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A NON-STANDARD SUBTRACTION OF INTERVALS

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A non-standard algebraical operation for subtraction of intervals is introduced in this paper. Then the properties of the set of all compact intervals together with this operation and the standard operations for addition and scalar multiplication are studied in some detail.

1. Introduction. The four standard algebraic operations in the set $I(R)$ of all compact intervals on the real line R are defined by

$$(1)\qquad (a*b)_s=\{\alpha*\beta:\alpha\in a,\ \beta\in b\},\quad *\in\{+,\ -,\ \times,\ /\}.$$

(The suffix s stands for "standard".) These operations define an algebraic structure on $I(R)$ called interval arithmetic [7; 10]. Although very simple interval arithmetic finds numerous applications in various numerical problems. As a typical example we shall mention the problem of finding the range of values of a rational function $\varphi(\alpha, \beta, \dots, \gamma)$ when the real variables $\alpha, \beta, \dots, \gamma$ vary in given intervals: $\alpha\in a,\ \beta\in b, \dots, \gamma\in c,\ a, b, \dots, c\in I(R)$. In some cases interval arithmetic gives a simple solution to this problem. Indeed, if we want to evaluate the set

$$F=\left\{\frac{\alpha-\beta}{\gamma-\delta}:\alpha\in a,\ \beta\in b,\ \gamma\in c,\ \delta\in d\right\}$$

we can simply write $F=(p/q)_s$, $p=(a-b)_s$, $q=(c-d)_s$, replacing thereby the variables in the expression for $\varphi(\alpha, \beta, \dots)$ by the corresponding intervals, and the operations in this expression by the corresponding operations (1) between intervals.

However, if we try to compute the interval

$$F=\left\{\frac{\alpha-\beta}{\alpha+\beta}:\alpha\in a,\ \beta\in b\right\}$$

we see that the above approach fails to work. In this case we can only state that $F\subset(p/q)_s$, $p=(a-b)_s$, $q=(a+b)_s$.

More generally, if some of the variables $\alpha, \beta, \dots$ occur more than once in the computation of the rational function $\varphi(\alpha, \beta, \dots)$, then the result may not be sharp; we obtain an inclusion relation, which is usually very rough.

The need of an extension of the interval arithmetic for achieving sharper results for wider class of problems is already discussed by several authors. Generalizations of the interval arithmetic on the base of the existing standard operations (1) are proposed in [1; 4].

In [5] we proposed an extension of the standard interval arithmetic by means of two new operations — a non-standard subtraction and a non-standard division of intervals. The algebraic structure thus obtained we called extended interval arithmetic. This arithmetic comprises the standard interval arithmetic. To clarify this, suppose that $a=[a_1, a_2]$ and $b=[b_1, b_2]$ are two itervals of non-zero width. The interval $(a*b)_s$, $*\in\{+, -, \times, /\}$, is the largest interval determined by the points $a_{11}=a_1*b_1$, $a_{12}=a_1*b_2$, $a_{21}=a_2*b_1$ and $a_{22}=a_2*b_2$, that is

$$(a*b)_s=[\min\{a_{11}, a_{12}, a_{21}, a_{22}\}, \max\{a_{11}, a_{12}, a_{21}, a_{22}\}].$$

Denote the end-points of this interval by α, β; $(a*\beta)_s=[\alpha, \beta]$, $\alpha, \beta\in\{a_{11}, a_{12}, a_{21}, a_{22}\}$. The remaining two points, say γ, δ, where $\gamma, \delta\in\{a_{11}, a_{12}, a_{21}, a_{22}\}$ determine a shorter interval $[\gamma, \delta]$. In applications we often need to express this interval in order to achieve sharper results. This is not possible in the standard interval arithmetic, and is possible in the extended interval arithmetic. In the notations of [5] we can state that extended interval arithmetic makes use of the following operations: $a+b$, $a+(-b)$, $a\times b$, $a\times(1/b)$, $a-(-b)$, $a-b$, $a/(1/b)$ and a/b, whereas the standard interval arithmetic makes use only of the first four of them: $(a+b)_s=a+b$, $(a-b)_s=a+(-b)$, $(a\times b)_s=a\times b$ and $(a/b)_s=a\times(1/b)$ (which always produce the wider interval).

In this paper we restrict our attention to the properties of the non standard subtraction of intervals and to the algebraic structure of the space $\langle I(R), +, \circ, -\rangle$. The non-standard difference of the intervals $a=[a_1, a_2]$, $b=[b_1, b_2]$ is the interval with end-points a_1-b_1 and a_2-b_2. This subtraction has the important property that $a-a=0$ for every interval a. It produces in general a narrower result than the standard subtraction. Both subtractions produce equal results only if one of the intervals a, b is of zero width ($a_1=a_2$ or $b_1=b_2$). As we already mentioned the non-standard subtraction is very useful in interval computations. This can be illustrated by the following examples:

$$\{a+a^2 : a\in x\}=\begin{cases} x+x^2, & x\geq 0,\\ x-(-x^2), & x\leq -1/2;\end{cases}$$

$$\{a-a^2 : a\in x\}=\begin{cases} x-x^2, & x\geq 1/2,\\ x+(-x^2), & x\leq 0, \qquad x\in I(R).\end{cases}$$

Finally, we shall note that operations which are similar to the non-standard subtraction are already studied. Some properties of the operation

$$a\ominus b=[a_1-b_1, a_2-b_2]$$

defined for $a=[a_1, a_2]$, $b=[b_1, b_2]$, such that $w(a)\geq w(b)$ ($w(a)$ is the width of a) are discussed in [2; 9]. For $a, b\in I(R)$ such that $w(a)<w(b)$ one can also define $a\ominus' b=[a_2-b_2, a_1-b_1]$ [2. p. 64]. The relation between this two operations and the non-standard subtraction is:

$$a-b=\begin{cases} a\ominus b, & \text{if} \quad w(a)\geq w(b),\\ a\ominus' b, & \text{if} \quad w(a)<w(b).\end{cases}$$

The differences $a\ominus b$, $a\ominus' b$ are known as Hukuhara-differences.

2. The space $\langle I(R), +, \circ\rangle$. The set of all compact intervals on the real line R is denoted by $I(R)$. Denote, as usual, the (standard) addition of $a=[a_1, a_2]$, $b=[b_1, b_2]\in I(R)$ by

(A) $$a+b=[a_1+b_1,\ a_2+b_2]$$

and the scalar multiplication of $\alpha\in R$ and $a=[a_1, a_2]\in I(R)$ by

(SM) $$\alpha\circ a=\alpha a=[\min\{\alpha a_1, \alpha a_2\},\ \max\{\alpha a_1, \alpha a_2\}].$$

The product $(-1)a$ is briefly denoted by $-a$, so that $(-1)a=-a=[-a_2, -a_1]$. The interval $[\alpha, \alpha]$ is denoted by α.

The operation (SM) is obviously a special case of the standard multiplication, $\alpha\circ b=(\alpha\times b)_s$, $\alpha\in R$. It is also easily seen that the standard subtraction (1) can be written as $(a-b)_s=a+(-b)$, that is, it is a composition of the operations (A) and (SM).

Denote the algebraic system consisting of the set $I(R)$ together with the operations (A) and (SM) by $\langle I(R), +, \circ\rangle$. Here are some well-known properties of the space $\langle I(R), +, \circ\rangle$ (in relations (R1)—(R6) $a, b, c, \ldots$ denote arbitrary intervals and $\alpha, \beta, \ldots$ are arbitrary reals):

(R1) $a+b=b+a$ and $(a+b)+c=a+(b+c)$, that is the set $I(R)$ is a commutative semigroup with respect to (A);

(R2) $$\alpha(b+c)=\alpha b+\alpha c;$$

(R3) $$(\alpha+\beta)c=\alpha c+\beta c \quad \text{for} \quad \alpha\beta\geq 0;$$

(R4) $$\alpha(\beta c)=(\alpha\beta)c;$$

(R5) $$1a=a;$$

(R6) $$0a=0.$$

It is to be remarked that we shall *not* denote the sum $a+(-b)$ by $a-b$ (as this is done in standard interval arithmetic). Thus we preserve the notation $a-b$ for the non-standard subtraction which properties are to be studied next.

3. The space $\langle I(R), +, \circ, -\rangle$. For every two intervals $a=[a_1, a_2]$, $b=[b_1, b_2]$ the interval $a-b$ is defined by

(S) $$a-b=[\min\{a_1-b_1, a_2-b_2\},\ \max\{a_1-b_1,\ a_2-b_2\}].$$

For example, $[1, 3]-[1, 3]=0$ (but $([1, 3]-[1, 3])_s=[-2, 2]$); $[-1, 1]-[-2, 2]=[-1, 1]$ (but $([-1, 1]-[-2, 2])_s=[-3, 3]$); $[-3, 3]-[-2, 2]=[-1, 1]$, etc.

In general $a-b\neq a+(-b)$; the inclusion $a-b\subset a+(-b)$ holds true.

Our further aim is to study the properties of the algebraic system $\langle I(R), +, \circ, -\rangle$ consisting of the set $I(R)$ together with the operations (A), (SM) and (S). The following relations ((R7)—(R12)) hold true in $\langle I(R), +, \circ, -\rangle$ in addition to relations (R1)—(R6):

Proposition 1. *For every $a, b\in I(R)$ it holds that*

(R7) $$(-a)-b=(-b)-a.$$

Verification. Assume first that $w(b)\leq w(a)$ where $w(b)$ denotes the width of b. Denoting $a=[a_1, a_2]$, $b=[b_1, b_2]$, this means that $b_2-b_1\leq a_2-a_1$ or that $-a_2-b_1\leq -a_1-b_2$. Therefore, we can write

$$(-a)-b=(-[a_1, a_2])-[b_1, b_2]=[-a_2, -a_1]-[b_1, b_2]=[-a_2-b_1, -a_1-b_2]$$
$$=[-b_1-a_2, -b_2-a_1]=[-b_2, -b_1]-[a_1, a_2]=(-b)-a.$$

In the case $w(a)\leqq w(b)$ we have $-a_1-b_2<-a_2-b_1$ and hence

$$(-a)-b=[-a_2, -a_1]-[b_1, b_2]=[-a_1-b_2, -a_2-b_1]$$
$$=[-b_2-a_1, -b_1-a_2]=[-b_2, -b_1]-[a_1, a_2]=(-b)-a.$$

P r o p o s i t i o n 2. *For every* $a, b\in I(R)$ *and* $\alpha\in R$,

(R8) $$\alpha(a-b)=\alpha a-\alpha b.$$

V e r i f i c a t i o n. Let $w(a)\geqq w(b)$ in which case $w(\alpha a)\geqq w(\alpha b)$ holds as well. Then for $\alpha\geqq 0$,

$$\alpha(a-b)=[\alpha(a_1-b_1), \alpha(a_2-b_2)]=[\alpha a_1-\alpha b_1, \alpha a_2-\alpha b_2]$$
$$=[\alpha a_1, \alpha a_2]-[\alpha b_1, \alpha b_2]=\alpha a-\alpha b.$$

In the case $\alpha<0$ we have

$$\alpha(a-b)=[\alpha(a_2-b_2), \alpha(a_1-b_1)]=[\alpha a_2-\alpha b_2, \alpha a_1-\alpha b_1]$$
$$=[\alpha a_2, \alpha a_1]-[\alpha b_2, \alpha b_1]=\alpha a-\alpha b.$$

The case $w(a)<w(b)$ is verified analogously, using again the definition of non-standard subtraction.

P r o p o s i t i o n 3. *For every interval c and reals* α, β, *such that* $\alpha\beta\leqq 0$ *we have*

(R9) $$(\alpha+\beta)c=\alpha c-(-\beta c)\quad (\textit{if}\ \alpha\beta\leqq 0).$$

V e r i f i c a t i o n. Let $\alpha\geqq 0$, $\beta\leqq 0$. Then:

i) in the case $\alpha\geqq|\beta|$ minding that $\alpha+\beta\geqq 0$ and $w(\alpha c)\geqq w(-\beta c)$ we have $(\alpha+\beta)c=[(\alpha+\beta)c_1, (\alpha+\beta)c_2]=[\alpha c_1+\beta c_1, \alpha c_2+\beta c_2]$ and, on the other side, $\alpha c-(-\beta c)=[\alpha c_1, \alpha c_2]-[-\beta c_1, -\beta c_2]=[\alpha c_1+\beta c_1, \alpha c_2+\beta c_2]$;

ii) in the case $0\leqq\alpha<|\beta|$ having in view that $\alpha+\beta\leqq 0$ and $w(\alpha c)\leqq w(-\beta c)$ we obtain $(\alpha+\beta)c=[(\alpha+\beta)c_2, (\alpha+\beta)c_1]=[\alpha c_2+\beta c_2, \alpha c_1+\beta c_1]$ and, on the other hand, $\alpha c-(-\beta c)=[\alpha c_1, \alpha c_2]-[-\beta c_1, -\beta c_2]=[\alpha c_2+\beta c_2, \alpha c_1+\beta c_1]$. which verifies the relation (R9) for $\alpha\geqq 0$, $\beta\leqq 0$. In order to verify the relation for $\alpha\leqq 0$, $\beta\geqq 0$ it is enough to put $\alpha=-\gamma$, $\beta=-\delta$ and to use the result just obtained.

Let us derive some simple corollaries from (R1)—(R9). Using relations (R3) and (R9) we obtain that every $a\in I(R)$ satisfies $a+0=a$, $0+a=a$, $a-0=a$, $0-a=-a$ and $a-a=0$. The last equality is very important. For the standard subtraction we have only $a+(-a)\ni 0$.

Properties (R3) and (R9) can be combined:

(2) $$(\alpha+\beta)c=\begin{cases}\alpha c+\beta c, & \text{if } \alpha\beta\geqq 0,\\ \alpha c-(-\beta c), & \text{if } \alpha\beta<0.\end{cases}$$

Relation (2) can be also written:

$$(\alpha-\beta)c=\begin{cases}\alpha c-\beta c, & \text{if } \alpha\beta\geqq 0,\\ \alpha c+(-\beta c), & \text{if } \alpha\beta<0.\end{cases}$$

Another useful relation is $a-b=-(b-a)$. Indeed, by means of (R7) and (R8) we obtain $a-b=(-b)-(-a)=-(b-a)$. Relation (R7) can be also written $a-(-b)=b-(-a)$.

In what follows we shall formulate three basic distributive relations as regards the operations (A) and (S). These relations are formulated by means of the width function $w([a_1, a_2])=a_2-a_1$.

We note first that w satisfies the following relations for arbitrary $\alpha\in R$, $a, b\in I(R)$:

(W1) $$w(a)\geq 0;$$

(W2) $$w(\alpha a)=|\alpha|\, w(a);$$

(W3) $$w(a)+w(b)=w(a+b);$$

(W4) $$|w(a)-w(b)|=w(a-b).$$

The following notations are used for brevity in relations (R10)—(R12):

$$W_1=(w(a)-w(c))\,(w(b)-w(d)),\quad W_2=(w(a)-w(b))\,(w(c)-w(d)).$$

Then the three basic relations in $\langle I(R), +, \circ, -\rangle$ are:

Proposition 4. *For every four intervals $a, b, c, d\in I(R)$ it holds true that*

(R10) $$(a+b)-(c+d)=\begin{cases}(a-c)+(b-d), & \text{if } W_1\geq 0,\\ (a-c)-(d-b), & \text{if } W_1<0.\end{cases}$$

Verification. Denote $m=[m_1, m_2]=(a+b)-(c+d)$, $n=[n_1, n_2]=(a-c)+(b-d)$, $p=[p_1, p_2]=(a-c)-(d-b)$.

Consider first the case $W_1\geq 0$. According to the definitions,

$$m_1=\min\{(a_1+b_1)-(c_1+d_1), (a_2+b_2)-(c_2+d_2)\},$$

$$n_1=\min\{a_1-c_1, a_2-c_2\}+\min\{b_1-d_1, b_2-d_2\},$$

$$m_2=\max\{(a_1+b_1)-(c_1+d_1), (a_2+b_2)-(c_2+d_2)\},$$

$$n_2=\max\{a_1-c_1, a_2-c_2\}+\max\{b_1-d_1, b_2-d_2\}.$$

Suppose that $w(a)-w(c)\geq 0$, $w(b)-w(d)\geq 0$. This implies $a_1-c_1\leq a_2-c_2$, $b_1-d_1\leq b_2-d_2$ and consequently $(a_1-c_1)+(b_1-d_1)\leq(a_2-c_2)+(b_2-d_2)$, i. e. $(a_1+b_1)-(c_1+d_1)\leq(a_2+b_2)-(c_2+d_2)$. Using this we obtain:

$$n_1=(a_1-c_1)+(b_1-d_1)=(a_1+b_1)-(c_1+d_1)=m_1,$$

$$n_2=(a_2-c_2)+(b_2-d_2)=(a_2+b_2)-(c_2+d_2)=m_2,$$

and hence $m=n$ in this case. The subcase $w(a)-w(c)\leq 0$, $w(b)-w(d)\leq 0$ is treated analogously.

Assume now that $W_1\leq 0$. Because of $W_1\leq 0$ it holds that either $w(a)\geq w(c)$, $w(b)\leq w(d)$ or $w(a)\leq w(c)$, $w(b)\geq w(d)$.

Let $w(a)\geq w(c)$ and $w(b)\leq w(d)$, so that $w(a)-w(c)\geq w(d)-w(b)$. Hence we have $w(a)+w(b)\geq w(c)+w(d)$ or $w(a+b)\geq w(c+d)$. Therefore in this case

$$m_1=(a_1+b_1)-(c_1+d_1),\ p_1=(a_1-c_1)-(d_1-b_1);$$

$$m_2=(a_2+b_2)-(c_2+d_2),\ p_2=(a_2-c_2)-(d_2-b_2),$$

so that $m=p$.

Let $w(a)<w(c)$ and $w(b)\geq w(d)$. In this case $w(c)-w(a)>w(b)-w(d)$, i. e. $w(c)+w(d)>w(a)+w(b)$, and we obtain

$$m_1=(a_2+b_2)-(c_2+d_2),\ p_1=(a_2-c_2)-(d_2-b_2);$$

$$m_2=(a_1+b_1)-(c_1+d_1),\ p_2=(a_1-c_1)-(d_1-b_1),$$

so that again $m=p$. This completes the verification of (R10).

Proposition 5. *For every $a, b, c, d\in I(R)$ it holds that:*

$$\text{(R11)}\qquad (a-b)+(c-d)=\begin{cases}(a-(-c))-(b-(-d)), & \text{if } W_2<0,\ W_1\geq 0,\\ (a-(-c))+((-b)-d), & \text{if } W_2<0,\ W_1<0,\\ (a+c)-(b+d), & \text{if } W_2\geq 0.\end{cases}$$

Proposition 6. *For every $a, b, c, d\in I(R)$ it holds that:*

$$\text{(R12)}\qquad (a-b)-(c-d)=\begin{cases}(a-c)-(b-d), & \text{if } W_2\geq 0,\ W_1\geq 0,\\ (a-c)+(-(b-d)), & \text{if } W_2\geq 0,\ W_1<0,\\ (a+(-c))-(b+(-d)), & \text{if } W_2<0.\end{cases}$$

We omit the verifications of relations (R11) and (R12), since they are completely analogous to the verification of (R10).

Some simple corollaries of relations (R10)–(R12) are in order.

Substituting $c-d=(-d)-(-c)$ in (R11) and (R12) we get immediately

$$\text{(R11)}'\qquad (a-b)+(c-d)=\begin{cases}(a-d)+(-(b-c)), & \text{if } W_2\geq 0,\ W_3\geq 0,\\ (a-d)-(b-c), & \text{if } W_2\geq 0,\ W_3<0,\\ (a+(-d))-(b+(-c)), & \text{if } W_2<0;\end{cases}$$

$$\text{(R12)}'\qquad (a-b)-(c-d)=\begin{cases}(a-(-d))+((-b)-c), & \text{if } W_2<0,\ W_3\geq 0,\\ (a-(-d))-(b-(-c)), & \text{if } W_2<0,\ W_3<0,\\ (a+d)-(b+c), & \text{if } W_2\geq 0,\end{cases}$$

wherein $W_3=(w(a)-w(d))(w(c)-w(b))$.

Relations (R10)–(R12) give us many opportunities to rearranging terms in algebraic expressions involving integrals. Consider for example the expressions $(a-b)+(c-d)$ and $(a-b)-(c-d)$ in the case $W_2\geq 0$. Using consecutively (R11) and (R10), resp. (R12) and (R10), we can write:

$$(a-b)+(c-d)=(a+c)-(d+b)=\begin{cases}(a-d)+(c-b), & \text{if } W_3\geq 0,\\ (a-d)-(b-c), & \text{if } W_3<0,\end{cases}$$

resp.,

$$(a-b)-(c-d)=(a+d)-(b+c)=\begin{cases}(a-c)-(b-d), & \text{if } W_1\geq 0,\\ (a-c)+(d-b), & \text{if } W_1<0.\end{cases}$$

Using (R10)—(R12) and the relation $a-b\subset a+(-b)$ it is easily seen that

$$(a+b)-(c+d)\subset(a-c)+(b-d);$$

$$(a-b)+(c-d)\subset\begin{cases}(a+c)-(d+b), \text{ if } W_2\geq 0,\\ (a-(-c))+((-b)-d), \text{ if } W_2<0;\end{cases}$$

$$(a-b)-(c-d)\subset\begin{cases}(a+d)-(b+c), \text{ if } W_2\geq 0,\\ (a-(-d))+((-b)-c), \text{ if } W_2<0.\end{cases}$$

Substituting $b=c=0$ in (R10) we obtain $a+(-d)=a-d$ iff $w(a)w(d)=0$, showing that the non-standard subtraction (S) coincides with the standard subtraction if and only if at least one of the intervals is of zero width.

Substituting $d=0$ in (R10)—(R12) we obtain:

$$(3_1)\qquad (a+b)-c=\begin{cases}(a-c)+b, \text{ if } w(a)\geq(c),\\ (a-c)-(-b), \text{ if } w(a)<w(c);\end{cases}$$

$$(3_2)\qquad (a-b)+c=\begin{cases}(a+c)-b, \text{ if } w(a)\geq w(b),\\ (a-(-c))+(-b), \text{ if } w(a)<w(b),\ w(a)<w(c),\\ (a-(-c))-b, \text{ if } w(a)<w(b),\ w(a)\geq w(c);\end{cases}$$

$$(3_3)\qquad (a-b)-c=\begin{cases}(a-c)-b, \text{ if } w(a)\geq w(b),\ w(a)\geq w(c),\\ (a-c)+(-b), \text{ if } w(a)\geq w(b),\ w(c)<w(c),\\ (a+(-c))-b, \text{ if } w(a)<w(b).\end{cases}$$

Similarly from (R11)′ and (R12)′ we obtain

$$(3_4)\qquad (a-b)+c=\begin{cases}a+(c-b), \text{ if } w(a)\geq w(b),\ w(c)\geq w(b),\\ a-(b-c), \text{ if } w(a)\geq w(b),\ w(c)<w(b),\\ a-(b+(-c)), \text{ if } w(a)<w(b);\end{cases}$$

$$(3_5)\qquad (a-b)-c=\begin{cases}a+((-b)-c), \text{ if } w(a)<w(b),\ w(c)\geq w(b),\\ a-(b-(-c)), \text{ if } w(a)<w(b),\ w(c)<w(b),\\ a-(b+c), \quad \text{ if } w(a)\geq w(b).\end{cases}$$

Substituting $c=a$ in (3_1) we obtain

$$(4)\qquad (a+b)-a=b, \text{ for every } a, b\in I(R).$$

Similarly, if we put in $(3_2)-(3_5)$ respectively $c=-a$, $c=a$, $c=b$ and $c=-b$, we obtain the following relations:

$$(4_1)\qquad (a-b)+(-a)=-b, \text{ if } w(a)\leq w(b);$$

$$(4_2)\qquad (a-b)-a=-b, \quad \text{ if } w(a)\geq w(b);$$

$$(4_3)\qquad (a-b)+b=a, \quad \text{ if } w(a)\geq w(b);$$

$$(4_4)\qquad (a-b)-(-b)=a, \quad \text{ if } w(a)\leq w(b).$$

There is a cancellation law in $\langle I(R), +, \circ\rangle$ with respect to the operations (A) and (SM), that is $a+x=a+y$ implies $x=y$ and $\alpha x=\alpha y$ implies

$x=y$ for every a, x, $y \in I(R)$, $\alpha \in R$. However, the cancellation law does not hold true as regards the subtraction (S), that is $x-a=y-a$ does not imply in general $x=y$. Indeed, $x-a=y-a$ is equivalent to $(x-a)+a=(y-a)+a$ which, according to (4_3) implies $x=y$ only in the case when $(w(x)-w(a))(w(y)-w(a)) \geqq 0$. For example, $[9, 13]-[1, 4]=[10, 12]-[1, 4]$, but $[9, 13] \neq [10, 12]$.

In some cases it is possible to transfer terms from one side of an equality to the other. In this respect the following proposition can be of some use:

Proposition 7. *If $w(a) \geqq w(b)$, then the equalities $a=b+c$ and $a-b=c$ are equivalent. If $w(a) \leqq w(b)$ and $w(c) \leqq w(b)$, then the equalities $a=b-c$ and $a-b=-c$ are equivalent.*

Proof. Suppose that $w(a) \geqq w(b)$. Then $a-b=c$ is equivalent to $(a-b)+b=c+b$, which in the case $w(a) \geqq w(b)$, according to (4_2) is equivalent to $a=c+b$.

Suppose that $w(a) \leqq w(b)$ and $w(c) \leqq w(b)$. In this case $a-b=c$ is equivalent to $(b-a)-b=c-b$, which according to (4_2), is equivalent to $-a=c-b$, that is $a=b-c$.

As it is well-known, the equality $a+x=b$, $a, b \in I(R)$, has a unique solution, if $w(a) \leqq w(b)$. In this case the solution is $x=b-a$.

The following two propositions are concerned with the solutions of the equalities $a-x=b$ and $x-a=b$.

Proposition 8. *The interval $x=a+(-b)$ is a solution of the equality $a-x=b$. In case that $w(a) \geqq w(b)$, this equality has one more solution: $x=a-b$.*

Proposition 9. *The interval $x=a+b$ is a solution of the equality $x-a=b$. In case that $w(a) \geqq w(b)$, the equality has one more solution: $x=a-(-b)$.*

4. Norm in $I(R)$. Consider the function $\|\cdot\|: I(R) \to [0, \infty)$, defined for every $a=[a_1, a_2] \in I(R)$ by

$$\|a\| = \max\{|a_1|, |a_2|\}.$$

This function is a norm in the sense of [6, p. 56], that is

(N1) $$\|a\| > 0, \text{ for } a \neq 0; \ \|0\| = 0,$$

(N2) $$\|\alpha a\| = |\alpha| \, \|a\|,$$

(N3) $$\|a+b\| \leqq \|a\| + \|b\|,$$

hold true for every $a, b \in I(R)$, $\alpha \in R$.

Moreover, it is easily seen that

(N4) $$\|a-b\| \leqq \|a\| + \|b\|.$$

The norm $\|\cdot\|$ generates the Hausdorff distance

$$r(a, b) = \max\{|a_1-b_1|, |a_2-b_2|\}$$

between the intervals $a=[a_1, a_2]$, $b=[b_1, b_2]$ as point sets in R, that is

$$\|a-b\| = r(a, b).$$

As it is well-known $\langle I(R), r\rangle$ is a complete metric space.

The neighbourhoods $U(a, \delta)$ of intervals are defined by

$$U(a, \delta)=\{x : x \in I(R), \ \| a-x \| <\delta\}, \ a \in I(R), \ \delta>0.$$

5. Basis of $I(R)$. We shall call two intervals $a, b \in I(R)$ independent, if $\alpha a-\beta b=0$, $\alpha\beta\geqq 0$, $\alpha, \beta \in R$, imply $\alpha=\beta=0$. We shall call every set of two independent intervals a basis of $I(R)$.

For example, the set $\{[0, 1], [-1, 0]\}$ is a basis of $I(R)$, whereas the set $\{[-1, 2], [-2, 4]\}$ is not.

Further, we shall show that if $\{a, b\}$ is a basis of $I(R)$, then every $x \in I(R)$ can be represented either as $x=\lambda a+\mu b$, $\lambda\mu\geqq 0$ or as $x=\lambda a-\mu b$, $\lambda\mu\geqq 0$.

Let us first remark that this representation is not unique. More precisely there are two such representations.

To clarify this, assume that, say, $x=\lambda a-\mu b$, $\lambda\mu\geqq 0$. Then we can find another representation $x=\lambda' a-\mu' b$, $\lambda'\mu'\geqq 0$, such that $(\lambda', \mu')\neq(\lambda, \mu)$ in general. Indeed, suppose that $\lambda(w(\lambda a)-w(\mu b))\geqq 0$ (the converse case is treated similarly). In this case we can write: $x_1=\lambda a_1-\mu b_1$, $x_2=\lambda a_2-\mu b_2$.

Now we can find λ', μ', such that $x=\lambda' a-\mu' b$, $\lambda'\mu'\geqq 0$ and $\lambda'(w(\lambda' a)-w(\mu' b))<0$. To this end it is enough to put $x_1=\lambda' a_2-\mu' b_2$, $x_2=\lambda' a_1-\mu' b_1$ and solve this system with respect to λ' and μ'. Since $\{a, b\}$ is a basis, the system gives such a solution for λ', μ'. A comparison between the pairs (λ, μ) and (λ', μ') produces

$$p'=[p(a_2b_2-a_1b_1)-(b_2^2-b_1^2)]/[p(a_2^2-a_1^2)-(a_2b_2-a_1b_1)],$$

where $p=\lambda/\mu$, $p'=\lambda'/\mu'$.

In order to have a unique representation, we have to require an additional assumption. For such an assumption one can use: $\lambda(w(\lambda a)-w(\mu b))\geqq 0$.

Proposition 10. *If $\{a, b\}$ is a basis of $I(R)$, and $x \in I(R)$, then there exist a pair (λ, μ) of reals with $\lambda\mu\geqq 0$, such that either $x=\lambda a+\mu b$ or $x=\lambda a-\mu b$.*

Proof. Fix $x=[x_1, x_2]$ and then consider the system

$$x_1=\alpha a_1+\beta b_1, \ x_2=\alpha a_2+\beta b_2.$$

Since $\{a, b\}$ is a basis, $a_1b_2-a_2b_1\neq 0$ and hence the system has a unique solution (α, β). The inequality $x_1\leqq x_2$ implies $\alpha a_1+\beta b_1\leqq \alpha a_2+\beta b_2$, that is $\alpha(a_2-a_1)+\beta(b_2-b_1)\geqq 0$, showing that the case $\alpha<0$, $\beta<0$ is impossible. Therefore we restrict our attention to the cases 1) $\alpha\geqq 0$, $\beta\geqq 0$; 2) $\alpha\geqq 0$, $\beta\leqq 0$ and 3) $\alpha\leqq 0$, $\beta\geqq 0$.

1) $\alpha\geqq 0$, $\beta\geqq 0$. In this case we put $\lambda=\alpha$, $\mu=\beta$. We then have $x=\lambda a+\mu b$.

2) $\alpha\geqq 0$, $\beta\leqq 0$. Now we let $\lambda=\alpha$, $\mu=-\beta$. The pair (λ, μ) satisfies

$$x_1=\lambda a_1-\mu b_1, \ x_2=\lambda a_2-\mu b_2,$$

and $\lambda\geqq 0$, $\mu\geqq 0$. Moreover, $x_1\leqq x_2 \Rightarrow \lambda a_1-\mu b_1\leqq \lambda a_2-\mu b_2 \Rightarrow \lambda w(a)\geqq \mu w(b) \Rightarrow w(\lambda a)\geqq w(\mu b) \Rightarrow \lambda(w(\lambda a)-w(\mu b))\geqq 0$. It is also easily verified that $x=\lambda a-\mu b$.

3) $\alpha\leqq 0$, $\beta\geqq 0$. We let again $\lambda=\alpha$, $\mu=-\beta$. The pair (λ, μ) satisfies

$$x_1=\lambda a_1-\mu b_1, \ x_2=\lambda a_2-\mu b_2,$$

and $\lambda\leqq 0$, $\mu\leqq 0$, so that $\lambda\mu\geqq 0$. Obviously $x=\lambda a-\mu b$. It is also easily seen that in this case the condition $\lambda(w(\lambda a)-w(\mu b))\geqq 0$ holds true.

Remark. The proof of proposition 10 gives a pair (λ, μ) with the desired properties. As we already mentioned, there is one more pair (λ', μ') with the same properties. This second pair can be obtained by considering the system

$$x_1 = \alpha a_2 - \beta b_2,\ x_2 = \alpha a_1 - \beta b_1,$$

and discussing the cases 1) $\alpha \leq 0$, $\beta \geq 0$, 2) $\alpha \leq 0$, $\beta \leq 0$ and 3) $\alpha \geq 0$, $\beta \leq 0$, in the same manner as we did above. We see that the pairs (λ', μ') obtained in cases 2) and 3) satisfy the inequality $\lambda'(w(\lambda' a) - w(\mu' b)) \leq 0$.

Consider now the basis $\{[0, 1], [-1, 0]\}$. In this case proposition 10 obtains the form:

Proposition 11. *For every interval $x \in I(R)$ there are two pairs (λ, μ), such that* 1) $\lambda\mu \geq 0$ *and* 2) *either* $x = \lambda[0, 1] - \mu[-1, 0]$ *or* $x = \lambda[0, 1] + \mu[-1, 0]$.

Indeed, if $x = [x_1, x_2] \not\ni 0$, then we can write $x = x_2[0, 1] - x_1[-1, 0]$.

We can also write $x = x_1[0, 1] - x_2[-1, 0]$, showing that in the case $0 \notin x$ we can choose for (λ, μ) either (x_1, x_2) or (x_2, x_1).

In the case $x = [x_1, x_2] \ni 0$ we can write $x = x_2[0, 1] + (-x_1)[-1, 0]$ or $x = x_1[0, 1] + (-x_2)[-1, 0]$ showing that for (λ, μ) we can choose either $(x_1, -x_2)$ or $(x_2, -x_1)$. The requirement $\lambda(w(\lambda a) - w(\mu b)) \geq 0$ leads in both cases ($x \not\ni 0$ and $x \ni 0$) to unique representation.

Once more, if α and β are the endpoints of $x \in I(R)$, then either $x = \alpha[0, 1] - \beta[-1, 0]$ or $x = \alpha[0, 1] + (-\beta)[-1, 0]$.

6. Interval spaces with non-standard subtraction. Algebraic spaces with two operations "+" and "o", satisfying relations (R1)—(R6) are called quasilinear spaces (see for example [5; 9]). Such speces arise not only in interval mathematics, but also when considering special classes of convex sets [3]. In this section we shall restrict our attention to the so-called interval spaces, that is spaces of interval objects (such as intervals, interval sequences, interval vectors and matrices, interval functions etc.).

By an interval space (cf. [5; 8; 9]) we shall understand the following. Consider a set $E = \{a, b, c, \ldots\}$ together with an addition "+", a scalar multiplication "o" over $R = \{\alpha, \beta, \gamma, \ldots\}$ and a partial ordering "$\leq$", such that the algebraic systems $\langle E, +, \circ\rangle$, $\langle E, \leq\rangle$ are respectively a vector space over R and a lattice; in addition we shall also require that the relation "$\leq$" satisfies for every $a \leq b$, $a, b \in E$:

$$a + x \leq b + x \text{ for every } x \in E,$$

$$\alpha a \leq \alpha b \text{ for every } \alpha \in R,\ \alpha \geq 0.$$

We shall briefly denote the system $\langle E, +, \circ, \leq\rangle$ by E.

We denote every ordered pair of two elements a, b of E, $a \leq b$, by $[a, b]$. (We can also think of $[a, b]$ as: $[a, b] = \{x : a \leq x \leq b$, if $a \leq b\}$). The pairs $[a, b]$ (such that $a, b \in E$, $a \leq b$) are called intervals in E.

Denote the set of all intervals in E by $I(E)$. Define addition and scalar multiplication in $I(E)$ by

$$\text{(A)} \qquad [a, b] + [c, d] = [a + c, b + d];$$

$$\text{(SM)} \qquad \alpha \circ [a, b] = \alpha[a, b] = \begin{cases} [\alpha a, \alpha b], & \alpha \geq 0, \\ [\alpha b, \alpha a], & \alpha < 0. \end{cases}$$

The algebraic structure $\langle I(E), +, \circ\rangle$ is called an interval space (cf. [5; 9]). It is a special case of quasilinear space, since relations (R1)—(R6) are satisfied. It is to be noted that the cancellation law with respect to (A) and to (SM) holds true as well [5; 9].

Our aim is to introduce a non-standard subtraction in the interval space, which will give more algebraic structure. The following remark gives a hint for the definition of this third operation.

Since $\langle E, \leq \rangle$ is a lattice, for every $a, b \in E$ there exist $\inf(a, b)$ and $\sup(a, b)$ defined by

$$c = \inf(a, b) \Longleftrightarrow \begin{cases} c \leq a, \ c \leq b; \\ d \leq a \,\&\, d \leq b \Longrightarrow d \leq c; \end{cases}$$

$$c = \sup(a, b) \Longleftrightarrow \begin{cases} a \leq c, \ b \leq c; \\ a \leq d \,\&\, b \leq d \Longrightarrow c \leq d. \end{cases}$$

Then the definition of the operations (A) and (SM) may be written:

(A) $$[a, b] + [c, d] = [\inf(a+c, b+d), \sup(a+c, b+d)];$$

(SM) $$\alpha[a, b] = [\inf(\alpha a, \alpha b), \sup(\alpha a, \alpha b)].$$

This suggests the following definition of subtraction:

(S) $$[a, b] - [c, d] = [\inf(a-c, b-d), \sup(a-c, b-d)].$$

The reader may verify that the subtraction (S) satisfies relations (R7)—(R9).

Further, consider the function $w: I(E) \to E$ defined by

$$w([a, b]) = b - a.$$

It is easily seen that w satisfies relations (W1)—(W4). (In relation (W4) we have $|w| = \sup(w, -w)$).

Further, relations (R10)—(R12) are satisfied (with the corresponding modifications in the formulations). For instance, we have:

For every $a, b, c, d \in E$ such that $w(a) \leq w(c)$ or $w(c) \leq w(a)$ and $w(b) \leq w(d)$ or $w(d) \leq w(b)$ it holds that

(R10) $$(a+b)-(c+d) = \begin{cases} (a-c)+(b-d), \text{ if } \begin{cases} w(a) \leq w(c) \text{ and } w(b) \leq w(d) \\ \text{or } w(c) \leq w(a) \text{ and } w(d) \leq w(b), \end{cases} \\ (a-c)+(-(b-d)), \text{ if } \begin{cases} w(a) \leq w(c) \text{ and } w(d) \leq w(b), \\ \text{or } w(c) \leq w(a) \text{ and } w(b) \leq w(d). \end{cases} \end{cases}$$

We may call the algebraic structure $\langle I(E), +, \circ, - \rangle$ extended interval space. An extended interval space is normed, if there is a function $\|.\|: I(E) \to R$, satisfying relations (N1)—(N4).

The space $\langle I(R), +, \circ, - \rangle$ with the norm $\|a\| = \max\{|a_1|, |a_2|\}$ is an example of a normed extended interval space. Here are two other examples.

Example 1. Let $E = R_n$, where R_n is the set of all real n-vectors $x = (x_1, x_2, \ldots, x_n)$ with operations $x+y = (x_1+y_1, \ldots, x_n+y_n)$, $\alpha x = (\alpha x_1, \ldots, \alpha x_n)$ and partial ordering "$\leq$": $(x_1', x_2', \ldots, x_n') \leq (x_1'', x_2'', \ldots, x_n'')$ if $x_i' \leq x_i''$ for all $i = 1, \ldots, n$.

$I(R_n)$ is the set of interval vectors: $x = [x', x''] = ([x_1', x_1''], \ldots, [x_n', x_n''])$, $x', x'' \in R_n$.

For $x', x'' \in R_n$ we have

$$\inf(x', x'') = (\min\{x_1', x_1''\}, \min\{x_2', x_2''\}, \ldots, \min\{x_n', x_n''\}),$$

$$\sup(x', x'') = (\max\{x_1', x_1''\}, \max\{x_2', x_2''\}, \ldots, \max\{x_n', x_n''\}).$$

The subtraction in $I(R_n)$ can be written

$$x - y = [x', x''] - [y', y''] = [\inf(x' - y', x'' - y''), \sup(x' - y', x'' - y'')$$
$$= ([\min\{x_1' - y_1', x_1'' - y_1''\}, \max\{x_1' - y_1', x_1'' - y_1''\}],$$
$$\ldots, [\min\{x_n' - y_n', x_n'' - y_n''\}, \max\{x_n' - y_n', x_n'' - y_n''\}]).$$

$I(R_n)$ satisfies relations (R1)—(R12). It can be normed by $\|x\|_n = \max_{1 \le i \le n} \|x_i\|$, where $\|.\|$ is the norm in $I(R)$.

Example 2. Let $E = A_\Omega$ is the set of all real-valued functions defined on some set Ω. A_Ω is a vector space under

$$(f+g)(x) = f(x) + g(x), \quad x \in \Omega, f, g \in A_\Omega,$$
$$(\alpha f)(x) = \alpha f(x), \quad x \in \Omega, f \in A_\Omega, \alpha \in R,$$

and a lattice under $f \le g \langle = \rangle f(x) \le g(x)$ for every $x \in \Omega$.

As in example 1 we see that $I(A_\Omega)$ is an interval space satisfying (R1)—(R12). For every $f = [f_1, f_2]$, $g = [g_1, g_2] \in A_\Omega$ the function $h = f - g$ is defined at $x \in \Omega$ by

$$h(x) = f(x) - g(x) = [\min\{f_1(x) - g_1(x), f_2(x) - g_2(x)\},$$
$$\max\{f_1(x) - g_1(x), f_2(x) - g_2(x)\}].$$

The interval space $I(A_\Omega)$ can be normed by $\|f\|_\Omega = \sup_{x \in \Omega} \|f(x)\|$ where $\|f(x)\| = \max\{|f_1(x)|, |f_2(x)|\}$, $f = [f_1, f_2]$.

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