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EXISTENCE OF EXTENDED MONOSPLINES OF LEAST DEVIATION

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Extended monosplines defined by an ETP kernel $K(x, t)$ are studied. The existence is proved of an extended monospline with minimal L_p norm $(1\leq p<\infty)$ among all monosplines of pre-assigned multiplicities of the knots. The problem is related to the existence of optimal quadrature formulae of fixed type.

1. Introduction. Let $K(x, t)$ be a real valued differentiable function defined on $J\times J$, where J is an open interval of the real line R. We postulate here that $K(x, t)$ is extended totally positive (ETP) on $J\times J$. For the definition of this and other terms employed in this paper the reader is referred to [1, 2]. We shall write

$$\mathbf{x}=\begin{pmatrix}x_0, & x_1, & \dots, & x_n, & x_{n+1}\\ \nu_0, & \nu_1, & \dots, & \nu_n, & \nu_{n+1}\end{pmatrix} \tag{1.1}$$

to denote that $\mathbf{x}$ is a system of $n+2$ distinct nodes $(x_n)_0^{n+1}$ in J of multiplicities $(\nu_k)_0^{n+1}$ respectively. Let $[a, b]\subset T\subset J$. Given the positive integers $(\nu_k)_0^{n+1}$, $\Omega((a, \nu_0), \nu_1, \dots, \nu_n, (b, \nu_{n+1}); T)$ will denote the set of all systems $\mathbf{x}$ of the form (1.1) with $x_0=a$, $x_{n+1}=b$, $x_k\in T$, $k=1, \dots, n$. In the case $T=J$ we shall omit the indication of T. Let $\omega(t)$ be a fixed positive summable function on $[a, b]$. We will further assume that all derivatives occurring in (1.2) below are continuous. The function

$$M(\mathbf{c}, \mathbf{x}; t)=\int_a^b K(x, t)\omega(x)dx-\sum_{k=0}^{n+1}\sum_{\lambda=1}^{\nu_k-1} c_{k\lambda}K_\lambda(x_k, t), \tag{1.2}$$

where $K_\lambda(x, t)=(\partial^\lambda/\partial x^\lambda)K(x, t)$ is called extended monospline (EM) with knots $\mathbf{x}$ and coefficients $\mathbf{c}=\{c_{k\lambda}\}$. For any fixed system of multiplicities $(\nu_k)_0^{n+1}$ we prove the existence of knots $\mathbf{x}\in\Omega((a, \nu_0), \nu_1, \dots, \nu_n, (b, \nu_{n+1}))$ and coefficients $\mathbf{a}$ such that

$$\|M(\mathbf{a}, \mathbf{x}; \cdot)\|_p=\inf\{\|M(\mathbf{c}, \mathbf{y}; \cdot)\|_p : \mathbf{c}, \mathbf{y}\in\Omega((a, \nu_0), \nu_1, \dots, \nu_n, (b, \nu_{n+1}))\}, \tag{1.3}$$

where $1\leq p<\infty$, $\|f\|_p=\{\int_a^b|f(t)|^p dt\}^{1/p}$. In the particular case $\nu_1=\dots=\nu_n=1$ we get a result announced by Karlin [2] and proved recently in [3; 4] (see also [5]).

2. Preliminaries and notations. Everywhere in this paper q denotes the conjugate number to p, i. e., $1/p+1/q=1$. We assume that $1\leq p<\infty$. The proof of our existence theorem is based on the relation between extended monosplines of least L_p deviation and best quadrature formulae in the class

$$K_q[a, b] := \{F : F(x) = F(f; x) = \int_a^b K(x, t) f(t) dt,\ f \in L_q[a, b],\ \|f\|_q \le 1\}.$$

We shall briefly recall this relation. Given the nodes (1.1) and coefficients $\mathbf{c} = \{c_{k\lambda}\}$ we define

$$S(\mathbf{c}, \mathbf{x}; F) = \sum_{k=0}^{n+1} \sum_{\lambda=0}^{\nu_k - 1} c_{k\lambda} F^{(\lambda)}(x_k).$$

The coefficients $\mathbf{a} = \mathbf{a}(\mathbf{x})$ are said to be best for the nodes $\mathbf{x}$ in the class $K_q[a, b]$ if

$$\sup\{|I(F) - S(\mathbf{a}, \mathbf{x}; F)| : F \in K_q[a, b]\}$$

$$= \inf_{\mathbf{c}} \sup\{|I(F) - S(\mathbf{c}, \mathbf{x}; F)| : F \in K_q[a, b]\} =: R_q(\mathbf{x}),$$

where $I(F) = \int_a^b F(x) \omega(x) dx$. As is easily verified,

$$R_q(\mathbf{x}) = \inf_{\mathbf{c}} \|M(\mathbf{c}, \mathbf{x}; \cdot)\|_p = \|M(\mathbf{a}(\mathbf{x}), \mathbf{x}; \cdot)\|_p. \tag{2.1}$$

The quadrature formula $I(F) \approx S(\mathbf{a}(\mathbf{x}), \mathbf{x}; F)$ is called best quadrature formula with fixed nodes $\mathbf{x}$ in the class $K_q[a, b]$. $R_q(\mathbf{x})$ is the error of this formula.

It follows from the strict convexity of the norm $\|\cdot\|_p$ $(1 < p < \infty)$ that the best coefficients $\mathbf{a}(\mathbf{x})$ are uniquely determined by $\mathbf{x}$. The uniqueness holds in the case $p = 1$, too, since the functions $\{K_\lambda(x_k, t)\}$ form a Haar system in $[a, b]$. Henceforth we set $M(\mathbf{x}; t) = M(\mathbf{a}(\mathbf{x}), \mathbf{x}; t)$. The equality (2.1) shows that there is one-to-one correspondence between monosplines $M(\mathbf{x}; t)$ and the best quadrature formulae in the class $K_q[a, b]$ with fixed nodes.

Let us introduce the functions

$$\varphi(\mathbf{x}; t) = \Big(\int_a^b |M(\mathbf{x}; t)|^p dt\Big)^{-1/q} |M(\mathbf{x}; t)|^{p-1} \operatorname{sign} M(\mathbf{x}; t), \tag{2.2}$$

$$\Phi(\mathbf{x}; x) = \int_a^b K(x, t) \varphi(\mathbf{x}; t)\, dt. \tag{2.3}$$

Evidently $\Phi(\mathbf{x}; \cdot) \in K_q[a, b]$. The converse of the Schwartz inequality implies that $\Phi(\mathbf{x}; \cdot)$ is the unique function in $K_q[a, b]$ for which $R_q(\mathbf{x}) = I(F) - S(\mathbf{a}(\mathbf{x}), \mathbf{x}; F)$. Furthermore, it follows from (2.1) that the best coefficients $\mathbf{a}(\mathbf{x})$ must satisfy the conditions

$$\frac{\partial}{\partial c_{k\lambda}} \|M(\mathbf{c}, \mathbf{x}; \cdot)\|_q \Big|_{\mathbf{c} = \mathbf{a}} = 0$$

for $k = 0, \ldots, n+1$, $\lambda = 0, \ldots, \nu_k - 1$. This gives $\int_a^b \varphi(\mathbf{x}; t) K_\lambda(x_k; t) dt = 0$, which is equivalent to $\Phi^{(\lambda)}(\mathbf{x}; x_k) = 0$. So we proved

Lemma 1. *For any fixed system* $\mathbf{x}$ *of nodes* (1.2) *there is a unique function* $\Phi(\mathbf{x}; \cdot) \in K_q[a, b]$ *such that* $\Phi^{(\lambda)}(\mathbf{x}; x_k) = 0$ *for* $k = 0, \ldots, n+1$, $\lambda = 0, \ldots, \nu_k - 1$ *and* $R_q(\mathbf{x}) = I(\Phi(\mathbf{x}; \cdot))$.

Definition. *Let the multiplicities* $(\nu_k)_0^{n+1}$ *be fixed and let* $T\subset J$. *We call the quadrature formula*

$$\int_a^b F(t)\omega(t)\,dt\approx\sum_{k=0}^{n+1}\sum_{\lambda=0}^{\nu_k-1}a_{k\lambda}F^{(\lambda)}(x_k) \tag{2.4}$$

optimal of the type $((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});T)$ *in the class* $K_q[a,b]$ *if* $\mathbf{x}\in\Omega((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});T)$ *and*

$$R_q(\mathbf{x})=\inf\{R_q(\mathbf{y}):\mathbf{y}\in\Omega(a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});T\}.$$

The nodes $\mathbf{x}$ *are said to be optimal of the same type.*

Now it is seen from (2.1) that our problem (1.3) will be solved if we prove the existence of optimal nodes of the type $((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1}))$. We conclude this section by showing some properties of the ETP kernels.

Let $Z(f;T)$ denote the number of the zeros of f in the interval T counting multiplicities.

Definition. *A system of differentiable functions* $\{u_1(t),\ldots,u_N(t)\}$ *defined on* $[a,b]$ *is termed "extended Tchebycheff system" (ETS) on* $[a,b]$ *iff for any non trivial* $f\in\operatorname{span}\{u_1,\ldots,u_N\}$, $Z(f;[a,b])\le N-1$.

Note that the ETP property of the kernel $K(x,t)$ entails [1] that the system $\{K_\lambda(x_k,t),\ k=1,\ldots,m,\ \lambda=0,\ldots,\ \mu_k-1\}$ is ETS in J for any $\mathbf{x}=\binom{x_1,\ldots,x_m}{\mu_1,\ldots,\mu_m}$. An analogous statement holds according to the second variable t of $K(x,t)$.

Lemma 2. *Let* $K(x,t)$ *be ETP on* $J\times J$. *Then for any set of nodes* (1.2) *such that* $x_0<\cdots<x_l\le a<x_{l+1}<\cdots<x_s<b\le x_{s+1}<\cdots<x_{n+1}$ *we have*

$$Z(\Phi(\mathbf{x};\cdot);J)\le\sum_{i=0}^{n+1}\nu_i+\sum_{i=l+1}^{s}\sigma_i,$$

where $\sigma_i=1$, *if* ν_i *is odd, and zero otherwise.*

Proof. Let $SC(f;T)$ denote the number of the sign changes of f in the interval T. Clearly our assertion follows immediately from the next two statements:

$$Z(F(g;\cdot);J)\le SC(g;(a,b)), \tag{2.5}$$

$$Z(f;J)\le\sum_{i=0}^{n+1}\nu_i+\sum_{i=l+1}^{s}\sigma_i=:N \tag{2.6}$$

for every piecewise continuous in J function g and every EM f of the form (1.2) respectively.

The estimation (2.6) is known one [3]. We shall give here a new simple proof. Let us assume that $\partial^\mu M(\mathbf{c},\mathbf{x};t_j)/\partial t^\mu=0$, $j=1,\ldots,m$, $\mu=0,\ldots,\mu_j-1$, $t_j\in J$ and $\mu_1+\cdots+\mu_m=N+1$. This implies that the quadrature formula

$$\int_a^b f(t)\omega(t)\,dt\approx\sum_{i=0}^{n+1}\sum_{\lambda=0}^{\nu_i-1}c_{i\lambda}f^{(\lambda)}(x_i) \tag{2.7}$$

is exact for the functions

$$(\partial^\mu/\partial t^\mu)K(x,t)|_{t=t_j},\quad j=1,\ldots,m,\ \mu=0,\ldots,\mu_j-1. \tag{2.8}$$

But the system (2.8) is ETS on J. Then the interpolation conditions

$$P^{(\lambda)}(x_i)=0 \text{ for } i=0,\ldots,n+1,\ \lambda=0,\ldots,\nu_i-1+\sigma_i,$$

$$P^{(\nu)}(a)=1 \text{ where } \nu=\nu_l \text{ if } a=x_l, \text{ and } \nu=0 \text{ otherwise},$$

determine the function $P\in \text{span}\{\partial^{\mu}K(x,t_j)/\partial t^{\mu},\ j=1,\ldots,m,\ \mu=0,\ldots,\mu_j-1\}$ uniquely. Evidently P has not other zeros excepting (x_i) in J. Then $P(x)\geq 0$ in $[a,b]$, since the zeros $x_{l+1},\ldots,x_s$ have even multiplicities and $P^{(\nu)}(a)=1$. Therefore $I(P)>0$. On the other hand, the quadrature (2.7) is exact for P, i. e. $I(P)=S(\mathbf{c},\mathbf{x};P)=0$. The contradiction proves (2.6).

In a similar way we show that (2.5) holds. Indeed let us assume that $SC(g;(a,b))=M$ and $F^{(\lambda)}(g;x_i)=0$ for $i=1,\ldots,n,\ \lambda=0,\ldots,\nu_i-1$, where $\Sigma_{i=1}^{n}\nu_i=M+1$. Then $\int_a^b K_\lambda(x_i,t)g(t)dt=0$ and consequently

$$\int_a^b P(t)g(t)dt=0 \tag{2.9}$$

for every function $P\in \text{span}\{K_\lambda(x_i,t),\ i=1,\ldots,n,\ \lambda=0,\ldots,\nu_i-1\}$. But according to our assumption $g(t)$ changes its sign M times in (a,b), say at the points $\xi_1,\ldots,\xi_M$. Since $K(x,t)$ is ETP, the interpolation problem

$$P(\xi_j)=0,\ j=1,\ldots,M,$$

$$P(t_0)=g(t_0) \text{ for some } t_0\notin(\xi_1,\ldots,\xi_M)$$

has a unique solution in span $\{K_\lambda(x_i,t)\}$. Denote it by $P_0(t)$. In addition $Z(P_0;J)\leq M$ since $\{K_\lambda(x_i,t)\}$ is ETS in J. Therefore $P_0(t)$ has precisely M zeros in J. Then sign $P_0(t)=$ sign $g(t)$ for $t\in(a,b)$ because $g(t_0)=P_0(t_0)$. This gives $\int_a^b P_0(t)\,g(t)\,dt>0$ which contradicts (2.9). The proof of the inequality (2.5) is complete. The assertion of the lemma then follows from (2.5) and (2.6) in view of (2.2) and (2.3).

3. **Continuity of the error**. We shall prove in this section the continuous dependence of the extremal function $\Phi(\mathbf{x};t)$ on the nodes $\mathbf{x}$.

Let us set

$$\Omega_N[a_1,b_1]=\{\mathbf{y}=(\tau_1,\ldots,\tau_N)\in(R^N: a_1\leq\tau_1\leq\cdots\leq\tau_N\leq b_1\}.$$

$\|\mathbf{y}\|$ will denote $\max_{1\leq k\leq N}|\tau_k|$ for every $\mathbf{y}=(\tau_1,\ldots,\tau_N)\in R^N$. Denote by $\mathbf{\Omega}((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[a_1,b_1])$ the closure of $\Omega((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[a_1,b_1])$, i. e., the set of all points $\mathbf{y}\in\Omega_N[a_1,b_1]$ for which there is a sequence $\{\mathbf{y}^{(i)}\}$ in $\Omega((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[a_1,b_1])$ such that $\lim_{i\to\infty}\|\mathbf{y}^{(i)}-\mathbf{y}\|=0$. As in [6] or [7] one could show that

$$\inf\{\|M(\mathbf{y};\cdot)\|_p:\mathbf{y}\in\Omega((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[a_1,b_1])\}$$
$$=\inf\{\|M(\mathbf{y};\cdot)\|_p:\mathbf{y}\in\mathbf{\Omega}((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[a_1,b_1])\}$$

for every interval $[a_1,b_1]\in J$. Next we shall slightly improve the above assertion.

Lemma 3. *Let $\{\mathbf{y}^{(i)}\}_1^\infty$ be a given sequence in $\Omega((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[a_1,b_1])$ and let*

$$\mathbf{x}=\begin{pmatrix}x_1,\ldots,x_m\\ \mu_1,\ldots,\mu_m\end{pmatrix}.$$

Suppose that $\lim_{i\to\infty}\|\mathbf{y}^{(i)}-\mathbf{x}\|=0$. Then the sequence $\{M(\mathbf{y}^{(i)};t)\}$ converges uniformly to $M(\mathbf{x};t)$ on $[a_1,b_1]$.

Proof. It is well-known [7] that the sequence $\{M(\mathbf{y}^{(i)};t)\}$ converges uniformly on $[a_1,b_1]$ to a certain EM $M(\mathbf{c},\mathbf{x};t)$. Without loss of generality we may assume that $\|\mathbf{y}^{(i)}-\mathbf{x}\|\leq \Delta\mathbf{x}$ for all i, where $\Delta\mathbf{x}=3^{-1}\min_{0\leq k\leq n}|x_{k+1}-x_k|$. Denote by $y_{k1}^{(i)},\ldots,y_{k,\mu_k}^{(i)}$ the coordinates τ of $\mathbf{y}^{(i)}$ yor which $|x_k-\tau|\leq \Delta\mathbf{x}$. Le. us rewrite $M(\mathbf{y}^{(i)};t)$ as

$$M_i(t)=\int_a^b K(x,t)\,\omega(x)\,dx-\sum_{k=1}^{m}\sum_{\lambda=0}^{\mu_k-1}\lambda!\,\alpha_{k\lambda}^{(i)}K[y_{k1}^{(i)},\ldots,y_{k,\lambda+1}^{(i)};t],$$

where $f[\tau_0,\ldots,\tau_k]$ denotes the divided difference of f based on the points $\tau_0\leq\cdots\leq\tau_k$. Evidently $\lambda!K[y_{ki}^{(i)},\ldots,y_{k,\lambda+1}^{(i)};t]$ converges uniformly to $K_\lambda(x_k,t)$ on $[a_1,b_1]$. It was shown in [6] that the number sequences $\{\alpha_{k\lambda}^{(i)}\}_{i=1}^{\infty}$ are bounded. Hence we may assume that $\lim_{i\to\infty}\alpha_{k\lambda}^{(i)}=c_{k\lambda}$. Let $\mathbf{a}=\{a_{k\lambda}\}$ be the best coefficients for the nodes $\mathbf{x}$. Obviously

$$\|M_i\|_p\leq\Big\|\int_a^b K(x,t)\omega(x)\,dx-\sum_{k=1}^{m}\sum_{\lambda=0}^{\mu_k-1}\lambda!\,a_{k\lambda}K[y_{k1}^{(i)},\ldots,y_{k,\lambda+1}^{(i)};t]\Big\|_p.$$

This gives $\|M(\mathbf{c},\mathbf{x};\cdot)\|_p\leq\|M(\mathbf{x};\cdot)\|_p$ since $\lim_{i\to\infty}M_i(t)=M(\mathbf{c},\mathbf{x};t)$. But the best coefficients $\mathbf{a}$ are unique. Therefore $\mathbf{c}=\mathbf{a}$. The lemma is proved.

Corollary 1. *Under the same assumption as in Lemma 3 we have*

$$\lim_{i\to\infty}\|\Phi^{(j)}(\mathbf{y}^{(i)};\cdot)-\Phi^{(j)}(\mathbf{x};\cdot)\|_{C[a_1,b_1]}=0$$

for $j=0,\ldots,\max(\mu_1,\ldots,\mu_m)$.

Indeed Lemma 3 implies $\lim_{i\to\infty}\|\varphi(\mathbf{y}^{(i)};\cdot)-\varphi(\mathbf{x};\cdot)\|=0$, where $\|\cdot\|=\|\cdot\|_\infty$ for $1<p<\infty$ and $\|\cdot\|=\|\cdot\|_1$ for $p=1$. The result then follows from (2.3) realizing that $K_j(x,t)$ is continuous.

Corollary 2. *$R_q(\cdot)$ is a continuous function in* $\mathbf{\Omega}((a,\nu_0),\nu_1,\ldots\nu_n,(b,\nu_{n+1}))$.

It is an immediately consequence of Corollary 1 and the equality $R_q(\mathbf{x})=I(\Phi(\mathbf{x};\cdot))$.

4. Characterization of the optimal quadrature formula

Theorem 1. *Let the multiplicities $(\nu_k)_1^n$ be even numbers and let the quadrature formula (2.4) be optimal of the type $((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1})[a_1,b_1])$ in the class $K_q[a,b]$, where $a_1<a<b<b_1$ and $1<q\leq\infty$. Then $a<x_1$, $x_n<b$, $a_{0,\nu_0-1}>0$, $(-1)^{\nu_{n+1}-1}a_{n+1,\nu_{n+1}-1}>0$ and*

$$a_{k,\nu_k-1}=0,\ a_{k,\nu_k-2}>0\ \textit{for}\ k=1,\ldots,n.$$

Proof. Consider the function $M_k(t)=\|M((\mathbf{a}(\mathbf{x}),\mathbf{y};\cdot)\|_p$ for

$$\mathbf{y}=\begin{pmatrix}x_0,\ldots,x_{k-1},t,x_{k+1},\ldots,x_{n+1}\\ \nu_0,\ldots,\nu_{k-1},\nu_k,\nu_{k+1},\ldots,\nu_{n+1}\end{pmatrix}.$$

It follows from the optimality of the nodes $\mathbf{x}$ that $M_k'(x_k)=0$ for $a_1<x_k<b_1$ $M_1'(a_1)\geq0$ if $x_1=a_1$ and $M_n'(b_1)\leq0$ if $x_n=b_1$. But $\operatorname{sign}M_k'(x_k)=-\operatorname{sign}\{a_{k,\nu_k-1}\Phi^{(\nu_k)}(\mathbf{x};x_k)\}$. Therefoer.

$$a_{k,\nu_k-1}\Phi^{(\nu_k)}(\mathbf{x};x_k)=0\ \text{for}\ a_1<x_k<b_1,\tag{4.1}$$

$$a_{1,\nu_1-1}\Phi^{(\nu_1)}(\mathbf{x};x_1)\leq0,\tag{4.2}$$

$$a_{n,\nu_n-1}\Phi^{(\nu_n)}(\mathbf{x};x_n)\geq0.\tag{4.3}$$

On the other hand, by Lemma 2, $Z(\Phi(\mathbf{x};\cdot);J)=\nu_0+\cdots+\nu_{n+1}$. This implies $\Phi^{(\nu_k)}(\mathbf{x};x_k)\neq 0$. Then in view of (4.1) we have $a_{k,\nu_k-1}=0$ for $a_1<x_k<b_1$.

Next we prove that $a_1<x_1$. Indeed, assuming that $a_1=x_1$ we obtain $\operatorname{sign}\Phi^{(\nu_1)}(\mathbf{x};x_1)=(-1)^{\nu_0}$ since the numbers $(\nu_k)_1^n$ are even and $\Phi(\mathbf{x};t)\geq 0$ in $[a,b]$. Then (4.2) gives

$$(4.4)\qquad (-1)^{\nu_0-1}a_{1,\nu_1-1}\geq 0.$$

Now, by virtue of Lemma 2, $Z(\Phi(\mathbf{x};\cdot))\leq SC(\varphi(\mathbf{x};\cdot);(a,b))$. Hence $M(\mathbf{x};t)$ has $N=\nu_0+\cdots+\nu_{n+1}$ distinct zeros at least in (a,b). Let us denote them by $t_1,\ldots,t_N$. The equalities $M(\mathbf{x};t_j)=0$ show that the quadrature formula (2.4) is exact for all functions of the set $\mathcal{P}:=\operatorname{span}\{K(x,t_j)\}_{j=1}^N$. Let us apply (2.4) to the function $P\in\mathcal{P}$ which is defined by the interpolation conditions

$$P^{(\nu_1-1)}(x_1)=1,\ P^{(\lambda)}(x_1)=0,\ \ \lambda=0,\ldots,\nu_1-2,$$
$$P^{(\lambda)}(x_k)=0,\ k\neq 1,\ \lambda=0,\ldots,\nu_k-1.$$

Since $K(x,t)$ is ETP, the function P is uniquely determined and $Z(P;J)=N-1$. Hence P does not vanish in other points excepting $x_0,\ldots,x_{n+1}$ Then $(-1)^{\nu_0}P(t)\geq 0$ for $t\in[a,b]$ and consequently

$$(-1)^{\nu_0}a_{1,\nu_1-1}=(-1)^{\nu_0}S(\mathbf{a},\mathbf{x};P)=(-1)^{\nu_0}I(P)>0.$$

The above inequality contradicts (4.4). Therefore $a_1<x_1$. Analogously, using (4.3), one can show that $x_n<b_1$. Now it is easy to see that $a<x_1$, $x_n<b$. Indeed, let us assume that $a>x_1$. Obviously the nodes $\mathbf{x}$ are optimal of the type $((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[x_1,b_1])$, too. Then, according to the necessary condition we just proved, x_1 must lie in the open interval (x_1,b_1) which is impossible. Hence $a<x_1$, since $a=x_0\neq x_1$ by assumption. A similar argument shows that $x_n<b$.

Next making use of the equalities $a_{k,\nu_k-1}=0$, $k=1,\ldots,n$ we see that $a_{k,\nu_k-2}=S(\mathbf{a},\mathbf{x};P)=I(P)>0$, where $P\in\operatorname{span}\{K(x,t_j)\}_{j=1}^{N-1}$ and

$$P^{(\nu_k-2)}(x_k)=1,\ P^{(\lambda)}(x_k)=0,\ \lambda=0,\ldots,\nu_k-3,$$
$$P^{(\lambda)}(x_i)=0,\ i\neq k,\ \lambda=0,\ldots,\nu_i-1.$$

Finally, $a_{0,\nu_0-1}=I(P)>0$ for $P\in\mathcal{P}$ and such that

$$P^{(\nu_0-1)}(a)=1,\ P^{(\lambda)}(a)=0,\ \lambda=0,\ldots,\nu_0-2,$$
$$P^{(\lambda)}(x_i)=0,\ i=1,\ldots,n+1,\ \lambda=0,\ldots,\nu_i-1.$$

Similarly $(-1)^{\nu_{n+1}-1}a_{n+1,\nu_{n+1}-1}>0$. The theorem is proved.

Theorem 2. *Let the multiplicities* $(\nu_k)_0^{n+1}$ *be fixed. Suppose that* $\nu_1,\ldots,\nu_n$ *are even numbers. If* $R_q(\mathbf{x})=\inf\{R_q(\mathbf{y}):\mathbf{y}\in\mathbf{\Omega}((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[a,b])\}$, $1<q\leq\infty$ *and* $\mathbf{x}\in\mathbf{\Omega}((a,\nu_0),\nu_1,\ldots,\nu_n,(b,\nu_{n+1});[a,b])$ *then the nodes* $\mathbf{x}$ *must be of the form*

$$\mathbf{x}=\begin{pmatrix} a, & x_1,\ldots,x_m, & b\\ \mu_0, & \mu_1,\ldots,\mu_m, & \mu_{m+1}\end{pmatrix},$$

where $\mu_0=\nu_0$, $\mu_{m+1}=\nu_{n+1}$.

Proof. Evidently $\mu_0 \geq \nu_0$, $\mu_{m+1} \geq \nu_{n+1}$. Let us assume that $\mu_0 > \nu_0$. Then $\mu_0 = \nu_0 + \nu_1 + \cdots + \nu_j$ for some j, $1 \leq j \leq n$. With any h, $0 \leqq h \leq x_1 - a$ we associate the nodes

$$\mathbf{x}_h = \begin{pmatrix} a, & a+h, & x_1, \ldots, & x_m, & b \\ \mu, & \nu_j, & \mu_1, \ldots, & \mu_m, & \mu_{m+1} \end{pmatrix},$$

where $\mu = \nu_0 + \cdots + \nu_{j-1}$. Since $\mathbf{x}_h \in \mathbf{\Omega}((a, \nu_0), \nu_1, \ldots, \nu_n, (b, \nu_{n+1}); [a, b])$ we have

$$R_q(\mathbf{x}) \leq R_q(\mathbf{x}_h) = \|M(\mathbf{x}_h; \cdot)\|_p \leq \|\widetilde{M}(\mathbf{x}_h; \cdot)\|_p, \tag{4.5}$$

where

$$\widetilde{M}(\mathbf{x}_h; t) = \int_a^b K(x, t)\omega(x)dx - \sum_{\lambda=0}^{\mu-1} a_{0\lambda} K_\lambda(a, t)$$

$$- \sum_{\lambda=\mu}^{\mu_0-1} a_{0\lambda} D_\lambda(K(\cdot; t); h) - \sum_{i=1}^{m+1} \sum_{\lambda=0}^{\mu_i-1} a_{i\lambda} K_\lambda(x_i, t).$$

Here $D_\lambda(f; h)$ denotes the divided difference of the function $\lambda! f(t)$ based on the points $\underbrace{a, \ldots, a}_{\mu}$, $\underbrace{a+h, \ldots, a+h}_{\lambda+1-\mu}$ and $\{a_{i\lambda}\}$ are the best coefficients for the nodes $\mathbf{x}$. Evidently $D_\lambda(K(\cdot; t); 0) = K_\lambda(a, t)$ and $\lim_{h\to 0} \|\widetilde{M}(\mathbf{x}_h; \cdot)\|_p = R_q(\mathbf{x})$. Then, in view of (4.5), we have

$$\frac{\partial}{\partial h} \int_a^b |\widetilde{M}(\mathbf{x}_h; t)|^p dt|_{h=0} = 0. \tag{4.6}$$

In order to evaluate the above derivative we observe that $D_\lambda(f; h) = \int_a^{a+h} u_1(t) f^{(\lambda)}(t)\, dt$, where $u_1(t)$ is the corresponding B-spline ([8; 9]) of degree $\lambda-1$ with knots a and $a+h$ of multiplicities μ and $\lambda+1-\mu$ respectively. It is not difficult to verify that $u_1(t) = (1/h)\, u((t-a)/h)$, where $u(\cdot)$ is the B-spline of degree $\lambda-1$ with knots at 0 and 1 of multiplicities μ and $\lambda+1-\mu$ respectively. Then

$$D_\lambda(f; h) = \frac{1}{h} \int_a^{a+h} u((t-a)/h) f^{(\lambda)}(t)\, dt = \int_0^1 u(\tau) f^{(\lambda)}(a+h\tau)\, d\tau$$

and consequently

$$\frac{\partial}{\partial h} D_\lambda(f; h)|_{h=0} = \int_0^1 \tau u(\tau) f^{(\lambda+1)}(a)\, d\tau = \alpha_\lambda f^{(\lambda+1)}(a), \tag{4.7}$$

where $\alpha_\lambda = \int_0^1 \tau u(\tau)\, d\tau$. It is well-known [8] that $u(\tau) \geq 0$ for $\tau \in [0, 1]$ and $u(\tau) \neq 0$. Therefore $\alpha_\lambda > 0$. Now, using (4.7), we get from (4.6)

$$\frac{\partial}{\partial h} \int_a^b |\widetilde{M}(\mathbf{x}_h; t)|^p\, dt|_{h=0}$$

$$= p \int_a^b |M(\mathbf{x}; t)|^{p-1} \operatorname{sign} M(\mathbf{x}; t) \{-\sum_{\lambda=\mu}^{\mu_0-1} a_{0\lambda} \alpha_\lambda K_{\lambda+1}(a, t)\}\, dt \geq 0.$$

This, together with Lemma 1, gives

$$-\sum_{\lambda=\mu}^{\mu_0-1} a_{0\lambda} \alpha_\lambda \Phi^{(\lambda+1)}(\mathbf{x}; a) = -a_{0,\mu_0-1}\, \alpha_{\mu_0-1} \Phi^{(\mu_0)}(\mathbf{x}; a) \geq 0.$$

But $\Phi^{(\mu_0)}(\mathbf{x};a)\neq 0$, as a consequence of Lemma 2. Moreover, $\Phi^{(\mu_0)}(\mathbf{x};a)>0$, since $\Phi(\mathbf{x};t)\geq 0$ for $t\in[a,b]$. Therefore $a_{0,\mu_0-1}\leq 0$ which contradicts Theorem 1. So we proved that $\mu_0=\nu_0$. In a similar way one can show that $\mu_{m+1}=\nu_{n+1}$. The theorem is proved.

5. Existence. First we prove an auxiliary result.

Lemma 4. *Let h be an arbitrary positive number and let $f\in C^r[\tau-h,\tau+h]$. Suppose that f has exactly r zeros in $[\tau-h,\tau+h]$ and $f(\tau-h)=f(\tau+h)=0$. If $0<m<f^{(r)}(t)<M$ for all $t\in[\tau-h,\tau+h]$ then $\beta-\alpha>\sqrt{m}(Mr!2^{r-2})^{-1/2}.h$, where α,β are the zeros of $f^{(r-2)}(t)$* in $[\tau-h,\tau+h]$.

Proof. By Rolle's Theorem $f^{(k)}(x)$ has exactly $r-k$ zeros in $[\tau-h,\tau+h]$. Denote them by $\{t_{kj}\}_{j=1}^{r-k}$ $(t_{k1}\leq\cdots\leq t_{k,r-k})$. Evidently $f^{(k)}(x)=\int_{t_{k2}}^{x} f^{(k+1)}(t)\,dt$ *for* $k\leq r-2$ and consequently

$$\max_{t_{k1}\leq x\leq t_{k,r-k}}|f^{(k)}(x)|\leq(t_{k,r-k}-t_{k1})\max_{t_{k+1,1}\leq x\leq t_{k+1,r-k-1}}|f^{(k+1)}(x)|. \tag{5.1}$$

Let ξ be the unique zero of $f^{(r-1)}(t)$ in $[\tau-h,\tau+h]$. Then

$$\max_{\alpha\leq t\leq\beta}|f^{(r-2)}(t)|=|f^{(r-2)}(\xi)|\leq M(\beta-\alpha)^2/4.$$

Since $t_{k1}<\xi<t_{k,r-k}$ for $k=0,\dots,r-2$, a repeated use of (5.1) gives

$$|f^{(k)}(\xi)|\leq(2h)^{r-k-2}(\beta-\alpha)^2M/4,\quad k=0,\dots,r-2. \tag{5.2}$$

Now suppose that $\xi\leq\tau$. By Taylor's formula

$$f(x)=\sum_{k=0}^{r-1}f^{(k)}(\xi)(x-\xi)^k/k!+\frac{1}{(r-1)!}\int_{\xi}^{x}(x-t)^{r-1}f^{(r)}(t)\,dt.$$

Making use of (5.2) and the assumptions of the lemma we get for $x\geq\tau$

$$f(x)\geq\frac{m}{(r-1)!}\int_{\xi}^{x}(x-t)^{r-1}dt-\sum_{k=0}^{r-2}|f^{(k)}(\xi)|\,|x-\xi|^k/k!$$
$$>m(x-\tau)^r/r!-M(\beta-\alpha)^2(2h)^{r-2}.$$

In the special case $x=\tau+h$ the above inequality gives $0=f(\tau+h)>mh^r/r!-M(\beta-\alpha)^2(2h)^{r-2}$ and our assertion follows immediately.

Now suppose that $\tau\leq\xi$. Let $x\leq\tau$. In a similar fashion as above we obtain

$$f(x)<-m(\tau-x)^r/r!+M(\beta-\alpha)^2(2h)^{r-2}$$

for odd r, and

$$f(x)>m(x-\tau)^r/r!-M(\beta-\alpha)^2(2h)^{r-2}$$

for even r. This together with the assumption $f(\tau-h)=0$ yields $M(\beta-\alpha)^2(2h)^{r-2}>mh^r/r!$. The lemma is proved.

Theorem 3. *Let $K(x,t)$ be an ETP kernel in $J\times J$ and let $[a,b]\subset J$. Suppose that $1<q\leq\infty$. Then for every system $(\nu_k)_0^{n+1}$ of multiplicities there exists an optimal quadrature formula of the type $((a,\nu_0),\nu_1,\dots,\nu_n,(b,\nu_{n+1}))$ in the class $K_q[a,b]$. The nodes $(x_k)_0^{n+1}$ and the coefficients $\mathbf{a}=\{a_{k\lambda}\}$ of this quadrature formula satisfy the relations*

$$a=x_0<x_1<\cdots<x_n<x_{n+1}=b,$$
$$a_{0,\nu_0-1}>0,\ (-1)^{\nu_{n+1}-1}a_{n+1,\nu_{n+1}-1}>0,$$

$$a_{k,\nu_k-1}=0,\ a_{kj}>0,\ j=0,2,\dots,\nu_k-2 \text{ for even } \nu_k,\ 1\le k\le n;$$

$$a_{kj}>0,\ j=0,2,\dots,\nu_k-1 \text{ for odd } \nu_k,\ 1\le k\le n.$$

Proof. First we consider the case when the multiplicities $(\nu_k)_1^n$ are even numbers. Let $\varrho=\inf\{R_q(\mathbf{y}):\mathbf{y}\in\Omega((a,\nu_0),\nu_1,\dots,\nu_n,(b,\nu_{n+1}))\}$. By virtue of Theorem 1 we have

$$\varrho=\inf\{R_q(\mathbf{y}):\mathbf{y}\in\Omega((a,\nu_0),\nu_1,\dots,\nu_n,(b,\nu_{n+1});[a,b])\}.$$

Since $R_q(\cdot)$ is a continuous function in $\mathbf{\Omega}((a,\nu_0),\nu_1,\dots,(b,\nu_{n+1});[a,b])$, there exists $\mathbf{x}\in\mathbf{\Omega}((a,\nu_0),\nu_1,\dots,\nu_n,(b,\nu_{n+1});[a,b])$ such that $\varrho=R_q(\mathbf{x})$. According to Theorem 2, $\mathbf{x}$ has a form

$$\mathbf{x}=\begin{pmatrix} a, & x_1,\dots, x_m, & b\\ \nu_0, & \mu_1,\dots,\mu_m, & \nu_{n+1}\end{pmatrix}.$$

Obviously $m\le n$. In the case $m=n$ we have $\mu_k=\nu_k$, $k=1,\dots,n$ and hence the nodes $\mathbf{x}$ are optimal of the type $(a,\nu_0),\nu_1,\dots,\nu_n,(b,\nu_{n+1}))$. Now assume that $m<n$. Then there is a node x_k for which $\mu_k=\nu_k+\nu_{k+1}+\dots+\nu_{k+j}$ and $j\ge 1$. With any h, $0\le h\le \Delta\mathbf{x}$, we associate the nodes

$$\mathbf{x}(h)=\begin{pmatrix} a, & x_1,\dots,x_{k-1}, & \tau-h, & \tau+h, & x_{k+1},\dots,x_m, & b\\ \nu_0, & \mu_1,\dots,\mu_{k-1}, & \nu_k, & \mu_k-\nu_k, & \mu_{k+1},\dots,\mu_m, & \nu_{n+1}\end{pmatrix},$$

where τ is a point from $[x_k-h,\ x_k+h]$ which we shall collocate below. It follows from the optimality of the nodes $\mathbf{x}$ that

$$R_q(\mathbf{x})\le R_q(\mathbf{x}(h)). \tag{5.3}$$

Next we shall note some properties of the function $\Phi(\mathbf{x}(h);t)$. First we observe that there exist numbers ε and h_0, $0<\varepsilon<\Delta\mathbf{x}$, $0<h_0<\varepsilon/2$ such that

$$\Phi^{(\mu_k)}(\mathbf{x}(h);t)>0 \tag{5.4}$$

for every $t\in[x_k-\varepsilon,\ x_k+\varepsilon]$ and $h\le h_0$. Indeed, as we saw before, $\Phi^{(\mu_k)}(\mathbf{x};x_k)>0$ Consequently $\Phi^{(\mu_k)}(\mathbf{x};t)>0$ in a neighbourhood of x_k. Then (5.4) follows from Corollary 1.

The inequality (5.4) and Rolle's Theorem yield that $\Phi^{(\lambda)}(\mathbf{x}(h);t)$ $(\lambda=0,\dots,\mu_k-1)$ has $\mu_k-\lambda$ zeros at most in $[x_k-\varepsilon,\ x_k+\varepsilon]$. On the other hand, $\Phi(\mathbf{x}(h);t)$ has μ_k zeros in $[x_k-\varepsilon,\ x_k+\varepsilon]$: $\tau-h$ and $\tau+h$ with multiplicities ν_k and $\mu_k-\nu_k$ respectively. Therefore

(5.5) $\Phi^{(\lambda)}(\mathbf{x}(h);t)$ has exactly $\mu_k-\lambda$ zeros in $[x_k-\varepsilon,\ x_k+\varepsilon]$ for every $h\le h_0$, $\lambda=0,\dots,\mu_k$.

Now suppose that h is fixed in the interval $[0,h_0]$. We choose the point $\tau=\tau(h)$ to satisfy the requirement

$$\Phi^{(\mu_k-1)}(\mathbf{x}(h);x_k)=0. \tag{5.6}$$

It may be done. Indeed, let $\xi(\tau)$ denote the unique zero of $\Phi^{(\mu_k-1)}(\mathbf{x}(h);t)$ in $[x_k-\varepsilon,\ x_k+\varepsilon]$. It is not difficult to verify that $\xi(\tau)$ is a continuous function of τ for fixed h. In addition $\xi(x_k-h)<x_k$ and $\xi(x+h)>x_k$. Therefore there exists $\tau\in(x_k-\varepsilon,\ x_k+\varepsilon)$ for which $\xi(\tau)=x_k$. In what follows we assume that the point τ is chosen in this way.

Since $\Phi^{(\mu_k)}(\mathbf{x}(h);t)$ is a sontinuous function in $[a, b]$, we conclude from (5.4) that there exist constants $C_1>0$, $C_2>0$ such that

$$C_2>\Phi^{(\mu_k)}(\mathbf{x}(h);t)>C_1, \quad t\in[x_k-\varepsilon,\ x_k+\varepsilon] \tag{5.7}$$

for every $h\leq h_0$. Lef $t_{\lambda,},\ldots, t_{\lambda,\mu_k-\lambda}$ be the zeros of $\Phi^{(\lambda)}(\mathbf{x}(h);t)$ in $[x_k-\varepsilon, x_k+\varepsilon]$. Then, by Newton's interpolation formula,

$$\Phi^{(\lambda)}(\mathbf{x}(h);t)=(t-t_{\lambda 1})\ldots(t-t_{\lambda,\mu_k-\lambda})\int_{\tau-h}^{\tau+h} u(x)\Phi^{(\mu_k)}(\mathbf{x}(h);x)\,dx. \tag{5.8}$$

Here $u(x)$ is the corresponding B-spline. Hence ([8], [9]) $u(x)\geq 0$ and $\int_{\tau-h}^{\tau+h} u(x)\,dx=1/(\mu_k-\lambda)!$ Using (5. 7) we get

$$|\Phi^{(\lambda)}(\mathbf{x}(h);x_k)|\leq Ch^{\mu_k-\lambda},\quad \lambda=0,\ldots,\mu_k \tag{5.9}$$

for all $h\leq h_0$, where C does not depend of h.

Let α, β be the zeros of $\Phi^{(\mu_k-2)}(\mathbf{x}(h);t)$ in $[x_k-\varepsilon,\ x_k+\varepsilon]$. In view of (5.6), (5.7) and (5.8)

$$|\Phi^{(\mu_k-2)}(\mathbf{x}(h);x_k)|=\max_{\alpha\leq t\leq\beta}|\Phi^{(\mu_k-2)}(\mathbf{x}(h);t)|\geqq C_1(\beta-\alpha)^2/4.$$

Then, according to Lemma 4, there exists a constant, denote it again by C_1 such that

$$|\Phi^{(\mu_k-2)}(\mathbf{x}(h);x_k)|\geq C_1h^2. \tag{5.10}$$

Finally note that

$$\Phi^{(\mu_k-2)}(\mathbf{x}(h);x_k)<0, \tag{5.11}$$

since $\alpha<x_k<\beta$ and $\Phi^{(\mu_k)}(\mathbf{x}(h);t)>0$ in $[x_k-\varepsilon,\ x_k+\varepsilon]$.

Clearly the nodes $\mathbf{x}$ are optimal of the type $((a, \nu_0), \mu_1,\ldots, \mu_m, (b, \nu_{n+1}); [a, b])$. Then, according to Theorem 1,

$$a_{k,\mu_k-1}=0,\ a_{k,\mu_k-2}>0,\ k=1,\ldots, n, \tag{5.12}$$

where $\{a_{k\lambda}\}$ are the best coefficients for the nodes $\mathbf{x}$. Let us apply the optimal quadrature formula with the nodes $\mathbf{x}$ to the function $\Phi(\mathbf{x}(h);t)$. We get

$$R_q(\mathbf{x}(h))-S(\mathbf{a}, \mathbf{x}; \Phi(\mathbf{x}(h);.))\leq R_q(\mathbf{x}). \tag{5.13}$$

Next, in view of (5.6), (5.10), (5.11) and (5.12), $R_q(\mathbf{x}(h))+a_{k,\mu_k-2}\, C_1h^2-\delta(h)\leq R_q(\mathbf{x})$, where

$$\delta(h)=\sum_{k=0}^{\mu_k-3} a_{k\lambda}|\Phi^{(\lambda)}(\mathbf{x}(h);x_k)|.$$

But, according to (5.9), $\delta(h)=O(h^3)$ when $h\to 0$. Hence

$$R_q(\mathbf{x}(h))<R_q(\mathbf{x}) \tag{5.14}$$

for a sufficiently small h. This contradicts (5.3). So $m=n$ and the existence part of our theorem is proved in the case of even multiplicities $(\nu_k)_1^n$.

Now consider the case of arbitrary multiplicities $(\nu_k)_0^{n+1}$. Denote $\mu_k=2[(\nu_k+1)/2]$, $k=1,\ldots, n$, where $[.]$ is the greatest integer function. Since the numbers $(\mu_k)_1^n$ are even, there exists a system of nodes $\mathbf{x}$ which are optimal of the type

$((a, \nu_0), \mu_1, \ldots, \mu_n, (b, \nu_{n+1}))$. But, according to Theorem 1, $a_{k,\mu_k-1}=0$ if $\mu_k>\nu_k$, where the coefficients $\{a_{k\lambda}\}$ are best for the nodes **x**. Then the nodes **x** are optimal of the type $((a, \nu_0), \nu_1, \ldots, \nu_n, (b, \nu_{n+1}))$ too.

In order to prove the last assertion of our theorem we observe that the monospline $M(\mathbf{x};t)$ has the maximal number of zeros. Denote them by $t_1, \ldots, t_N$, $N=\sum_{k=0}^{n+1}\nu_k+\sum_{k=1}^{n}\sigma_k$. Then $\int_a^b K(x, t_j)\omega(x)\,dx$

$$=\sum_{\lambda=0}^{\nu_0-1} a_{0\lambda}K_\lambda(a, t_j)+\sum_{\lambda=0}^{\nu_{n+1}-1} a_{n+1,\lambda}K_\lambda(b, t_j)+\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1-\sigma_k} a_{k\lambda}K_\lambda(x_k, t_j)$$

for $j=1, \ldots, N$. But $\{K(x, t_j)\}_{j=1}^{N}$ is an extended complete Tchebycheff (ECT) system in $[a, b]$, because $K(x, t)$ is ETP in J and $[a, b]\subset J$. Then $a_{kj}>0$, $j=0$, $2, \ldots, \nu_k-1-\tau_k$, $k=1, \ldots, n$, according to an observation due to Karlin and Pinkus [10, Proposition 2]. The proof is completed.

The following is a consequence of the relation between optimal quadrature formulae in $K_q[a, b]$ and monosplines of least L_p deviation.

Theorem 4. *Let $K(x, t)$ be an ETP kernel in $J\times J$ and $[a, b]\subset J$. Suppose that $1\leq p<\infty$. Then for any system $(\nu_k)_0^{n+1}$ of multiplicities there exists a monospline $M(\mathbf{a}, \mathbf{x}; t)$ with nodes $a=x_0<x_1<\cdots<x_n<x_{n+1}=b$ such that $\|M(\mathbf{a}, \mathbf{x};\cdot)\|_p\leq\|M(\mathbf{b}, \mathbf{y};\cdot)\|_p$ for all $\mathbf{y}\in\Omega((a, \nu_0), \nu_1, \ldots, \nu_n, (b, \nu_{n+1}))$ and all real coefficients* **b**. *Moreover,* $a_{0,\nu_0-1}>0$, $(-1)^{\nu_{n+1}-1}a_{n+1,\nu_{n+1}-1}>0$, $a_{k,\nu_k-1}=0$, $a_{kj}>0$, $j=0, 2, \ldots, \nu_k-2$ *for even* ν_k, $1\leq k\leq n$, $a_{kj}>0$, $j=0, 2, \ldots, \nu_k-1$ *for odd* ν_k, $1\leq k\leq n$.

It can be derived from (5.14) that $R_q(\mathbf{x})<R_q(\mathbf{y})$ if **x** and **y** are optimal of the type $((a, \nu_0), \nu_1, \ldots, \nu_n, (b, \nu_{n+1}))$ and $((a, \nu_0), \nu_1, \ldots, \nu_{k-2}, \nu_{k-1}+\nu_k, \nu_{k+1}, \ldots, \nu_n, (b, \nu_{n+1}))$ respectively. This fact and Theorem 3 imply the following refirement of Karlin's result [2] (see also [3]).

Corollary 3. *Let the nodes* **x** *be optimal of the type* $((a, \nu), \underbrace{1, \ldots, 1}_{m}, (b, \mu))$ *in the class* $K_q[a, b]$ $(1\leq p<\infty)$. *Then* $\|M(\mathbf{x};\cdot)\|_p\leq\|M(\mathbf{y};\cdot)\|_p$ *for all* **y** *of the form* $\mathbf{y}=\begin{pmatrix} a, & y_1, \ldots, & y_n, & b \\ \nu, & \nu_1, \ldots, & \nu_n, & \mu\end{pmatrix}$ *where* $[(\nu_1+1)/2]+\cdots+[(\nu_n+1)/2]\leq m$. *The equality holds only for* $n=m$ *and* $\nu_1=\cdots=\nu_n=1$.

The main result of this paper was announced in [11]. The method used in the proof of Theorem 3 is a modification of the method employed by the author in [12; 13] to show the existence of optimal quadrature formulae of a pre-assigned type in the classes $W_q^r[a, b]$ $(1<q\leq\infty)$.

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