ(\bar{\Omega})}$ satisfying*

(58) $$a_0>1+(4n+2)C_1(v)+(2n\varkappa_2+2n^2+n+1)C_2(v)$$

be non-empty. Then the second derivatives of the solutions $u=u(v\,;\varepsilon)$ of the boundary value problem (21), (22) *are uniformly bounded in $\bar{\Omega}$ for $v\in V''$ and $\varepsilon>0$.*

Proof. Let

(59) $$Q=\sum_{k,l=1}^n u^2_{kl}.$$

If $S_\varepsilon(Q)=\Sigma^n_{k,l=1}S_\varepsilon(u_{kl})$ then (36) with $N=2$ implies

(60) $$L_\varepsilon(v\,;Q)=2u_{kl}L_\varepsilon(v\,;u_{kl})+aQ+2S_\varepsilon(Q).$$

By differentiation with respect to x_k and x_l, $k,l=1,\dots,n$, (21) implies

(61) $$L_\varepsilon(v\,;u_{kl})=f_{kl}-a^{ij}_k u_{ijl}-a^{ij}_l u_{ijk}-\bar{a}^{ij}_{kl}u_{ij}$$
$$-\bar{a}^i_k u_{il}-\bar{a}^i_l u_{ik}-\bar{a}^i_{kl}u_i+\bar{a}_k u_l+\bar{a}_l u_k+\bar{a}_{kl}u.$$

Now (12), (18) and (39) imply

(62) $$|2u_{kl}(\bar{a}^{ij}_k u_{ijl}+\bar{a}^{ij}_l u_{ijk})|\leqq 4\delta n\varkappa_2 C_2(v)S_\varepsilon(Q)+\delta^{-1}Q.$$

From (17), (18) and (12) with $\delta=1$ follow

(63) $$2|f_{kl}u_{kl}|\leq \sum_{|\alpha|=2}\|D^\alpha f\|^2_{C^0(\Omega)}+Q,$$

(64) $$2|a^{ij}_{kl}u_{ij}u_{kl}|\leq |2n^2C_2(v)Q,$$

(65) $$2|u_{kl}(a^i_k u_{il}+a^i_l u_{ik})|\leq 4nC_1(v)Q,$$

(66) $$2|u_{kl}a^i_{kl}u_i|\leq nC_2(v)Q+n^2C_2(v)P,$$

(67) $$2|u_{kl}(a_k u_l+a_l u_k)|\leq 2C_1(v)Q+2nC_1(v)P,$$

(68) $$2|u_{kl}a_{kl}u|\leq C_2(v)Q+n^2C_2(v)u^2$$

where $P=P^1$ in (66) and (67), cf. (46). From (60)—(68) follows

$$L_\varepsilon(v,Q)\geq(2-4\delta n\varkappa_2C_2(v))S_\varepsilon(Q)$$
$$+\{a_0-1^{-1}-\delta-2(2n+1)C_1(v)-(2n^2+n+1)C_2(v)\}Q$$
$$-\{n^2C_2(v)+2nC_1(u)\}P-n^2C_2(v)u^2-\sum_{|\alpha|=2}\|D^\alpha f\|^2_{C^0(\Omega)}.$$

From this inequality with $\delta=1/2n\varkappa_2C_2(v)$, (53) with $\delta=2/n\varkappa_2C_2(v)$, $m=1$ and $L_\varepsilon(v;u^2)\geq(a_0-1)u^2-\|f\|_{C^0(\bar\Omega)}$ follows

$$L_\varepsilon(v;u^2+P+Q)\geq\{a_0-1-2(2n+1)C_1(v)-(2n\varkappa_2+2n^2+n+1)C_2(v)\}Q$$
$$+\{a_0-1-(4n+1)C_1(v)-(n\varkappa_2/2+n^2)C_2(v)\}P$$
$$+\{a_0-1-nC_1(v)-n^2C_2(v)\}u^2-\|f\|^2_{C^2(\Omega)}.$$

Since the coefficients of u^2, P and Q are nonnegative according to (58), the maximum principle and the above inequality, as in the proof of lemma 2, imply

(69) $$u^2+P+Q\leq\max\{a_0^{-1}\|f\|^2_{C^2(\Omega)},\quad \max_{\partial\Omega}(u^2+P+Q)\}$$
$$\leq a_0^{-1}\|f\|^2_{C^2(\Omega)}+\max_{\partial\Omega}(u^2+P+Q).$$

Now (69), (22), (45) and (46) with $m=1$ imply

(70) $$u^2+P+Q\leq a_0^{-1}\|f\|^2_{C^2(\Omega)}+4nK^2\gamma_1^2\|f\|^2_{C^0(\Omega)}+\max_{\partial\Omega}Q.$$

Let us estimate the second derivatives on the boundary. To this end le

(71) $$q=(\max_{\partial\Omega}Q)^{1/2}$$

and $w_2=\exp[-\gamma_2 t+\beta_2\psi]$ in the region $0<t<\beta_2\psi/\gamma_2$, where β_2 is such that

(72) $$e^{\beta_2}>1+|\partial u/\partial y_m|,\qquad m=1,\ldots,n-1$$

in the region under consideration. Obviously the following considerations take' place in a neighbourhood of arbitrary boundary point, straightened by means of the transformations (24). Lemma 2 implies that (72) holds for instance if

(73) $$\beta_2=\ln(2+K\|f\|_{C^1(\Omega)}+2n\gamma_1K^2\|f\|_{C^0(\Omega)}).$$

Let us consider the functions $w_2\pm\partial u/\partial y_m$, $m=1,\ldots,n-1$. By differentiation with respect to y_m, $m=1,\ldots,n-2$, (28) implies

(74)
$$\widetilde{L}_\varepsilon(v;\partial u/\partial y_m)=-\frac{\partial A^{kl}}{\partial y_m}u_{kl}-2\frac{\partial A^k}{\partial y_m}u_{kt}-\frac{\partial \bar{A}}{\partial y_m}u_{tt}$$
$$-\frac{\partial B^k}{\partial y_m}u_k-\frac{\partial B}{\partial y_m}u_t+\frac{\partial C}{\partial y_m}u+\frac{\partial F}{\partial y_m},\qquad m=1,\dots,n-1.$$

It is easily seen that

$$|\partial A^{kl}/\partial y_m|\leq M_1(n, m_1, K)(1+|D^1v|)$$
$$\leq M_1(u, m_1, K)(1+\|f\|_{C^1(\Omega)}+2K\gamma_1\|f\|_{C^0(\Omega)})=M_1,$$

where the constant M_1 does not depend on $v\in V''$ and $\varepsilon>0$. Similar inequalities hold also for the derivatives of the remaining coefficients, i. e.

$$M_1\geq\max\{|\partial A^{kl}/\partial y_m|,\ |\partial A^k/\partial y_m|,\ |\partial A/\partial y_m|,\ |\partial B^k/\partial y_m|,\ |\partial B/\partial y_m|,\ |\partial C/\partial y_m|\}.$$

Now (74) and the last inequality imply

$$|\widetilde{L}_\varepsilon(v;\partial u/\partial y_m)|\leq M_1\{\sum_{k,l=1}^{n-1}|u_{kl}|+2\sum_{k=1}^{n-1}|u_{kt}|+|u_{tt}|$$
$$+\sum_{k=1}^{n-1}|u_k|+|u_t|+|u|\}+K\|f\|_{C^1(\Omega)}\leq(n+1)^2KM_1(u^2+P+Q)^{1/2}+K\|f\|_{C^1(\Omega)}.$$

From (70), (71) and the above inequality follows

(75)
$$|\widetilde{L}_\varepsilon(v;\partial u/\partial y_m)|\leq(n+1)^2KM_1q+K\{(n+1)^2M_1[\|f\|_{C^1(\bar{\Omega})}$$
$$+a_0^{-1/2}\|f\|_{C^2(\Omega)}+2\sqrt{n}K\gamma_1\|f\|_{C^0(\bar{\Omega})}\}.$$

For brevity let

$$T(f)=K\{(n+1)^2M_1[a_0^{-1/2}\|f\|_{C^2(\bar{\Omega})}+2\sqrt{n}K\gamma_1\|f\|_{C^0(\Omega)}]+\|f\|_{C^1(\bar{\Omega})}\}.$$

From (31), (33) and (75) it follows that if

(76)
$$\gamma_2=\gamma_0(\beta_2)+((m_0+a_0)/a_0)^{1/2}+a_0^{-1/2}[(n+1)^2KM_1q+T(f)]^{1/2},$$

then

(77)
$$\widetilde{L}_\varepsilon(v;w_2\pm\partial u/\partial y_m)\geq 0,\qquad m=1,\dots,n=1$$

since the minimum of the function w_2 is equal to 1. Similarly as in the proof of lemma 2, (72) and (77) imply that the maximum of the functions $w_2\pm\partial u/\partial y_m$ $(m=1,\dots,n-1)$ is attained on that part of the boundary, where $t=0$, $\psi(y)=1$. Again as in the proof lemma 2 one obtains

(78)
$$|\partial^2u/\partial y_m\partial t|\leq\gamma_2e^{\beta_2}\qquad m=1,\dots,n-1,$$

for $t=0$, $\psi(y)=1$. The equation (28), the first of the equalities (29), (78)' (43) and

(79)
$$\partial^2u/\partial y_k\partial y_l|_{t=0}=\partial u/\partial y_k|_{t=0}=u|_{t=0}=0\qquad k, l=1,\dots,n-1$$

imply

(80)
$$\partial^2u/\partial t^2|_{t=0}\leq(2a)^{-1}(\|f\|_{C^0(\bar{\Omega})}+2m_0\gamma_1\|f\|_{C^0(\Omega)}+2(n-1)m_0\gamma_2e^{\beta_2}).$$

Since (78) and (80) hold in vicinity of the images of every boundary point of Ω,

(81) $$q \leq K\Big(\sum_{k,l=1}^{n-1} |u_{kl}| + 2\sum_{k=1}^{n-1} |u_{kt} + u_{tt}|\Big)\Big|_{t=0} + K\Big(\sum_{k=1}^{n-1} |u_k| + |u_t|\Big)\Big|_{t=1}$$

and (43), (78)—(81) imply

$$0 \leq q \leq 2K(n-1)(1+m_0/2a_0)\gamma_2 e^{\beta_2} + K[2\gamma_1 + (1+2m_0\gamma_1)/2a_0]\|f\|_{C^0(\bar\Omega)}.$$

Now (73), (76) and the last inequality imply $0 \leq q \leq A\sqrt{q}+B$, where

(82) $$A = 2K(n^2-1)(1+m_0/2a_0)[2+K\|f\|_{C^1(\bar\Omega)} + 2K\gamma_1\|f\|_{C^0(\bar\Omega)}]\sqrt{KM_1/a_0},$$

(83) $$B = 2K(n^2-1)(1+m_0/2a_0)[2+K\|f\|_{C^1(\bar\Omega)} + 2K\gamma_1\|f\|_{C^0(\bar\Omega)}][\gamma_0(\beta_2) + \sqrt{(m_0+a_0)/a_0} + \sqrt{T(f)/a_0}] + K[2\gamma_2 + (1+2m_0\gamma_1)/a_0]\|f\|_{C^0(\bar\Omega)}.$$

Elementary calculations show that $0 \leq q \leq A\sqrt{q}+B$ implies the following inequaity for q:

(84) $$0 \leq q \leq B + (A^2 + A\sqrt{A^2+4B})/2 \leq A^2 + 2B.$$

Since A and B do not depend on v and ε, the uniform boundedness of the second derivatives is thus proved. More precisely (59), (69)—(71) and (84) imply

(85) $$|D^2u| \leq (u^2+P+Q)^{1/2} \leq \max\{a_0^{-1/2}\|f\|_{C^2(\bar\Omega)},\ 2\sqrt{n}K\gamma_1\|f\|_{C^0(\bar\Omega)} + A^2 + 2B\} \leq a_0^{-1/2}\|f\|_{C(\bar\Omega)} + 2\sqrt{n}K\gamma_1\|f\|_{C^0(\bar\Omega)} + A^2 + 2B.$$

Remark 1. The inequalities (57) and (85) may be written also in the form

(86) $$|D^1(u)| \leq \max\{\sum_{|\alpha|=1}\|D^\alpha f\|_{C^0(\bar\Omega)},\ \mu_1\|f\|_{C^0(\bar\Omega)}\} \leq \sum_{|\alpha|=1}\|D^\alpha f\|_{C^0(\bar\Omega)} + \mu_1\|f\|_{C^0(\bar\Omega)},$$

(87) $$|D^2u| \leq (u^2+P+Q)^{1/2} \leq \max\{a_0^{-1/2}\|f\|_{C^2(\bar\Omega)},\ \mu_2 + \mu_3\sqrt{a_0/a_0}\} \leq a_0^{-1/2}\|f\|_{C^0(\bar\Omega)} + \mu_2 + \mu_3\sqrt{a_0/a_0}$$

respectively, where the constants μ_1, μ_2 and μ_3 depend on m_0, m_1, m_2, a_0, K, f and a_0, but do not increase with a_0 and are independent from $v \in V''$ and ε.

Remark 2. In view of our further aims let us emphasize, that because of (34) with $\beta_1 = a_0^{-1}$ and β_2 defined by (73), from (41), (76), (82) and (83) follows, that for large a_0 ($a_0 \geq 1$) the constants μ_1 and μ_2 may be written in the form $\mu_1 = \mu_1'/\sqrt{a_0}$, $\mu_2 = \mu_2'/\sqrt{a_0}$, where μ_1' and μ_2' do not increase with a_0.

Theorem 1. *If the operator* (1) *with* (2) *and the region* $\Omega \subset R^n$ *satisfy the hypotheses* a) — c) *and the function* $f \in C^4(\Omega)$ *and the constant* a_0 *from* (15) *satisfy the inequalities*

(88) $$a_0 > 2 + (12n+6)m_1[1 + \sum_{|\alpha|=1}\|D^\alpha f\|_{C_0(\Omega)} + \mu_1\|f\|_{C^0(\bar\Omega)}]$$

$$+\max\{\sqrt{12n^5}, 9n\varkappa_2+12n^2+12n+6\}\{m_1(a_0^{-1/2}\|f\|_{C^2(\Omega)}+\mu_2+\mu_3\sqrt{a_0/a_0})$$
$$+m_2\sum_{\nu=0}^{2}(\sum_{|\alpha|=1}\|D^\alpha f\|_{C^0(\overline{\Omega})}+\mu_1\|f\|_{C(\Omega)})^\nu\}$$

and

(89) $$\varkappa_1 a_0^{-1}\|f\|_{C^0(\Omega)}<M,$$

then the boundary value problem

(90) $$L(u)=f$$

in Ω and

(91) $$u|_{\partial\Omega}=0$$

has a classical solution.

Remark 3. The inequalities (88), (89) show, that for any given $f\in C^4(\overline{\Omega})$ the boundary value problem (90), (91) has a classical solution, provided the constant a_0 is sufficiently large, see remark 1.

Proof of theorem 1. Let $\{\varepsilon_\nu\}_{\nu=1}^\infty$ be a sequence of real numbers with

(92) $$0<\varepsilon_{\nu+1}\leq\varepsilon_\nu \qquad \nu=1, 2, \ldots$$

and

(93) $$\lim_{\nu\to\infty}\varepsilon_\nu=0.$$

Lemma 1 and (89) imply, that a sequence of functions $\{u_\nu\}_{\nu=0}^\infty$ mav be defined. satisfying the following conditions: $u_0=0$ anjd provided (20), $u_{\nu+1}$ denotes the unique solution of the boundary value prob em

(94) $$L_{\varepsilon_{\nu+1}}(u_\nu;\ u_{\nu+1})=f$$

in Ω and

(95) $$u_{\nu+1}|_{\partial\Omega}=0.$$

More precisely lemma 1 implies the inequalities

(96) $$\|u_\nu\|_{C^0(\overline{\Omega})}\leq a_0^{-1}\|f\|_{C^0(\overline{\Omega})} \qquad \nu=1, 2, \ldots$$

and (89) imply, that the equation (94) is meaningful; the existence and the uniqueness of the solution or the boundery value problem follows from the well-known theorem of Schauder [10]. Lemma 2 implies the inequalities

(97) $$|D^1 u_\nu|\leq\sum_{|\alpha|=1}\|D^\alpha f\|_{C^0(\Omega)}+\mu_1\|f\|_{C^0(\Omega)}$$

($\nu=0, 1, \ldots$). Indeed (88) and (89) imply the validity of lemma 2 for $v=0$, i. e. (97) holds for $\nu=1$. Let (97) hold for some ν. Now (88), (89), (17) and (96) imply that the set V' is non empty and that (40) holds for $v=u_\nu$, i. e. the hypotheses of lemma 2 are satisfied for this ν. Hence (57), (86) imply (97) for $u=u_{\nu+1}$. Similarily applying lemma 3 one proves the validity of the inequalities

(98) $$|D^2 u_\nu|\leq(u_\nu^2+P_\nu+Q_\nu)^{1/2}\leq_0{}^{-1/2}\|f\|_{C^2(\overline{\Omega})}+\mu_2+\mu_3\sqrt{a_0/a_0}$$

$\nu=0, 1, \ldots,$ where $P_\nu=\Sigma_{k=1}^n(\partial u_\nu/\partial x_k)$ and $Q_\nu=\Sigma_{k,l=1}^n(\partial^2 u_\nu/\partial x_k\,\partial x_l)^2$.

Further on the proof of theorem 1 is based on the next two lemmas.

Lemma 4. *The third derivatives of the functions* u_ν, $\nu=1,2,\ldots$, *are uniformly bounded in* Ω.

Proof. Let

$$R_\nu=\sum_{k,l,m=1}^{n}(\partial^3 u_\nu/\partial x_k\partial x_l\partial x_m)^2. \tag{99}$$

For the sake of brevity we shall put $u_\nu=u$, $u_{\nu-1}=v$, $\varepsilon_\nu=\varepsilon$ for all ν. If $S_\varepsilon(R_\nu)=\Sigma_{k,l,m=1}^n S_\varepsilon(u_{klm})$, then (36) with $N=2$ implies

$$L_\varepsilon(v;R_\nu)=2u_{klm}L_\varepsilon(v;u_{klm})+2S_\varepsilon^2(R_\nu)+aR_\nu. \tag{100}$$

By differentiation with respect to x_k, x_l and x_m $(k,l,m=1,\ldots,n)$ (21) implies

$$\begin{aligned}L_\varepsilon(v;u_{klm})&=f_{klm}-a_k^{ij}u_{ijlm}-a_l^{ij}u_{ijkm}-\bar a_m^{ij}u_{ijkl}-a_{kl}^{ij}u_{ijm}-a_{km}^{ij}u_{ijl}\\&-a_{lm}^{ij}u_{ijk}-a_k^i u_{ilm}-a_l^i u_{ikm}-a_m^i u_{ikl}-a_{klm}^{ij}u_{ij}-\bar a_{kl}^i u_{im}-\bar a_{lm}^i u_{ik}-a_{km}^i u_{il}\\&+a_k u_{lm}+\bar a_l u_{km}+\bar a_m u_{kl}-a_{klm}^i u_i+a_{km}u_l+a_{kl}u_m+a_{lm}u_k+a_{klm}u.\end{aligned} \tag{101}$$

Now (12) (39) and (18) imply

$$2|u_{klm}(a_k^{ij}u_{ijlm}+a_l^{ij}u_{ijkm}+a_m^{ij}n_{ijkl})|\leqq 9\delta n\varkappa_2 C_2(v)S_\varepsilon(R_\nu)+\delta^{-1}R_\nu. \tag{102}$$

From (12) and (17)—(19) follow

$$2|u_{klm}(\bar a_{kl}^{ij}u_{ijm}+\bar a_{lm}^{ij}u_{ijk}+\bar a_{km}^{ij}u_{ijm})|\leqq 6n^2C_2(v)R_\nu, \tag{103}$$

$$2|u_{klm}(a_k^i u_{ilm}+a_l^i u_{ikm}+a_m^i u_{ikl})|\leqq 6n\,C_1(v)R_\nu, \tag{104}$$

$$2|u_{klm}(a_{kl}^i u_{im}+a_{lm}^i u_{ik}+a_{km}^i u_{il})|\leqq 3n\,C_2(v)R_\nu+3n^2C_2(v)Q_\nu, \tag{105}$$

$$2|u_{klm}(a_k u_{lm}+\bar a_l u_{km}+\bar a_m u_{kl})|\leqq 3C_1(v)R_\nu+3n\,C_1(v)Q_\nu, \tag{106}$$

$$2|u_{klm}(a_{kl}u_m+a_{km}u_l+a_{lm}u_k)|\leqq 3C_2(v)R_\nu+3n^2C_2(v)P_\nu, \tag{107}$$

$$2|u_{klm}a_{klm}^{ij}u_{ij}|\leqq \eta_1 n^2C_3(v)R_\nu+n^3\eta_1^{-1}C_3(v)Q_\nu, \tag{108}$$

$$2|u_{klm}a_{klm}^i u_i|\leqq \eta_2 n C_3(v)R_\nu+n^3\eta_2^{-1}C_3(v)P_\nu, \tag{109}$$

$$2|u_{klm}a_{klm}u|\leqq \eta_3 C_3(v)R_\nu+n^3\eta_3^{-1}C_3(v)u_\nu^2, \tag{110}$$

$$2|u_{klm}f_{klm}|\leqq \sum_{|\alpha|=3}\|D^\alpha f\|^2_{C^0(\Omega)}+R_\nu. \tag{111}$$

Now (100)—(111) imply

$$\begin{aligned}L_\varepsilon(v;R_\nu)&\geqq(2-9\delta n\varkappa_2C_2(v))S_\varepsilon(R_\nu)+\{a_0-1+(\eta_1n^2+\eta_2n+\eta_3)C_3(v)-\delta^{-1}\\&-(6n^2+3n+3)C_2(v)-(6n+3)C_1(v)\}R_\nu-(3n^2C_2(v)+3nC_1(v)+n^3\eta_3^{-1}C_3(v))Q_\nu\\&-(3n^2C_2(v)+n^3\eta_2^{-1}C_3(v))P_\nu-n^3\eta_3^{-1}C_3(v)u_\nu^2-\sum_{|\alpha|=3}\|D^\alpha f\|^2_{C^0(\Omega)}.\end{aligned}$$

From here with $\delta=2/9n\varkappa_2C_2(v)$ and $\eta_1=\eta_2=\eta_3=a_0/6n^2C_3(v)$ follows

$$L_\varepsilon(v;R_\nu)\geqq\{a_0/2-1-(9n\varkappa_2/2+6n^2+6n+3)C_2(v)-(6n+3)C_1(v)\}R_\nu$$
$$-a_0^{-1}6n^5C_3^2(v)(u_\nu^2+P_\nu+Q_\nu)-(3n^2C_2(v)+3nC_1(v))Q_\nu-3n^2C_2(v)P_\nu-\sum_{\alpha=3}|D^\alpha f|^2_{C^0(\Omega)}.$$

From (19) with $v=u_{\nu-1}$ and (99) follows $C_3^2(v)\leqq 2m_1^2R_{\nu-1}+M_2$, where the constant M_2 does not depend on ν according to (97) and (98). From (88) follows that the coefficient of R_ν is nonnegative, whence

$$L_\varepsilon(v;R_\nu)\geqq -a_0^{-1}(12n^5m_2^2\max R_{\nu-1}\max(u^2+P_\nu+Q_\nu)-M_3),$$

where the constant M_3 does not depend on ν according to (97), (98). The maximum principle and the last inequality imply

(112) $$R_\nu\leqq a_0^{-2}12n^5\, m_1^2\max(u_\nu^2+P_\nu+Q_\nu)\max R_{\nu-1}+M_3a_0^{-1}+\max_{\partial\Omega}R_\nu.$$

From (88) follows

(113) $$a_0^2>12n^5m_1^2(a_0^{-1/2}\|f\|_{C^2(\Omega)}+\mu_2+\mu_3\sqrt{a_0/\alpha_0})^2.$$

Now (98), (112) and (113) imply

114) $$\max R_\nu\leqq \alpha R_{\nu-1}+M_3/a_0+\max_{\partial\Omega}R_\nu,$$

where $0<\alpha<1$. We shall estimate $\max_{\partial\Omega}R_\nu$. Let

(115) $$r_\nu=(\max_{\partial\Omega}R_\nu)^{1/2}$$

and $w_3=\exp[-\gamma_3t+\beta_3\psi]$ in the region $0<t<\beta_3\gamma_3^{-1}\psi$, where $e^{\beta_3}>1+|\partial^2u_\nu/\partial y_r\partial y_s|$, $r, s=1,\dots,n-1$. According to (97), (98) β_3 may be determined independently of ν. By differentiation with respect to y_r, y_s $(r,s=1,\dots,n-1)$ (29) implies

$$\tilde L_\varepsilon(v;u_{y_ry_s})=F_{rs}-A_s^{kl}u_{klr}-A_r^{kl}u_{kls}-2\bar A_s^k u_{krt}-2\bar A_r^k u_{kst}-\bar A_r u_{stt}-\bar A_s u_{rtt}$$
$$-A_{rs}^{kl}u_{kl}-2\bar A_{rs}^k u_{kt}-\bar A_{rs}u_{tt}-B_s^k u_{kr}-B_r^k u_{ks}-B_s^s u_{rt}-\bar B_r u_{st}-$$
$$-B_{rs}^k u_k-\bar B_{rs}u_t+\bar C_s u_r+\bar C_r u_s+C_{rs}u.$$

Lemmas 1–3 and the last equality imply

(116) $$|\tilde L_\varepsilon(v;u_{y_ry_s})|\leqq M_4|D^3u_\nu|+M_5,$$

where the constants M_4 and M_5 do not depend on ν. Now (114) and (115) imply

(117) $$|D^3u_\nu|\leqq\sqrt{\alpha R_{\nu-1}}+\sqrt{M_3/a_0}+r_\nu.$$

From (116), (117) follows

(118) $$|\tilde L_\varepsilon(v;u_{y_r y_s})|\leqq M_4\sqrt{\alpha R_{\nu-1}}+M_4r_\nu+M_6,$$

where the constant M_6 does not depend on ν. If

$$\gamma_3=\gamma_0(\beta_3)+((m_0+a_0)/\alpha_0)^{1/2}+\alpha^{-1/2}(M_4\sqrt{\alpha R_{\nu-1}}+M_4r_\nu+M_6)^{1/2},$$

then (31), (33) and (118) imply $L_\varepsilon(v;w_3\pm u_{y_ry_s})\geqq 0$ since the minimum of w_3 is equal to 1. Similarily as in the proof of lemmas 2 and 3 one obtains

(119) $$|\partial^3 u/\partial y_r\,\partial y_s\,\partial t| \leq \gamma_3\, e^{\beta_3} \leq M_7 \sqrt[4]{a R_{\nu-1}} + M_8 \sqrt{r_\nu} + M_9,$$

with $t=0$, $\psi=1$, where the constants M_7-M_9 do not depend on ν. By differentiation with respect to y_r $(r=1, \ldots, n-1)$ (28) implies

(120) $$\bar{A}^{kl} u_{klr} + 2\bar{A}^k u_{krt} + A\, u_{rtt} + B^k u_{kr} + \bar{B}\, u_{rt} - \bar{C}\, u_r$$
$$= F_r - \bar{A}_r^{kl} u_{kl} - 2\bar{A}_r^k u_{kl} - \bar{A}_r u_{tt} - \bar{B}_r^k u_k - \bar{B}_r u_t + \bar{C}_r u.$$

From

(121) $$\partial^3 u/\partial y_k\, \partial y_l \partial y_r\,|_{t=0} = 0,$$

(119), (120) and (29) it follows

(123) $$|\partial^3 u/\partial y_r\,\partial t^2|_{t=0} \leq M_{10} \sqrt[4]{a R_{\nu-1}} + M_{11}\sqrt{r_\nu} + M_{12}.$$

Similarly by differentiation with respect to t and substituting (119) and (122), (28) implies

(123) $$|\partial^3 u/\partial t^3|_{t=0} \leq M_{13} \sqrt[4]{a R_{\nu-1}} + M_{14}\sqrt{r_\nu} + M_{15}$$

and (119), (121)—(123) imply $r_\nu \leq M_{16} \sqrt[4]{R_{\nu-1}} + M_{17}\sqrt{r_\nu} + M_{18}$, where none of the constants $M_{10}-M_{18}$ depend on ν. From the last inequality, similarly to (84), follows

(124) $$r_\nu \leq M_{17}^2 + 2M_{18} + 2M_{16} \sqrt[4]{R_{\nu-1}}.$$

Now (114), (115) and (124) imply

$$\max R_\nu \leq a \max R_{\nu-1} + M_{19} \sqrt{\max R_{\nu-1}} + M_{20}.$$

Since none of the constants in this inequality depends on ν, the uniform boundedness of the third derivatives is thus established.

Indeed, if for a sequence of numbers $\{a_\nu\}_{\nu=1}^\infty$ the inequality

(125) $$0 \leq a_{\nu+1} \leq a a_\nu + \beta \sqrt{a_\nu} + \gamma \quad 0<a<1\,;\ \beta, \gamma>0$$

holds, then this sequence is bounded. In order to prove this statement let ξ be a real solution of the equation $x = ax + \beta\sqrt{x} + \gamma$ $(x>0)$; such a solution exists since $a<1$. For every $x>\xi$ the inequality

(126) $$ax + \beta\sqrt{x} + \gamma < x$$

holds and for every $x \leq \xi$ the inequality

(127) $$x \leq ax + \beta\sqrt{x} + \gamma$$

holds. We shall show that $a_\nu \leq \max\{\xi, a_1\}$ $(\nu=1, 2, \ldots)$. For $\nu=1$ this statement is obviously true. Let it hold for some ν. Now either

(128) $$a_\nu \leq \xi,$$

or

(129) $$\xi < a_\nu \leq a_1$$

holds. Since the function $\alpha x+\beta\sqrt{x}+\gamma$ $(x>0)$ increases monotonously, (128), (125) and (127) imply

$$a_{\nu+1}\leqq\alpha a_\nu+\beta\sqrt{a_\nu}+\gamma\leqq\alpha\xi+\beta\sqrt{\xi}+\gamma=\xi$$

and (129), (125) and (126) imply

$$a_{\nu+1}\leqq\alpha a_\nu+\beta\sqrt{a_\nu}+\gamma<a_\nu\leqq a_1.$$

Now the last two inequalities imply $a_{\nu+1}\leqq\max\{\xi, a_1\}$.

Lemma 5. *The sequence* $\{u_\nu\}_{\nu=1}^{\infty}$ *is uniformly convergent in* $\bar{\Omega}$.

Proof. From (94) follows

(130) $$L_{\varepsilon_{\nu+1}}(u_\nu;\, u_{\nu+1})-L_{\varepsilon_\nu}(u_{\nu-1};\, u_\nu)=0,\quad \nu=1, 2, \ldots$$

From (21) it follows

(131) $$L_{\varepsilon_{\nu+1}}(u_\nu;\, u_{\nu+1})-L_{\varepsilon_\nu}(u_{\nu-1};\, u_\nu)=L_{\varepsilon_{\nu+1}}(u_\nu;\, u_{\nu+1}-u_\nu)$$
$$+(\varepsilon_{\nu+1}-\varepsilon_\nu)\,\Delta u_\nu+(a^{ij}(x, u_\nu)-a^{ij}(x, u_{\nu-1}))\,\partial^2 u_\nu/\partial x_i\,\partial x_j$$
$$+(a^i(x, u_\nu)-a^i(x, u_{\nu-1}))\,\partial u_\nu/\partial x_i-(a(x, u_\nu)-a(x, u_{\nu-1}))\,u_\nu$$

$\nu=1, 2, \ldots$. The mean value theorem and (16) imply

(132) $$|a^{ij}(x, u_\nu)-a^{ij}(x, u_{\nu-1})|\,|\partial^2 u_\nu/\partial x_i\,\partial x_j|\leqq m_1|\partial^2 u_\nu/\partial x_i\,\partial x_j|\,|u_\nu-u_{\nu-1}|,$$

(133) $$|a^i(x, u_\nu)-a^i(x, u_{\nu-1})|\,|\partial u_\nu/\partial x_i|\leqq m_1|\partial u_\nu/\partial x_i|\,|u_\nu-u_{\nu-1}|,$$

(134) $$|a(x, u_\nu)-a(x, u_{\nu-1})|\,|u_\nu|\leqq m_1|u_\nu|\,|u_\nu-u_{\nu-1}|.$$

Now (130)—(134) and (98) imply

(135) $$|L_{\varepsilon_{\nu+1}}(u_\nu;\, u_{\nu+1}-u_\nu)|\leqq(\varepsilon_\nu-\varepsilon_{\nu+1})\,n\,|D^2 u_\nu|$$
$$+m_1(n^2+n+1)^{1/2}(u_\nu^2+P_\nu+Q_\nu)^{1/2}\,\|u_\nu-u_{\nu-1}\|_{C^0(\bar{\Omega})}$$
$$\leqq\{a_0^{-1/2}\|f\|_{C^2(\bar{\Omega})}+\mu_2+\mu_3\sqrt{a_0/\alpha_0}\}\,[m_1(n+1)\,\|u_\nu-u_{\nu-1}\|_{C^0(\bar{\Omega})}+n(\varepsilon_\nu-\varepsilon_{\nu+1})].$$

From (95) it follows

(136) $$(u_{\nu+1}-u_\nu)|_{\partial\Omega}=0.$$

Similarly as in the proof of lemma 1, (135), (136) and the maximum principle imply

$$\|u_{\nu+1}-u_\nu\|_{C^0(\bar{\Omega})}\leqq\{\alpha_0^{-1}(a_0^{-1/2}\|f\|_{C^0(\bar{\Omega})}+\mu_2$$
$$+\mu_3\sqrt{a_0/\alpha_0})\,m_1(n+1)\}\,[(\varepsilon_\nu-\varepsilon_{\nu+1})/m_1+\|u_\nu-u_{\nu-1}\|_{C^0(\bar{\Omega})}].$$

The series $\Sigma_{\nu=1}^{\infty}(u_\nu-u_{\nu-1})$ is uniformly convergent. Indeed the above inequality implies

$$\|u_{\nu+1}-u_\nu\|_{C^0(\bar{\Omega})}\leqq\alpha\,((\varepsilon_\nu-\varepsilon_{\nu+1})/m_1+\|u_\nu-u_{\nu-1}\|_{C^0(\bar{\Omega})}),$$

where $0<\alpha<1$ according to (88). Let a sequence of numbers $\{a_\nu\}_{\nu=1}^{\infty}$ be given, such that

$$(137) \qquad 0 \leq a_{\nu+1} \leq a(a_\nu + b_\nu), \qquad (b_\nu \geq 0)$$

where $0 < a < 1$ and the series $\Sigma_{\nu=1}^{\infty} b_\nu$ is convergent. Then the series $\Sigma_{\nu=1}^{\infty} a_\nu$ is also convergent. Indeed (137) implies by induction

$$(138) \qquad 0 \leq a_{\nu+1} \leq a^\nu a_1 + \sum_{i=1}^{\nu} a^{\nu+1-i} b_i.$$

The series with a general term $a^\nu a_1 + \Sigma_{i=1}^{\nu} a^{\nu+1-i} b_i$ is convergent, since it is the sum of a convergent geometric series and the product of the convergent series $\Sigma_{\nu=1}^{\infty} b_\nu$ with a convergent geometric series. Now the convergency of the series $\Sigma_{\nu=1}^{\infty} a_\nu$ follows from (138). The convergency of $\Sigma_{\nu=1}^{\infty}(u_\nu - u_{\nu-1})$ follows now for $a_\gamma = \| u_\nu - u_{\nu-1} \|_{C^0(\overline{\Omega})}$ and $(\varepsilon_\nu - \varepsilon_{\nu+1})/m_1$, the convergency of $\Sigma_{\nu=1}^{\infty} b_\nu$ following from (92) (93). This accomplishes the proof of lemma 5.

Completion of the proof of theorem 1. The Arzelá—Ascoli theorem and lemmas 1—4 imply, that a subsequence $\{u_{\nu_\mu}\}_{\mu=1}^{\infty}$ of the sequence $\{u_\nu\}_{\nu=0}^{\infty}$ may be chosen, which is uniformly convergent in Ω together with its derivatives up to second order. Let

$$(139) \qquad u = \lim_{\mu\to\infty} u_{\nu_\mu}.$$

Then

$$(140) \qquad \lim_{\mu\to\infty} \partial u_{\nu_\mu}/\partial x_i = \partial u/\partial x_i \quad i = 1, \ldots, n.$$

and

$$(141) \qquad \lim_{\mu\to\infty} \partial^2 u_{\nu_\mu}/\partial x_i \partial x_j = \partial^2 u/\partial x_i \partial x_j, \quad i, j = 1, \ldots, n,$$

Lemma 5 implies that the sequence $\{u_{\nu_\mu - 1}\}_{\mu=1}^{\infty}$ is uniformly convergent also and $\lim_{\mu\to\infty} u_{\nu_\mu - 1} = u$. Taking limits in the equalities

$$L_{\varepsilon_{\nu_\mu}}(u_{\nu_\mu - 1};\, u_{\nu_\mu}) = f \quad \mu = 1, 2, \ldots,$$

one concludes from (20) and (94) that u satisfies in Ω equation (1) and from (95) that $u|_{\partial\Omega} = 0$. This accomplishes the proof of theorem 1.

Let us note that (139)—(141), (98) and (10) imply the inequality

$$(142) \qquad \| u \|_{C^2(\overline{\Omega})} \leq a_0^{-1/2} \| f \|_{C^0(\overline{\Omega})} + \mu_2 + \mu_3 \sqrt{a_0/a_0}.$$

Theorem 2. *Under the conditions of theorem 1 the boundary value problem* (90), (91) *has exactly one solution.*

Proof. Let u be the solution of the boundary value problem (90), (91), constructed in theorem 1, and let v be an arbitrary solution of the same problem. Then

$$(143) \qquad L(u) - L(v) = 0$$

in Ω and

$$(144) \qquad (u - v)|_{\partial\Omega} = 0.$$

Now (1), (143), (144), $w = u - v$ and

$$\Phi(x) = -(a^{ij}(x, u) - a^{ij}(x, v))u_{ij} - (a^i(x, u) - a^i(x, v))u_i + (a(x, u) - a(x, v))u$$

imply

$$a^{ij}(x, v)w_{ij} + a^i(x, v)w_i - a(x, v)w = \Phi(x)$$

and $w|_{\partial\Omega} = 0$. The maximum principle, applied to the last two equalities as in lemma 1, implies

$$\|w\|_{C^0(\overline{\Omega})} \leq a_0^{-1} \max_{\overline{\Omega}} |\Phi(x)|. \tag{145}$$

Calculations, similar to those in the proof of lemma 5, together with the mean value theorem and (142) imply

$$\|\Phi\|_{C^0(\overline{\Omega})} \leq m_1(n+1)(a_0^{-1/2}\|f\|_{C^2(\overline{\Omega})} + \mu_2 + \mu_3\sqrt{\overline{a}_0/a_0})\|w\|_{C^0(\overline{\Omega})}. \tag{146}$$

Now (145), (146) imply

$$[1 - m_1(n+1)a_0^{-1}(a_0^{-1/2}\|f\|_{C^2(\overline{\Omega})} + \mu_2 + \mu_3\sqrt{\overline{a}_0/a_0})]\|w\|_{C^0(\overline{\Omega})} \leq 0.$$

According to (88) the first term in the left hand side of the above inequality is positive, whence $\|w\|_{C^0(\overline{\Omega})} = 0$. This accomplishes the proof of theorem 2.

Remark 4. The assumptions for four times continuous differentiability of the coefficients and the right hand side of the operator (1) are too strong. The same results remain valid under the assumption that the coefficients and the right hand side are only two times differentiable, the second derivatives being Lipschitz continuous. In this case however the proofs are to be carried out by approximating the coefficients and the right hand side with sequences of sufficiently smooth functions, whose derivatives up to third order are uniformly bounded. Since the corresponding considerations increase the dead load of the construction in a purely technical aspect, we shall not discuss this more general case here.

Now applying theorems 1 and 2 we are in a position to study boundary value problems for the operator (1) for regions with $\partial\Omega \neq S_3$ contrary to theorem 1. We shall use technique, similar to that used in [2; 4] in the linear case for continuation of the coefficients of (1) in a larger region.

Theorem 3. *Let $G \subset R^n$ be a region with a piecewise smooth boundary and let there exist regions Ω and Ω' in R^n, satisfying the following hypotheses:*

A) $\overline{\Omega} \subset \Omega'$.

B) *The coefficients of the operator* (1) *and the right hand side of* (1) *are defined and four times continuously differentiable in $\Omega' \times [-M, M]$ and*

$$a^{ij}(x, y)\xi_i\xi_j \geq 0 \quad (\xi \in R^n, (x, y) \in \overline{\Omega}' \times [-M, M]). \tag{147}$$

C) *The boundary $\partial\Omega$ is at least of the class C^6 and*

$$a^{ij}(x, y)\nu_i(x)\nu_j(x) > 0, \quad x \in \partial\Omega, y \in [-M, M],$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_m(x))$ *is an unit exterior normal vector at $\partial\Omega$.*

D) *All interior points with respect to ∂G of S_1 and S_2 are interior points of Ω and the remaining boundary points of G are boundary points of Ω: in particularity $S_3 \subset \partial\Omega$; for the definition of $S_1 - S_3$ cf.* (5)—(9).

E) *The components* $\Omega_\lambda (\lambda = 1, \ldots, l)$ *of* $\Omega \setminus G$ *do not have simultaneously interior points of* S_1 *and* S_2 *on their boundaries.*

F) *The function* $f \in C^4(\overline{\Omega})$ *and the constant* a_0 *from* (15) *are such that provided* (15) *and* (16), *the inequalities*

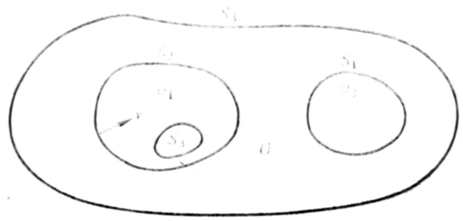

Fig. 1

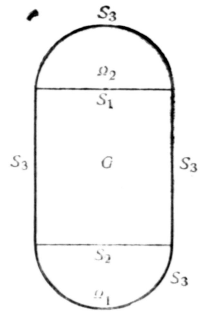

Fig. 2a

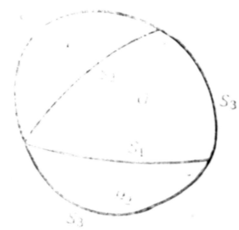

Fig. 2b

$$(148)\qquad a_0 > 2 + (12n+6)\, m_1 [1 + \sum_{|\alpha|=1} \| D^\alpha f \|_{C^0(\overline{\Omega})} + \mu_1 \| f \|_{C^0(\overline{\Omega})}]$$

$$+ \max\{\sqrt{12n^5}, 9n\varkappa_2 + 12n^2 + 12n + 6\} \{ m_1 (a_0^{-1/2} \| f \|_{C^0(\overline{\Omega})} + \mu_2 + \mu_3 \sqrt{a_0/\alpha_0})$$

$$+ m_2 \sum_{\nu=0}^{2} (\sum_{|\alpha|=1} \| D^\alpha f \|_{C^0(\overline{\Omega})} + \mu_1 \| f \|_{C^0(\overline{\Omega})})^\nu \}$$

and $a_0^{-1} \varkappa_1 \| f \|_{C^0(\overline{\Omega})} < M$ *hold, where the constants* $\mu_1, \mu_2, \mu_3, \alpha_0, \varkappa_1$ *and* $\varkappa_2$ *are connected with the operator* (1) *and with the region* Ω *as in theorem* 1 *and* $f(x) = 0$ *on those components* Ω_λ, *which contain on their boundaries interior points of* S_2.

Then the boundary value problem $L(u) = f$ *in* G *and* $u|_{S_2 \cup S_3} = 0$ *has exactly one classical solution.*

Remark 5. The conditions of theorem 3 are satisfied for instance when the coefficients of (1) are defined in a region $G' \supset \overline{G}$, the sets S_1, S_2 and S_3 are mutually disjoint (Fig. 1) or intersect on "edges" of the region G (Fig. 2, a and b) (cf. [4] for the linear case) and a_0 is a sufficiently large positive constant. The last situation naturally arises when we consider parabolic quasilinear equations in cylindrical regions.

Proof of theorem 3. From A) — C) it follows, that the hypotheses a) — c) of theorem 1 are satisfied and from F) it follows that (88) and (89) hold, i. e. all the hypotheses of theorem 1 are satisfied. Hence the boundary value problem $L(u) = f$ in Ω and $u|_{\partial\Omega} = 0$ has exactly one classical solution. Since $G \subset \Omega$, then $L(u) = f$ in G and $S_3 \subset \partial\Omega$ implies $u|_{S_3} = 0$. It remains to prove that $u|_{S_2} = 0$. We shall accomplish this as in [4] or [7]. Let the component Ω_λ contain on its boundary interior points of S_2. Then F) implies that $f \equiv 0$ in Ω_λ and by twofold integration by parts we obtain

$$(149)\qquad 0 = \int_{\Omega_\lambda} fu\,dx = \int_{\Omega_\lambda} L(u) u\,dx = \int_{\Omega_\lambda} (a^{ij} u_i u_j + (2a + a^i_i - a^{ij}_{ij}) u^2/2) dx$$

$$-\int_{\partial\Omega_\lambda} a^{ij} u_i\, u\, \nu_j ds - \frac{1}{2}\int_{\partial\Omega_\lambda} (a^i - a^{ij}_j)\, u^2 \nu_i\, ds.$$

The surface integrals in (149) are nonpositive. Indeed, since $\partial\Omega_\lambda$ is a subset of the boundary of $\Omega\diagdown G$ and the last is contained in $\partial\Omega\cup\partial G$, $\partial\Omega_\lambda$ may be decomposed as $\Gamma'_\lambda\cup\Gamma''_\lambda$, where $\Gamma'_\lambda=\partial\Omega\cap\partial\Omega_\lambda$ and $\Gamma''_\lambda=\partial\Omega_\lambda\diagdown\Gamma'_\lambda$. Now D) and E) imply that $\Gamma''\subset S_2$. From $u|_{\partial\Omega}=0$ follows, that the surface integrals on Γ'_λ vanish, i. e. it remains to prove that

(150) $$\int_{\Gamma''_\lambda} a^{ij} u_i\, u\, \nu_j\, ds + \frac{1}{2}\int_{\Gamma''_\lambda}(a^i - a^{ij}_j)\, u^2 \nu_i\, ds \leq 0.$$

From (6) follow

(151) $$a^{ij}(x, y)\,\nu_j(x)=0, \quad i=1, \ldots, n,$$

and

(152) $$a^{ij}(x, y)\,\nu_i(x)=0, \quad j=1, \ldots, n,$$

on S. Now (151) implies that the first integral in (150) vanishes. From (152) by differentiation with respect to y one obtains $a^{ij}_y(x, y)\,\nu_i(x)=0$, $j=1, \ldots, n$. Since ν_i in (150) are the components of the exterior for Ω_λ normal, i. e. of the nterior for G normal, (7) and (9) imply

$$\frac{1}{2}\int_{\Gamma''_\lambda}(a^i - a^{ij}_j)\,\nu_i\, u^2\, ds \leq 0.$$

Since u is a solution of the boundary value problem $L(u)=f$ in Ω and $u|_{\partial\Omega}=0$ obtained by application of theorem 1, (142) and (148) imply the inequality $(2a+a^i_i-a^{ij}_{ij})/2\geq 1$. From (147) follows $a^{ij}u_iu_i\geq 0$. Now (149) and (150) imply $\int_{\Omega_\lambda} u^2 dx\leq 0$, i. e. $u=0$ in Ω_λ, whence in particularity follows that $u|_{S_2}=0$.

The uniqueness of the solution thus obtained now follows as in the proof of theorem 2, by using Fichera estimation, instead of lemma 1.

In the special case $\partial\Omega=S_2$ theorem 3 is a slight improvement of the result in [7], as it is easily seen by extending the equation in a sufficiently large region Ω and taking advantage of the fact that in this case we dispose of the constant a_0; see remark 2.

Let us note also, that since the results obtained do depend not on the dimensions of the regions considered, but on their "shape" only, these results may be used for the study of the Cauchy problem in unbounded regions [7; 13].

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