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PHASE SPACE AND THE WIGNER EQUATION

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This paper is concerned with the Wigner equation, through the theory of the Quantum-Mechanical foundation of F. Bopp in the phase space. The Wigner equation is similar to the Quantum Statistical equation of von Neumann and its solutions can be expressed in a similar way. For the case of the stationary distribution the solution of the Wigner equation is given in form of a power series in terms of $(\hbar/2)^2$.

1. Introduction. Apart of the known representations p and q of Quantum-Mechanics in the configuration space, F. Bopp [1] gave a new quantum-mechanical description in the phase space, i. e. the space of $6N$ coordinates. That is there exists a function $f(q, p, t)$ — *average density of distributions* from which one can form the averages of all the observable quantities. This distribution is different from the probability density of Classical Mechanics and even more it accepts negative values. Bopp stated 16 requests, which must be fulfilied by the average density of distribution $f(p, q, t)$. The results of his whole theory [2] are separated in two basic parts.

First, for the function $f(p, q, t)$ the following differential equation holds

$$i\hbar\,\partial f/\partial t=\{H(P, Q)-H(P^*, Q^*)\}f, \tag{1.1}$$

where $P=p-(i\hbar/2)\partial/\partial q$, $Q=q+(i\hbar/2)\partial/\partial p$, P^*, Q^* are the complex conjugates of P, Q, and $H(P, Q)$ is the Hamiltonian operator.

Second, a series of rules and formulas is used which in their greater part are due to Wigner [3] and the results obtained correspond to the usual results of Quantum-Mechanics (Ψ-functions). Equation (1.1) is called Wigner equation and for $\hbar\rightarrow 0$ coincides with the Liouville's equation in classica Statistical Mechanics, which is

$$\partial f/\partial t=\{H, f\}. \tag{1.2}$$

Also, when the operator of Hamilton consists of quadratic expressions of q, p, the Wigner equation coincides with the Liouville's equation.

Here we study general properties of the Wigner equation (1.1), bacause it is similar to the von Neumann equation and its solution can be expressed in a similar way. We also find approximately the everage stationary distribution, i. e. the time independent one, expressed in power series of $(\hbar/2)^2$.

2. Remarks on the Wigner equation. We note first that the new Hamilton's operator $H(P, Q)$ through a transformation [4] can be reduced to an equivalent Schrödinger operator in the p or q representation and then the eigenvalues of the operator $H(P, Q)$ coincide with those of Schrödinger.

Through the known solution of the motion equation of von Neumann $\partial f/\partial t=(1/i\hbar)[H, f]$, which is of the form $f=\exp[iHt/\hbar]\,f_0\exp[-iHt/\hbar]$ we obtain a similar solution of the Wigner equation (1.1), i. e.

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$$f=\exp\left\{\frac{1}{i\hbar}[H(P,Q)-H(P,Q)]t\right\}f_0, \tag{2.1}$$

where $f_0(p,q)\equiv f(p,q,0)$ is an initial distribution.

The solution (2.1), for $\hbar\to 0$, gives the known solution of the Liouville's equation [5], i. e.

$$f(p,q,t)=\exp\left[-t\left(\frac{\partial H}{\partial p(0)}\frac{\partial}{\partial q(0)}-\frac{\partial H}{\partial q(0)}\frac{\partial}{\partial p(0)}\right)\right]f(p(0),q(0)).$$

3. Connection between the equations of Wigner and Schrödinger. The Wigner equation (1.1) has the structure of a Schrödinger equation, but instead of having Hamilton's operator in it one has the Wigner's operator

$$H(P,Q)-H(P^*,Q^*)\equiv W(q,p,\partial/\partial q,\partial/\partial p) \tag{3.1}$$

and the number of independent variables has been doubled [6].

For the Wigner propagator we take the initial condition $f(q,p;q',p',0)=\delta(q-q')\delta(p-p')$, and then the Wigner propagator can be expressed in the following form:

$$f(p,q;q',p',t)=\sum_{i,j}\varphi^*_{i,j}(q',p')\exp[-i(\omega_i-\omega_j)t]\varphi_{i,j}(q,p),$$

where $\omega_j=E_j/\hbar$ are the eigenvalues of the operator (3.1) and $\varphi_{i,j}(q,p)$ its eigenfunctions, coinciding to the phase space eigenfunctions [7]. Because of the above correspondence we can apply the well-known procedures for the Schrödinger equation [8] and we can determine the Wigner propagator in several known expressions of Hamilton's operator. Further more, we are also able to transform the Wigner equation (1.1) to an integral one, which for the operator $H=H_0+H_1$ is

$$f(q,p;q',p',t)=f_0(q,p;q',p',t)-\int_0^t dt'\int f_0(q,p;q'',p'',t-t')[H(P,Q)-H(P^*,Q^*)]f(q'',p'';q',p',t')dq''dp'', \tag{3.2}$$

where $f_0(q,p;q',p',t)$ represents the Wigner propagator of the operator $W_0=H_0(P,Q)-H_0(P^*,Q^*)$. The last term of (3.2) is small if H_1 is small and it is a correction term to the approximate equation $f\approx f_0$. If the correction term is small we can use an approximate f in order to have a better approximation for f. Proceeding in that way, we get finally the following result for the Wigner propagator:

$$f(q,p;q',p',t)=f_0(q,p;q',p',t)-\int_0^t dt'\int dq''dp''f_0(q,p;q'',p'',t-t')[H_1(P'',Q'')-H_1(P^{*''},Q^{*''})]f_0(q'',p'';q',p',t')+\cdots$$

The above series may be used in several cases, when the operator H_1 is considered as perturbation.

4. Approximate solution to the Wigner equation in form of power series of $(\hbar/2)^2$. The sufficient and necessary condition for a system to be in Quantum-Statistical equilibrium is $[H,f]=0$, or, equivalently,

$$\partial f/\partial t=0. \tag{4.1}$$

The last equation means that the mean values of the quantities, which determine the ensemble are necessarily time independent.

If we apply condition (4.1) to the Wigner equation, we may say that the average density distribution $f(q, p)$ is time-independent and that the ensemble is in equilibrium or we may say that we will have a stationary distribution, which for $\hbar \to 0$ tends to the classical one.

The Wigner's equation (1.1) for the stationary distribution $\partial f/\partial t = 0$ is written as

$$\left[H\left(p - \frac{i\hbar}{2}\frac{\partial}{\partial q},\; q + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) - H\left(p + \frac{i\hbar}{2}\frac{\partial}{\partial q},\; q - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)\right] f(p, q) = 0. \tag{4.2}$$

The above equation by expansion of the operators in terms of $(\hbar/2)^2$, when

$$H(P, Q) = \sum_{J=1}^{k} \frac{P_J^2}{2m} + V(Q_1, Q_2, \ldots, Q_k),$$

is written as

$$\left\{-\frac{1}{m}\,\mathrm{p}\nabla_q + \frac{1}{2}\sum_{n_1, n_2, \ldots, n_k}^{\infty} (i\hbar/2)^{n_1+n_2+\ldots+n_k-1}\right.$$

$$\left.\times \frac{[1-(-1)^{n_1+n_2+\ldots+n_k}]}{n_1!\, n_2! \cdots n_k!} \frac{\partial^{n_1+n_2+\ldots+n_k} V}{\partial q_1^{n_1} \partial q_2^{n_2} \ldots \partial q_k^{n_k}} \cdot \frac{\partial^{n_1+n_2+\ldots+n_k}}{\partial p_1^{n_1} \partial p_2^{n_2} \ldots \partial p_k^{n_k}}\right\}.$$

For simplicity we consider here the case $k = 1$ and the last expression takes the form

$$\left\{-\frac{1}{m} p \frac{\partial}{\partial q} + \frac{V}{\partial q}\frac{\partial}{\partial p} - \left(\frac{\hbar}{2}\right)\frac{1}{3!}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3}{\partial p^3} + \cdots + (-1)^n \left(\frac{\hbar}{2}\right)^n \frac{1}{(2n+1)!}\frac{\partial^{2n+1}}{\partial q^{2n+1}}\frac{\partial^{2n+1}}{\partial p^{2n+1}}\right\} f = 0.$$

If for the solution of the above equation we set $f(q, p) = \Sigma_{n=0}^{\infty} (\hbar/2)^{2n} f_{2n}(q, p)$, then for the calculation of the new functions $f_{2n}(q, p)$ we are lead to the following partial differential system of first order:

$$-\frac{p}{m}\frac{\partial f_0}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial f_0}{\partial p} = 0$$

$$-\frac{p}{m}\frac{\partial f_2}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial f_2}{\partial p} - \frac{1}{3!}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 f_0}{\partial p^3} = 0$$

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$$-\frac{p}{m}\frac{\partial f_{2n}}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial f_{2n}}{\partial p} + (-1)^n \frac{1}{(2n+1)!}\frac{\partial^{2n+1} V}{\partial q^{2n+1}}\frac{\partial^{2n+1} f_0}{\partial p^{2n+1}} + \cdots + \frac{1}{3!}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 f_{2(n-1)}}{\partial p^3} = 0$$

. . . .

The first equation of the above system coincides with the Liouville equation (1.2) and accepts a solution $f_0(q, p) = F_0(H)$, $H = (2m)^{-1}p^2 + V(q) = \text{const.}$ We can easily find the solution of the second equation, namely

$$f_2(q, p) = -F_0(H)\frac{1}{3!}\int \frac{\partial p}{\partial H/\partial q}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 F_0(H)}{\partial p^3}.$$

Working similarly, we find that the function $f_{2n}(q, p)$ is given by the relation

$$f_{2n}(q, p) = -F_0(H)\int \frac{\partial p}{\partial H/\partial q}\frac{1}{(2n+1)!}\left[\frac{\partial^{2n+1}V}{\partial q^{2n+1}}\frac{\partial^{2n+1}F(H)}{\partial p^{2n+1}} + \cdots + \frac{1}{3!}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 f_{2(n-1)}}{\partial p^3}\right]$$

Using the same procedure, we find a solution of (4.2) in the following form

$$f(q, p) = F_0(H)\left\{1 - \sum_{n=1}^{\infty}\left(\frac{\hbar}{2}\right)^{2n}\int \frac{\partial p}{\partial H/\partial q}\frac{1}{(2n+1)!}\left[\frac{\partial^{2n+1}V}{\partial p^{2n+1}}\frac{\partial^{2n+1}F_0(H)}{\partial p^{2n+1}} - \cdots + \frac{1}{3!}\frac{\partial^3 V \partial^3 f_{2(n-1)}}{\partial q^3 \quad \partial p^3}\right]\right.,$$

which can be written in the following simpler form

$$f(q, p) = F_0(H)\{1 - \left(\frac{\hbar}{2}\right)^2 \frac{1}{3!}\int \frac{\partial p}{\partial H/\partial q}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 F_0(H)}{\partial p^3} + o\left(\left(\frac{\hbar}{2}\right)^4\right) + \cdots$$

The above results verify that in fact a stationary solution of the Wigner equation has been found, expressed in form of power series in terms of $(\hbar/2)^2$. One can extend these results to the phase space of $6N$-dimensions.

5. Wigner distribution for square forms. From [8, part 3] we have that when one knows the propagator of the Wigner operator, the Wigner distribution is given as $f(q, p, t) = \int\int f(\mathbf{q}, \mathbf{p}', \mathbf{q}', \mathbf{p}', t)\Phi(\mathbf{q}', \mathbf{p}')dq'dp'$, where $\Phi(q', p')$ is a distribution [6].

The propagator $f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t)$ is expressed as a Fourier integral, namely

$$f(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}', t) = \int\int \exp\left[\frac{i\,\mathbf{p}\cdot\boldsymbol{\tau}}{\hbar} - \frac{i\,\mathbf{p}'\cdot\boldsymbol{\tau}'}{\hbar}\right]\Psi^*\left(\mathbf{q}+\frac{\boldsymbol{\tau}}{2}, \mathbf{q}'+\frac{\boldsymbol{\tau}'}{2}, t\right)\Psi\left(\mathbf{q}'-\frac{\boldsymbol{\tau}'}{2}, \mathbf{q}-\frac{\boldsymbol{\tau}}{2}, t\right)d\boldsymbol{\tau}'d\boldsymbol{\tau}$$

where $\Psi(\mathbf{q}, \mathbf{p}, t)$ is the corresponding Schrödinger's propagator of the Hamilton's operator [9].

From the above one can easily study the case of quadratic forms, for which the Wigner equation (1.1) coincides with the Liouville's equation. For all quadratic forms the propagators of Wigner are expressed through δ-functions and are independent of $\hbar$.

In the case of free particles the Wigner distribution is of the form

$$f(\mathbf{q}, \mathbf{p}, t) = \Phi(\mathbf{q} - \mathbf{p}\,t/m, \mathbf{p}). \tag{5.1}$$

Also in the case where a uniform electric field is presented, the Wigner distribution is of the form

$$f(\mathbf{q}, \mathbf{p}, t) = (\mathbf{q} - \frac{\mathbf{p}}{m}t + \frac{e\,\mathbf{F}}{2m}t^2, \mathbf{p} - e\,\mathbf{F}t). \tag{5.2}$$

Finally, in the case of a spherical harmonic oscillator the Wigner distribution is given by the relation

$$f(\mathbf{q}, \mathbf{p}, t) = \Phi(\mathbf{q}\cos\omega t - \frac{\mathbf{p}}{m\omega}\sin\omega t, m\omega\,\mathbf{q}\sin\omega t + \mathbf{p}\cos\omega t). \tag{5.3}$$

The Wigner equation for quadratic forms coincides with the Liouville's equation. Therefore, the following result stands [10].

For every quadratic form of the Hamilton operator, the Wigner distribution results at once from the function $\Phi(\mathbf{q}_0 . \mathbf{p})$, if $\mathbf{q}_0$ and $\mathbf{p}_0$ are replaced by the initial values of the canonical variables, i. e. the solutions of the canonical Hamilton's equations.

In the case of free particles we have the solutions $\mathbf{q}=\mathbf{q}_0+t\mathbf{p}/m$, $\mathbf{p}=\mathbf{p}_0$ and the distribution coincides with (5.1).

The same happens in the case when a uniform electric field is present. Then the classical equations of motion are of the form $d\mathbf{q}/dt=\mathbf{p}/m$ $d\mathbf{p}/dt=e\mathbf{F}$, which by integration become

$$\mathbf{p}=e\mathbf{F}t+\mathbf{p}_0,\ \mathbf{q}=\mathbf{q}_0+\frac{e\mathbf{F}}{2m}t^2+\frac{\mathbf{p}_0 t}{m}=\mathbf{q}_0+\frac{\mathbf{p}}{m}t-\frac{e\mathbf{F}}{2m}t^2,$$

or

$$\mathbf{p}_0=\mathbf{p}-e\mathbf{F}t,\ \mathbf{q}_0=\mathbf{q}-\frac{\mathbf{p}}{m}t+\frac{e\mathbf{F}}{2m}t^2.$$

Consequenty, the distribution $\Phi(\mathbf{q}_0, \mathbf{p}_0)$ coincides with (5.2).

Also in the case of the simple harmonic oscilator, the solutions of the equations of Hamilton are of the form

$$\mathbf{q}=\mathbf{q}_0\cos\omega t+\frac{\mathbf{p}_0}{m\omega}\sin\omega t,\ \mathbf{p}=-m\omega\,\mathbf{q}_0\sin\omega t+\mathbf{p}_0\cos\omega t,$$

form which the values of $(\mathbf{q}_0, \mathbf{p}_0)$ result easily, that is

$$\mathbf{q}_0=\mathbf{q}\cos\omega t-\frac{\mathbf{p}}{m\omega}\sin\omega t,\ \mathbf{p}_0=\mathbf{p}\cos\omega t+m\omega\,\mathbf{q}\sin\omega t.$$

So, the distribution, in the case of the simple harmonic oscillator, coincdes with (5.3).

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