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WIGNER OPERATOR AND WIGNER DISTRIBUTION

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In the present paper we find the eigenvalues and eigenfunctions of the Wigner operator in the phase space. It is also proved that the Wigner operator describes the Quantum-Mechanical phase space, as exactly the Hamilton operator does in the Schrödinger theory. With the help of eigenfunctions of the Wigner operator we obtain the known Wigner distribution and we calculate its propagator and Green's function in the phase space. For the cases of the harmonic oscillator and the free-electrons in a uniform magnetic and electric field we calculate the eigenvalues and eigenfunctions of the Wigner operator.

1. Introduction. F. Bopp [1] and R. Kubo [2] introduced the Quantum-mechanical formulation in phase space which is determined as follows:

$$P=p-\frac{i\hbar}{2}\frac{\partial}{\partial q},\quad Q=q+\frac{i\hbar}{2}\frac{\partial}{\partial p}. \tag{1.1}$$

For the determination of the mean density $f(p, q, t)$ Bopp and Kubo obtained the Wigner equation

$$i\hbar\,\partial f/\partial t=[H(P, Q)-H(P^*, Q^*)]f. \tag{1.2}$$

The quantities P^*, Q^* are the complex conjugates of (1.1) and the operator

$$H(P, Q)-H(P^*, Q^*)=W(q, p, \partial/\partial q, \partial/\partial p) \tag{1.3}$$

is called Wigner operator. Because of (1.3) Wigner equation (1.2) is written as $i\hbar\,\partial f/\partial t=Wf$ which has the form of a Schrödinger equation.

In a recent paper by the authors [3] it has been proved that the eigenvalues of the Wigner operator are given by the differences of the eigenvalues of the equivalent Schrödinger equations and the corresponding eigenfunctions are expressed with the help of Fourier integral

$$f_{n,m}(\mathbf{p}, \mathbf{q}, t)=(2\pi)^{-3}\int \exp(i\,\mathbf{p}\,.\,\boldsymbol{\tau}/\hbar)\,\Psi_n^*(\mathbf{q}+\boldsymbol{\tau}/2, t)\Psi_m(\mathbf{q}-\boldsymbol{\tau}/2, t)\,d\boldsymbol{\tau},$$

where $\Psi_n(\mathbf{q}, t)=\exp(-iE_nt/\hbar)U_n(\mathbf{q})$ and the eigenfunctions $U_n(\mathbf{q})$ fulfil the Schrödinger equation $H(\mathbf{q}, -i\hbar\nabla)U_n(\mathbf{q})=E_nU_n(\mathbf{q})$.

Because of the above results the eigenfunctions of the Wigner operator are given by the relation

$$\Phi_{n,m}(\mathbf{q}, \mathbf{p})=(2\pi)^{-3}\int \exp(i\mathbf{p}\,.\,\boldsymbol{\tau}/\hbar)\,U_n^*(\mathbf{q}+\boldsymbol{\tau}/2)U_m(\mathbf{q}-\boldsymbol{\tau}/2)\,d\boldsymbol{\tau}, \tag{1.4}$$

which coincide with the eigenfunctions in phase space [4], and the corresponding eigenvalues are

$$w_{nm}=E_m-E_n. \tag{1.5}$$

For the case of free electrons the eigenvalue equation for Wigner operator is of the form $m^{-1}i\hbar\mathbf{p}\nabla_q\Phi(\mathbf{q}, \mathbf{p})=w\Phi(\mathbf{q}, \mathbf{p})$ and accepts the solutions

(1.6) $$\Phi(\mathbf{p}, \mathbf{q}) \sim \exp[i(\mathbf{k}-\mathbf{k}')\mathbf{q}]\,\delta(\mathbf{k}+\mathbf{k}'+2\mathbf{p}/\hbar)$$

with the eigenvalues

(1.7) $$w = \hbar^2(k'^2-k^2)/2m.$$

Before we apply (1.4) and (1.5) to special cases, as the harmonic oscillator and others, we shall use the eigenfunctions (1.4) for the derivation of the Wigner distribution [4, 5].

2. Wigner Distribution. The Wigner distribution [4, 5] as it is referred in the literature can be derived also from the eigenfunctions (1.4) because these are eigenfunctions in phase space. The important fact here is that these are eigenfunctions of Wigner operator with the corresponding eigenvalues $E_m - E_n$. The case $n = m$ corresponds to the eigenvalue 0 which is degenerate.

If we multiply the eigenfunctions (1.4) with the coefficients $a_m(t)$, $a_n^*(t)$ and sum over n, m with the assumption that the Schrödinger eigenfunctions $U_n(\mathbf{q})$ form a complete orthonormal set, then the resulting function is of the form

$$f(\mathbf{p}, \mathbf{q}, t) = (2\pi)^{-3} \int \exp(i\mathbf{p}\,.\,\boldsymbol{\tau}/\hbar)\, \Psi(\mathbf{q}-\mathbf{t}/2, t)\, \Psi^*(\mathbf{q}+\boldsymbol{\tau}/2, t)\, d\boldsymbol{\tau},$$

which coincides with the definition of the Wigner distribution [4, 5].

The equation of Wigner (1.2) has been considered above as a Schrödinger equation in the phase space with the known boundary conditions. This equation can be compared to the results of the theory of propagators, as it is given in [6], and then it is easy to form the propagator in phase space with the following boundary condition

(2.1) $$\lim_{t\to 0} f(\mathbf{q}, \mathbf{p};\, \mathbf{q}', \mathbf{p}', t) = \delta(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}').$$

Consequently the propagator corresponding to the Wigner operator (1.3) is of the form

(2.2) $$f(\mathbf{q}, \mathbf{p};\, \mathbf{q}', t) = \sum_{n,m} \Phi^*_{n,m}(\mathbf{q}', \mathbf{p}') \exp[-it(\omega_n - \omega_m)]\Phi_{n,m}(\mathbf{q}, \mathbf{p}),$$

where $\omega_n = E_n/\hbar$. Because the eigenfunctions $\Phi_{n,m}(\mathbf{q}, \mathbf{p})$ form a complete orthonormal set, which can be easily proved from (1.4) and the $U_n(q)$ it follows that eqs. (2.2) for $t \to 0$ fulfil (2.1).

Once we know the eigenfunctions and eigenvalues of the Wigner operator, it is easy to determine Green's function of the Wigner operator in the phase space, namely

$$G(\mathbf{q}, \mathbf{p};\, \mathbf{q}', \mathbf{p}', E) = \lim_{\eta\to 0} \sum_{n,m} \frac{\Phi^*_{n,m}(\mathbf{q}', \mathbf{p}')\Phi_{n,m}(\mathbf{q}, \mathbf{p})}{E - E_m + E_n + i\eta}.$$

In a new paper [7] it was proved that Green's function for Wigner operator is the Fourier transform of Wigner propagator, that is

(2.3) $$G(\mathbf{q}', \mathbf{p}';\, \mathbf{q}, \mathbf{p}, E) = \lim_{\eta\to 0} \frac{1}{i\hbar} \int \exp\left(\frac{it}{\hbar}[E + i\eta]\right) f(\mathbf{q}', \mathbf{p}', \mathbf{q}, \mathbf{p}, t)\, dt,$$

where the propagator $f(\mathbf{q}', \mathbf{p}', \mathbf{q}, \mathbf{p}, t)$ of Wigner operator is given in the following part.

Working in the same way we can also determine other corresponding physical quantities of the Schrödinger theory in phase space.

3. Propagator of Wigner Operator. Because of the definition (2.2) of propagator in the phase space, we see that the function $f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t)$ by the help of (1.4) and (1.5) is written as

$$f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t)$$
$$= \sum_{n,m} \left(\frac{1}{2\pi}\right)^6 \int\int \exp\left[i\frac{\mathbf{p}\cdot\boldsymbol{\tau}}{\hbar} - i\frac{\mathbf{p}'\cdot\boldsymbol{\tau}'}{\hbar}\right] U_n^*\left(\mathbf{q}'+\frac{\boldsymbol{\tau}'}{2}\right) U_m\left(\mathbf{q}'-\frac{\boldsymbol{\tau}'}{2}\right) \exp[-it(\omega_n$$
$$-\omega_m)]\, U_n\left(\mathbf{q}+\frac{\boldsymbol{\tau}}{2}\right) U_m^*\left(\mathbf{q}-\frac{\boldsymbol{\tau}}{2}\right) d\boldsymbol{\tau}\, d\boldsymbol{\tau}'.$$

Because of the definition of the Schrödinger propagator [6] the last equation is written as

(3.1) $$f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t)$$
$$= \left(\frac{1}{2\pi}\right)^6 \int\int \exp\left[i\frac{\mathbf{p}\cdot\boldsymbol{\tau}}{\hbar} - i\frac{\mathbf{p}'\cdot\boldsymbol{\tau}'}{\hbar}\right] \Psi\left(\mathbf{q}'+\frac{\boldsymbol{\tau}'}{2};\ \mathbf{q}+\frac{\boldsymbol{\tau}}{2}, t\right) \Psi^*\left(\mathbf{q}'-\frac{\boldsymbol{\tau}'}{2};\ \mathbf{q}-\frac{\boldsymbol{\tau}}{2}, t\right) d\boldsymbol{\tau}\, d\boldsymbol{\tau}',$$

where $\Psi(\mathbf{q}', \mathbf{q}, t)$ is the corresponding Schrödinger propagator. The above propagator of Wigner operator coincides with the propagator of the Wigner distribution functions [8].

The function $f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t)$ fulfils the condition (2.1) for $t \to 0$; it becomes $\Psi(\mathbf{q}'+\boldsymbol{\tau}'/2;\ \mathbf{q}+\boldsymbol{\tau}/2, 0) = \delta(\mathbf{q}-\mathbf{q}'+(\boldsymbol{\tau}-\boldsymbol{\tau}')/2)$ and (3.1) gives

$$f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', 0) = \left(\frac{1}{2\pi}\right)^6 \int\int \exp\left[i\frac{\mathbf{p}\cdot\boldsymbol{\tau}}{\hbar} - i\frac{\mathbf{p}'\cdot\boldsymbol{\tau}'}{\hbar}\right] \delta\left(\mathbf{q}-\mathbf{q}'+\frac{\boldsymbol{\tau}-\boldsymbol{\tau}'}{2}\right) \delta\left(\mathbf{q}-\mathbf{q}'\right.$$
$$\left.-\frac{\boldsymbol{\tau}-\boldsymbol{\tau}'}{2}\right) d\boldsymbol{\tau}\, d\boldsymbol{\tau}' = \delta(\mathbf{q}-\mathbf{q}') \exp\left[-\frac{2i\mathbf{p}'}{\hbar}(\mathbf{q}-\mathbf{q}')\right] \left(\frac{1}{2\pi}\right)^3 \int \exp\left[\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\boldsymbol{\tau}\right] d\boldsymbol{\tau}$$
$$= \delta(\mathbf{q}-\mathbf{q}')\, \delta(\mathbf{p}-\mathbf{p}').$$

A simple application of formula (3.1) for the case of free electrons, where the propagator $\Psi(\mathbf{q}', \mathbf{q}, t)$ is given by the relation $\Psi(\mathbf{q}, \mathbf{q}', t) = (m/2\pi i\hbar t)^{3/2} \times \exp\left[-\frac{im}{2\hbar t}(\mathbf{q}-\mathbf{q}')^2\right]$ gives

$$f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t) \sim \left(\frac{m}{2\pi\hbar t}\right)^3 \left(\frac{1}{2\pi}\right)^6 \int\int \exp\left[\frac{im}{\hbar t}(\mathbf{q}-\mathbf{q}')(\boldsymbol{\tau}-\boldsymbol{\tau}') + \frac{i\mathbf{p}\cdot\boldsymbol{\tau}}{\hbar} - \frac{i\mathbf{p}'\cdot\boldsymbol{\tau}'}{\hbar}\right] d\boldsymbol{\tau}\, d\boldsymbol{\tau}'$$
$$= \delta\left(\mathbf{q}-\mathbf{q}'-\frac{t}{m}\mathbf{p}'\right) \delta(\mathbf{p}-\mathbf{p}').$$

We get the same result with the help of (1.6) and (1.7), namely

$$f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t) \sim \left(\frac{1}{2\pi}\right)^6 \int\int \exp\left[i(\mathbf{k}-\mathbf{k}') + i\frac{t\hbar}{2m}(\mathbf{k}^2-\mathbf{k}'^2)\right] \delta\left(\mathbf{k}+\mathbf{k}'+\frac{2\mathbf{p}}{\hbar}\right) \delta\left(\mathbf{k}+\mathbf{k}'\right.$$
$$\left.+\frac{2\mathbf{p}}{\hbar}\right) d\mathbf{k}\, d\mathbf{k}' = \delta(\mathbf{q}-\mathbf{q}'-t\mathbf{p}'/m)\, \delta(\mathbf{p}-\mathbf{p}').$$

From the above statements we see that the propagator for free electrons in phase space is independent of $\hbar$, which was expected, as the Wigner equation, which is of the form $m^{-1}\mathbf{p}\nabla_q f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t) = \partial f/\partial t$, coincides with the Liouville equation of Classical Mechanics [9].

With the help of the propagator $f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t)$ in phase space we are able to find the general solution of Wigner equation (1.2) with the initial condition

(3.2) $$f(\mathbf{q}, \mathbf{p}, 0) = \Phi(\mathbf{q}, \mathbf{p}),$$

where $\Phi(\mathbf{q}, \mathbf{p})$ is a distribution. The solution will be of the form

(3.3) $$f(\mathbf{q}, \mathbf{p}, t) = \int\int f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t)\, \Phi(\mathbf{q}', \mathbf{p}')\, d\mathbf{q}' d\mathbf{p}'.$$

In this way we have found the general solution (3.3) of the Wigner equation with the initial condition (3.2). The solution (3.3) for free electrons becomes $f(\mathbf{p}, \mathbf{q}, t) = F(\mathbf{q} - t\mathbf{p}/m, \mathbf{p})$.

Now we can easily apply (3.3) also to other cases such as the harmonic oscillator, free electrons in uniform fields, etc. We need only to know the Schrödinger propagator.

4. Calculation of the Wigner eigenfunctions of the harmonic oscillator. The normalized Schrödinger eigenfunctions of the harmonic oscillator are of the form $U_n(g) = (\sqrt{\pi}\, 2^n n! l)^{-1/2} \exp(-q^2/2l^2) H_n(q/l)$, where $l = \sqrt{\hbar/m_0\omega}$ and $H_n(\mathbf{q}/l)$ are the Hermite polynomials.

The eigenfunctions of the Wigner operator for the harmonic oscillator because of (1.4) will be

(4.1) $$\Phi_{n,m}(q, p) = (l\sqrt{\pi}\sqrt{2^{n+m} n!\, m!})^{-1} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\{\frac{ip\tau}{\hbar} - \frac{1}{2l^2}[(q+\frac{\tau}{2})^2 + (q-\frac{\tau}{2})^2]\} H_n(\frac{p+\tau/2}{l}) H_m(\frac{q-\tau/2}{l})\, d\tau.$$

After some algebra and transformations the above integral is written as

(4.2) $$\Phi_{n,m}(q, p) = (-1)^m (\sqrt{\pi}\sqrt{2^{n+m} n! m!})^{-1} \frac{1}{\pi} \exp[-(\frac{q^2}{l^2} + \frac{l^2p^2}{\hbar^2})] \int_{-\infty}^{+\infty} e^{-u} H_n(u + \frac{q}{l} + i\frac{lp}{\hbar}) H_m(u - (\frac{q}{l} - i\frac{lp}{\hbar}))\, du.$$

One can find the integral (4.2) in [10] from which we find

(4.3) $$\Phi_{n,m}(q, p) = \frac{(-1)^m}{\pi} \sqrt{n! 2^{m-n}/m!} \exp[-(\frac{q^2}{l^2} + \frac{l^2p^2}{\hbar^2})] (-\frac{q}{l} + i\frac{lp}{\hbar})^{m-n} L_n^{m-n}[2(\frac{q^2}{l^2} + \frac{l^2p^2}{\hbar^2})]$$

and with the relations

$$\frac{q^2}{l^2} + \frac{l^2p^2}{\hbar^2} = \frac{H(q, p)}{\hbar\omega/2} = [\frac{p^2}{2m_0} + \frac{m_0\omega^2 q^2}{2}] / \frac{\hbar\omega}{2}$$

the eigenfunctions (4.3) take finally the form

(4.4) $$\Phi_{n,m}(q, p) = \frac{2(-1)^m}{\pi\hbar^{1/2}} \sqrt{n!/m!} (i(\sqrt{2/\hbar m_0\omega})^{m-n} (p + im_0\omega q)^{m-n} \exp[-\frac{2H(q, p)}{\hbar\omega}] L_n^{m-n}(\frac{4H(q, p)}{\hbar\omega}),$$

where $L_n^{m-n}(x)$ are the generalized Laguere polynomials [10]. The eigenvalues which correspond to the eigenfunctions (4.4) are of the form $w_{nm} = \hbar\omega(m-n)$.

Of interest is the case $n = m$ for which we have $w = 0$ and

$$\Phi_{n,n}(q, p) = \frac{2(-1)^n}{\pi\hbar^{1/2}} \exp[-\frac{H(q, p)}{\hbar\omega/2}] L_n(2\frac{H(p, q)}{\hbar\omega/2}).$$

Finally for $n = 0$ we obtain the function

$$\Phi_{0,0}(q, p) = \frac{2}{\pi\hbar^{1/2}} \exp\left[-\frac{H(q,p)}{\hbar\omega/2}\right],$$

which if we ignore the normalization factor coincides with the distribution of Boltzmann of the harmonic oscillator with the difference that instead of the thermal energy KT we have the ground state energy of the harmonic oscillator $\hbar\omega/2$.

5. Calculation of the Wigner eigenfunctions of the free electrons in mutually perpendicular uniform magnetic and electric fields. The eigenfunctions and eigenvalues of the Schrödinger equation with the vector potential $(0, Hx, 0)$ and electric potential energy eFx can be easily calculated and are of the form

$$(5.1)\quad \Psi_{\mathbf{k},n}(x-x_0, y, z) = (\sqrt{\pi}\, 2^n n! l)^{-1/2} \exp[ik_y y + ik_z z] \exp\left[-\frac{1}{2}\frac{(x-x_0)^2}{l^2}\right] H_n\left(\frac{x-x_0}{l}\right),$$

$$E = \hbar\omega\left(n+\frac{1}{2}\right) + \frac{\hbar^2}{2m}k_z^2 + \frac{\hbar Fc}{H}k_y - \frac{mc^2F^2}{H^2},$$

where $x_0 = \hbar c(k_y - mcF/\hbar H)/eH$, $\omega = eH/mc$, $l = \sqrt{\hbar/m\omega}$. $H_n((x-x_0)/2)$ are the Hermite polynomials and n is the Landau quantum number.

The eigenfunctions of the Wigner operator for (5.1) because of (1.4) and (1.6) for the coordinates y and z will be

$$\Phi_{\mathbf{k},\mathbf{k},n,m}(q, p) = (l\sqrt{\pi}\sqrt{2^{n+m}\, n!\, m!})^{-1} \exp[i(k_y - k'_y)q + i(k_z - k'_z)q] / \delta(k_y + k'_y$$
$$+ 2p_2/\hbar)\, \delta(k_z + k'_z + 2p_3/\hbar) \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2l^2}\left[\left(q_1 + \frac{\tau_1}{2} - x_0\right)^2 + \left(q_1 - \frac{\tau_1}{2} - x'_0\right)^2\right]\right\}$$
$$\times H_n\left(\frac{1}{l}\left(q_1 + \frac{\tau_1}{2} - x_0\right)\right) H_m\left(\frac{1}{l}\left(q - \frac{\tau_1}{2} - x'_0\right)\right) d\tau_1.$$

In the above relation the integral is proportional to the integral in (4.1). After some algebra and integrations we find

$$(5.2)\quad \Phi_{\mathbf{k},\mathbf{k}',n,m}(q, p) = \frac{2(-1)^m}{\pi\hbar^{1/2}} \sqrt{n!\, 2^{m-n}/m!} \exp[i(k_y - k'_y)q_2 + i(k_z - k'_z)q^3]\delta(k_y + k'_y$$
$$+ \frac{2p_2}{\hbar})\, \delta(k_z + k'_z + \frac{2p_3}{\hbar})\left[-\frac{q_1}{l} + \frac{x_0 + x'_0}{2l} + \frac{ilp_1}{\hbar}\right]^{m-n} \exp\left[i\frac{p_1}{\hbar}(x - x_0) - \left\{\left[\frac{q_1}{l} - \frac{x_0 + x'_0}{2l}\right]^2\right.\right.$$
$$\left.\left. + \frac{l^2 p_1^2}{\hbar^2}\right\}\right] L_n^{m-n}\left[2\left\{\frac{q_1}{l} - \frac{x_0 + x'_0}{2l}\right\}^2 + \frac{l^2p^2}{\hbar^2}\right].$$

The corresponding eigenvalues in the above functions are the following:

$$(5.3)\quad W = \hbar\omega(m-n) + \frac{\hbar Fc}{H}(k_y - k'_y) + \frac{\hbar^2}{2m}(k_z'^2 - k_z^2).$$

For the case in which there is no electric field, namely $F=0$, and we have only magnetic field, the parameter $x_0 = \hbar c k_y/eH$ and the eigenfunctions are given from (5.2). For the eigenvalues (5.3) the second term in the right-hand side vanishes.

Of interest is the case $\mathbf{k} = \mathbf{k}'$, $n = m$ when the eigenvalues give $w = 0$ and the eigenfunctions (5.2) take the form

$$\Phi_{\mathbf{k},\,\mathbf{k}',\,m,\,n}(q,\,p)=\frac{2(-1)^n}{4\pi\hbar^{1/2}}\,\delta(k_y+\frac{p_2}{\hbar})\,\delta(k_z+\frac{p_3}{\hbar})\ \exp\{-[(\frac{q_1-x_0}{l})^2$$

$$+\frac{l^2p_1^2}{\hbar^2}]\}\ L_n[2\{(\frac{q_1-x_0}{l})^2+\frac{l^2p_1^2}{\hbar^2}\}].$$

In the same way we may study other cases, too; we only need to know the eigenfunctions and eigenvalues of the corresponding Schrödinger equation.

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