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PERMUTATIONALLY CONVERGENT MATRICES

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Some necessary and sufficient conditions are established for the following relations $\lim_{i\to\infty}\lim_{j\to\infty}x_{ij}=\lim_{j\to\infty}\lim_{i\to\infty}x_{ij}=\lim_{i,j\to\infty}x_{ij}$ to hold.

1. By a permutationally convergent matrix we mean any matrix $\{x_{ij}\}$ $(i,j\in N)$ such that all limits in the equalities

$$\lim_{i\to\infty}\lim_{j\to\infty}x_{ij}=\lim_{j\to\infty}\lim_{i\to\infty}x_{ij}=\lim_{i,j\to\infty}x_{ij} \tag{1}$$

exist and the equalities hold.

We establish some necessary and sufficient conditions for the permutational convergence of matrices. It turns out that a matrix is permutationally convergent if for each subsequence of columns there exists a subsequence such that sums of rows form a Cauchy sequence. The sufficient condition given here derives the Banach theorem on equicontinuity of sequences of mappings. It is noteworthy that in the Banach theorem the assumption on the completeness of a normed space can be replaced by a weaker assumption that requires any sequence tending to zero to have a subsequence that is summable. Other sufficient conditions are concerned with the permutation of summation and limits and with the equality of iterated sums (see Theorems 5 and 6).

At first our results are established for number matrices and next they are extended to matrices of elements from a quasi-normed group and an Abelian group endowed with a family of quasi-norms.

2. Throughout this and next sections we deal with square infinite matrices $\{x_{ij}\}$, $i,j\in N$, whose elements are numbers.

We say that a matrix $\{x_{ij}\}$ is separately convergent iff its rows and columns are convergent, i. e.

$$\lim_{j\to\infty}x_{ij}=x_i,\ i\in N,\ \lim_{i\to\infty}x_{ij}=y_j,\ j\in N.$$

We recall the Diagonal Theorem (for non-negative matrices) proved in [1].

Diagonal Theorem (for non-negative matrices). *If $\{x_{ij}\}$, $i,j\in N$, is a separately convergent to zero matrix and its diagonal tends to zero, i. e.*

$$\lim_{j\to\infty}x_{ij}=0,\ i\in N,\ \lim_{i\to\infty}x_{ij}=0,\ j\in N,\ \textit{and}\ \lim_{i\to\infty}x_{ii}=0,$$

then there exists an infinite set I, $I\subset N$, such that

$$\sum_{i,j\in I}x_{ij}<\infty.$$

Hence, the elements of I can be arranged into an increasing sequence $\{p_i\}$ such that

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} x_{p_i p_j} = 0 \text{ and } \lim_{j\to\infty} \sum_{i=1}^{\infty} x_{p_i p_j} = 0.$$

By using the Diagonal Theorem we prove the following

Theorem 1. *If $\{z_{ij}\}$, $i, j \in N$, is a separately convergent to zero matrix, i.e.*

$$\text{(A)} \qquad \lim_{j\to\infty} z_{ij} = 0,\ i \in N, \text{ and } \lim_{i\to\infty} z_{ij} = 0,\ j \in N,$$

then the following conditions are equivalent:

(A_1) *for each infinite increasing sequence $\{m_i\}$ there exists a subsequence $\{n_i\}$ such that* $\lim\{\Sigma\{z_{n_i n_j} \mid 1 \leq j \leq \infty\} \mid i \to \infty\} = 0$;

$$(A_2) \qquad \lim_{i\to\infty} z_{ii} = 0;$$

(A_3) *there exists an infinite set I, $I \subset N$, such that* $\Sigma\{|z_{ij}| \mid i, j \in I\} < \infty$.

Proof. Let $x_{ij} = |z_{ij}|$ for $i \neq j$ and $x_{ii} = 0$. Then $\lim_{i\to\infty} x_{ii} = 0$, $\lim_{j\to\infty} x_{ij} = 0$ and $\lim_{i\to\infty} x_{ij} = 0$. Thus, by the Diagonal Theorem there exists a sequence $\{p_i\}$ of positive integers such that

$$(1) \qquad \lim_{i\to\infty} \sum_{j=1}^{\infty} x_{p_i p_j} = 0.$$

If (A_1) holds, then there exists a subsequence $\{q_i\}$ of $\{p_i\}$ such that

$$(2) \qquad \lim_{i\to\infty} \sum_{j=1}^{\infty} z_{q_i q_j} = 0.$$

Since $|a| \leq |a+b| + |b|$ and $|\Sigma\{b_j \mid 1 \leq j \leq \infty\}| \leq \Sigma\{|b_j| \mid 1 \leq j \leq \infty\}$, we can write

$$\left|\sum_{j=1}^{\infty} z_{q_i q_j}\right| + \sum_{j=1}^{\infty} |z_{q_i q_j}| \geq 2\,|z_{q_i q_i}|.$$

Hence, we have

$$\left|\sum_{j=1}^{\infty} z_{q_i q_j}\right| + \sum_{j=1}^{\infty} x_{q_i q_j} \geq |z_{q_i q_i}|.$$

From (1) and (2) it follows that $\lim_{i\to\infty} z_{q_i q_i} = 0$. Hence we get that from each subsequence of $\{z_{ii}\}$ we can select a subsequence tending to zero. As the convergence of number sequences is an Uryshon convergence, that is if each subsequence of a sequence possesses a subsequence tending to x then the sequence is tending to x, we have $\lim_{i\to\infty} z_{ii} = 0$. This shows that ($A_1$) implies ($A_2$). From the Diagonal Theorem it follows that if (A_2) holds, then also (A_3) holds. Evidently (A_3) implies (A_1). Thus the proof is complete.

Any separately convergent to zero matrix satisfying one of the conditions from (A_1) to (A_3) is called a vanishing matrix.

Theorem 2. *Let $\{x_{ij}\}$, $i, j \in N$, be a matrix such that*

(B) $\lim_{j\to\infty} x_{ij} = x_i$, $i \in N$, $\lim_{i\to\infty} x_i = x$ *and columns are Cauchy sequences.*

Then the following conditions are equivalent:

(B$_1$) *for any sequences* $\{m_i\}$ *and* $\{n_i\}$ *of positive integers the matrix* $\{x_{m_{i+1}n_{j+1}}-x_{m_{i+1}}+x_{m_i}-x_{m_i n_{j+1}}\}$, $i,j\in N$, *is vanishing;*

(B$_2$) *all limits in the equality* $\lim_{i\to\infty} x_i=\lim_{i,j\to\infty} x_{ij}$ *exist and the equality holds;*

(B$_3$) *for any increasing sequences* $\{m_i\}$ *and* $\{n_i\}$ *of positive integers the matrix* $\{x_{m_{i+1}n_{j+1}}-x_{m_{i+1}n_{j+2}}+x_{m_i n_{j+2}}-x_{m_i n_{j+1}}\}$, $i,j\in N$, *is vanishing;*

(B$_4$) $\lim_{i,j\to\infty} x_{ij}$ *exists;*

(B$_5$) $\lim_{j\to\infty} x_{ij}=x_i$ *uniformly in* i *on* N.

P r o o f. At first we note that the rows are convergent sequences. Assume that (B$_1$) holds and let $\{m_i\}$ and $\{n_i\}$ be increasing sequences of positive integers. Since $\lim_{j\to\infty} x_{ij}=x_i$ and $\lim_{i\to\infty} x_i=\lim_{i\to\infty}\lim_{j\to\infty} x_{ij}=x$, there exists an increasing sequence $\{p_i\}$ of positive integers such that

(3) $$\lim_{i\to\infty} x_{m_{p_i}n_{p_{i+1}}}=x.$$

Let $z_{ij}=x_{m_{p_{i+1}}n_{p_{j+1}}}-x_{m_{p_{i+1}}}+x_{m_{p_i}}-x_{m_{p_i}n_{p_{j+1}}}$ for i,j from N. Then $\{z_{ij}\}$ is a vanishing matrix and consequently $\lim_{i\to\infty} z_{ii}=0$. Hence, by (3) we see that $\lim_{i\to\infty} x_{m_{p_{i+1}}n_{p_{i+1}}}=x$. This implies that $\lim_{i,j\to\infty} x_{ij}=x$. Therefore (B$_2$) holds.

Next assume that (B$_2$) holds and let $\{m_i\}$ and $\{n_i\}$ be any increasing sequences of positive integers and let

$$z_{ij}=x_{m_{i+1}n_{j+1}}-x_{m_{i+1}n_{j+2}}+x_{m_i n_{j+2}}-x_{m_i n_{j+1}},\quad i,j\in N.$$

For columns that are Cauchy sequences (see (B)) we have $x_{m_{i+1}n_{j+1}}-x_{m_i n_{j+1}}\to 0$ as $i\to\infty$, $j\in N$, and $x_{m_{i+1}n_{j+2}}-x_{m_i n_{j+2}}\to 0$ as $i\to\infty$, $j\in N$. Consequently, we have $\lim_{i\to\infty} z_{ij}=0$, $j\in N$. This shows that the matrix $\{z_{ij}\}$ is separately convergent to zero. Besides we have $\lim_{i\to\infty} z_{ii}=0$. This proves that the matrix $\{z_{ij}\}$ is vanishing. This means that (B$_2$) implies (B$_3$).

That (B$_3$) implies (B$_4$) is evident from the fact that for any increasing sequences $\{m_i\}$ and $\{n_i\}$ of positive integers there exists an increasing sequence $\{p_i\}$ of positive integers such that

$$\lim_{i\to\infty} x_{m_{p_i}n_{p_{i+1}}}=x \text{ and } \lim_{i\to\infty} x_{m_{p_i}n_{p_{i+2}}}=x.$$

It is clear that (B$_4$) implies (B$_5$).

Assume (B$_5$) and let $\{m_i\}$ and $\{n_i\}$ be increasing sequences of positive integers and let

$$z_{ij}=x_{m_i n_{j+1}}-x_{m_{i+1}}+x_{m_i}-x_{m_i n_{j+1}},\quad i,j\in N.$$

From (B) it follows that $\{z_{ij}\}$ is a separately convergent to zero matrix. Since $\lim_{i\to\infty} x_i$ exists and $\lim_{j\to\infty} x_{ij}=x_i$ uniformly in i on N, we have $\lim_{i\to\infty}(x_{m_i}-x_{m_{i+1}})=0$ and $\lim_{i\to\infty} x_{m_{i+1}n_{i+1}}=\lim_{i\to\infty} x_{m_i n_{i+1}}=\lim_{i\to\infty} x_i$. Hence, $\lim z_{ii}=0$ and consequently $\{z_{ij}\}$ is a vanishing matrix. This proves that (B$_5$) implies (B$_1$) and completes the proof.

Any matrix $\{x_{ij}\}$, $i, j \in N$, satisfying (B) and one of the conditions from (B_1) to (B_5) is called Δ_i-vanishing. The matrix is said to be Δ_j-vanishing iff it is Δ_i-vanishing after changing columns with rows.

Theorem 3. *Let $\{x_{ij}\}$, $i, j \in N$, be a separately convergent matrix such that*

$$\text{(C)} \qquad \lim_{j\to\infty} x_{ij} = x_i,\ i \in N, \text{ and } \lim_{i\to\infty} x_{ij} = y_j,\ j \in N.$$

Then the following conditions are equivalent:

(C_1) $\{x_{ij}\}$ *is Δ_i-vanishing matrix;*

(C_2) *all limits in the equalities* $\lim_{i\to\infty} x_i = \lim_{j\to\infty} y_j = \lim_{i,j\to\infty} x_{ij}$ *exist and the equalities hold;*

(C_3) $\{x_{ij}\}$ *is Δ_j-vanishing matrix;*

(C_4) $\lim^{i}{}_{,j\to\infty} x_{ij}$ *exists;*

(C_5) $\lim_{j\to\infty} x_{ij} = x_i$ *uniformly in i on N,*
$\lim_{i\to\infty} x_{ij} = y_j$ *uniformly in j on N.*

Proof. Assume (C_1) and let $\{n_j\}$ be an increasing sequence of positive integers such that $\lim_{j\to\infty}(x_{n_j j} - y_j) = 0$. Hence, from ($B_2$) it follows that ($C_2$) holds.

That (C_2) implies (C_3) is evident from the fact that (B_2) implies (B_3) after changing the columns with the rows in the matrix $\{x_{ij}\}$. For the similar reasons (C_3) implies (C_4). Evidently (C_4) implies (C_5).

From (C) and (C_5) it follows that the matrix $\{x_{ij}\}$ satisfies the conditions (B) and (B_5) which means that $\{x_{ij}\}$ is Δ_i-vanishing. Therefore (C_5) implies (C_1). This completes the proof.

Any matrix $\{x_{ij}\}$, $i, j \in N$, which satisfies (C) and one of the conditions from (C_1) to (C_5), is said to be a permutationally convergent matrix.

3. In this section some sufficient conditions for the permutational convergence of matrices are proved and some applications of the results of the paper are discussed.

Theorem 4. *Let $\{x_{ij}\}$, $i, j \in N$, be a separately convergent matrix such that $\lim_{j\to\infty} x_{ij} = x_i$, $i \in N$, and $\lim_{i\to\infty} x_i = x$. If for each increasing sequence $\{n_i\}$ of positive integers, there exists a subsequence $\{p_i\}$ such that*

$$\Big\{\sum_{j=1}^{\infty}(x_{ip_j} - x_i)\Big\},\quad i = 1, 2, \ldots,$$

is a Cauchy sequence, then all limits in the equalities

$$\lim_{j\to\infty}\lim_{i\to\infty} x_{ij} = \lim_{i\to\infty}\lim_{j\to\infty} x_{ij} = \lim_{i,j\to\infty} x_{ij}$$

exist and the equalities hold, i.e. the matrix $\{x_{ij}\}$ is permutationally convergent.

Proof. Let $\{m_i\}$ and $\{n_i\}$ be increasing sequences of positive integers and let

$$z_{ij}=x_{m_{i+1}n_{j+1}}-x_{m_{i+1}}+x_{m_i}-x_{m_i n_{j+1}},\ i,j\in N.$$

Let $\{p_j\}$ be an increasing sequence of positive integers. By the hypothesis there exists a subsequence $\{r_j\}$ of $\{p_j\}$ such that $\{\Sigma_{j=1}^{\infty}(x_{iq_j}-x_i)\}$, $i=1,2,\ldots,$ is a Cauchy sequence with $q_j=m_{r_j+1}$. This implies that

$$\lim_{i\to\infty}\sum_{j=1}^{\infty}z_{r_i r_j}=0$$

and consequently $\{z_{ij}\}$ is a vanishing matrix. For $\lim_{i\to\infty}x_i=x$, $\{x_{ij}\}$ is Δ_i-vanishing. Thus $\{x_{ij}\}$ is permutationally convergent and by (C_2) we obtained our assertion.

Let X be a metrizable vector space satisfying the following condition:

(K) for any sequence $\{x_n\}$ tending to zero there exists a subsequence $\{x_{m_n}\}$ and an element x such that $x=\Sigma\{x_{m_n}\mid 1\leq n\leq\infty\}$.

As a simple consequence of the Theorem 4 we have the following: *If $\{f_n\}$ is a sequence of additive and continuous mappings on X which possesses the property (K) and $f(x)=\lim_{n\to\infty}f_n(x)$, $x\in X$, then f is additive and continuous.*

This statement is equivalent to the Banach theorem on equicontinuity when X is a complete metric space.

If m and n are positive integers and $m<n$, then the set of all integers p such that $m\leq p\leq n$ is called a segment of positive integers. A sequence $\{I_n\}$ of segments of positive integers is said to be increasing iff each element of I_n is less than any element of I_{n+1}.

Theorem 5. *Let $\{x_{ij}\}$, $i,j\in N$, be a separately convergent matrix such that the limit $\lim\{\Sigma\{x_{ij}\mid 1\leq j\leq\infty\}\mid i\to\infty\}$ exists. If for any increasing sequence of segments $\{I_j\}$ of integers there exists a subsequence $\{I_{m_j}\}$ of $\{I_j\}$ such that*

$$\{\sum_{j=1}^{\infty}\sum_{k\in I_{m_j}}x_{ik}\},\ i=1,2,\ldots,$$

is a Cauchy sequence, then all limits in the equalities

$$\lim_{i\to\infty}\sum_{j=1}^{\infty}x_{ij}=\sum_{j=1}\lim_{i\to\infty}x_{ij}=\lim_{i,j\to\infty}\sum_{k=1}^{j}x_{ik}$$

exist and the equalities hold.

Proof. Let $s_{ij}=\Sigma\{x_{ik}\mid 1\leq k\leq j\}$, $i,j\in N$. For $\{x_{ij}\}$ is separately convergent and sums $\Sigma\{x_{ij}\mid 1\leq j\leq\infty\}$, $i\in N$, exist, we infer that $\{s_{ij}\}$ is a separately convergent matrix. Thus (C) holds for $\{s_{ij}\}$. We shall prove that $\{s_{ij}\}$ is Δ_i-vanishing. In fact, let $\{m_i)$ and $\{n_i\}$ be any increasing sequences of positive integers and let

$$z_{ij}=s_{m_{i+1}n_{j+1}}-s_{m_{i+1}n_{j+1}}+s_{m_i n_{j+2}}-s_{m_i n_{j+1}},\ i,j\in N.$$

We note that $z_{ij}=\Sigma\{x_{m_{i+1}k}\mid k\in I_j\}-\Sigma\{x_{m_i k}\mid k\in I_j\}$ with $I_j=[n_{j+1},n_{j+2}]$. Let $\{p_i\}$ be an arbitrary increasing sequence of positive integers. By the hypothesis there exists a subsequence $\{r_i\}$ of $\{p_i\}$ such that

$$\{\sum_{j=1}^{\infty} \sum_{k \in I_{r_j}} x_{ik}\}, \quad i=1, 2, \ldots,$$

is a Cauchy sequence. This implies that

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} z_{r_i r_j} = 0$$

and consequently $\{s_{ij}\}$ is Δ_i-vanishing. Hence, for $\{s_{ij}\}$ is separately convergent, $\{s_{ij}\}$ is a permutationally convergent matrix. Thus by (C_2) all limits in the equalities

$$\lim_{i\to\infty} \lim_{j\to\infty} s_{ij} = \lim_{j\to\infty} \lim_{i\to\infty} s_{ij} = \lim_{i,j\to\infty} s_{ij}$$

exist and the equalities hold which is equivalent to our assertion. This completes the proof of the theorem.

Let X be a collection of sets satisfying the following condition:

($\bar{\text{K}}$) for any disjoint sequence $\{E_n\}$ of sets in X there exists a subsequence $\{E_{m_n}\}$ of $\{E_n\}$ such that $\cup\{E_{m_n}\ 1 \leq n \leq \infty\} \in X$.

An additive set function μ on X is said to be countable additive iff $\Sigma_{n=1}^{\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$ provided $\{E_n\}$ is a disjoint sequence of sets and $\bigcup_{n=1}^{\infty} E_n \in X$.

Let $\{E_j\}$ be a disjoint sequence of sets in X and let μ_n be a sequence of countable additive set functions on X such that $\mu(E) = \lim_{i\to\infty} \mu_i(E)$ for each E in X. The matrix $x_{ij} = \mu_i(E_j)$ satisfies the conditions of the Theorem 5, if $E = \bigcup_{n=1}^{\infty} E_n$ and $E \in X$. Consequently, we have

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j)$$

which means that μ is a countable additive set function. This theorem is usually proved under the assumption that X is a σ-ring of sets.

Theorem 6. *Let $\{x_{ij}\}$ be a matrix such that the limits*

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} x_{ij} \quad \textit{and} \quad \lim_{j} \sum_{i=1}^{\infty} x_{ij}$$

exist. If for any increasing sequence $\{I_j\}$ of segments of positive integers there exists a subsequence $\{I_{n_j}\}$ such that

$$\{\sum_{j=1}^{\infty} \sum_{k=1}^{i} \sum_{l \in I_{n_j}} x_{kl}\}, \quad i=1, 2, \ldots,$$

is a Cauchy sequence, then all sums in the equalities

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_{ij} = \sum_{i,j=1}^{\infty} x_{ij}$$

exist and the equalities hold.

Proof. To prove the theorem we take the matrix $\{s_{ij}\}$ such that

$$s_{ij} = \sum_{k=1}^{i} \sum_{l=1}^{j} x_{kl}$$

for i, j from N. Similarly, as in the proof of the theorem 5, we obtain

$$z_{ij} = \sum_{k \in J_i} \sum_{l \in I_j} x_{kl}$$

with $I_j = [n_{j+1}, n_{j+2}]$ and $J_i = [m_i, m_{i+1}]$. Let $\{p_i\}$ be an increasing sequence of positive integers. By the hypothesis there exists a subsequence $\{r_i\}$ of $\{p_i\}$ such that

$$\{\sum_{j=1}^{\infty} \sum_{k=1}^{i} \sum_{l \in I_{r_j}} x_{kl}\}, \; i = 1, 2, \ldots,$$

is a Cauchy sequence. This implies that

$$\lim_{i \to \infty} \sum_{j=1} z_{r_i r_j} = 0,$$

and consequently $\{s_{ij}\}$ is A_i-vanishing. Hence, for $\{s_{ij}\}$ is separately convergent, we see that $\{s_{ij}\}$ is permutationally convergent. Thus by (C_2) all limits in the equalities

$$\lim_{i \to \infty} \lim_{j \to \infty} s_{ij} = \lim_{j \to \infty} \lim_{i \to \infty} s_{ij} = \lim_{i,j \to \infty} s_{ij}$$

exist and the equalities hold. These equalities are equivalent to our assertion.

4. One could observe that in all the previous considerations we used only group properties of the real number system and the properties of the module, $|-x| = |x|$, $|x+y| \leq |x| + |y|$. We did not need the property of completeness of the real line. This observation enables us to generalize all considerations to matrices whose elements are in a guasi-normed group X and to matrices whose elements are in an Abelian group endowed with a family of quasi-norms.

By a quasi-normed group we mean any Abelian group endowed with a functional (called a quasi-norm) on X such that $|-x| = |x|$ and $|x+y| \leq |x| + |y|$. A sequence $\{x_n\}$ of elements from a quasi-normed group is Cauchy iff for any sequence $\{p_i\}$ of positive integers $x_{p_{n+1}} - x_{p_n} \to 0$ as $n \to \infty$.

If X is an Abelian group endowed with a family Q of quasi-norms on X, then a sequence in X is convergent to x if it converges to x with respect to each norm in Q, and it is a Cauchy sequence if it is a Cauchy sequence with respect to each norm. Each topological group can be considered as a group endowed with a family of quasi-norms.

REFERENCES

1. P. Antosik. A diagonal theorem for non-negative matrices and equicontinuous sequences of mappings. *Bull. Acad. Polon. Sci., Sér. math. astronom. phys.*, **24**, 1976, No. **10**, 855—860.

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