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SOME PROPERTIES OF GENERALIZED THERMAL POTENTIALS IN A NON-CYLINDRICAL DOMAIN

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Properties of generalized thermal potentials in non-cylindrical domains were examined by A. Piskorek [7; 8; 9], L. J. Kamynin [5; 6], E. A. Baderko [1] and A. Borzymowski [2]. In this paper we investigate some further properties of such potentials. An application of our results to a boundary value problem is given in [3].

1. Introduction. Let D_T be a non-cylindrical domain in the space-time of the points (X, t), where $X(x_1, \ldots, x_n) \in E_n, n \geqq 2$, and $t \in [0, T]$ with T being a finite positive number. The boundary of D_T consists of domains Ω_0 and Ω_T placed in the planes $t=0$ and $t=T$ respectively, and of a n-dimensional lateral surface σ_T included in the layer $0 \leqq t \leqq T$. We denote by S_τ the intersection of σ_T and the plane $t=\tau$, $\tau \in [0, T]$, by Ω_τ the n-dimensional domain of the plane $t=\tau$ bounded by S_τ (in particular, for $\tau=0$ and $\tau=T$ we have the aforesaid domains Ω_0 and Ω_T respectively) and by σ_τ the product $\sigma_T \cap [0, \tau]$.

Consider the normal parabolic equation

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(X, t) \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{j=1}^{n} b_j(X, t) \frac{\partial u}{\partial x_j} + c(X, t) u - \frac{\partial u}{\partial t} = 0, \tag{1}$$

whose coefficients $a_{\alpha\beta}$, b_j and c are defined in a closed bounded cylinder $D_*: = \bar{\Omega}_* \times [0, T]$ containing the set $D_T \cup \sigma_T$, and satisfy Hölder's condition with respect to X (exponent $h \in (0, 1]$). Moreover, we assume that $a_{\alpha\beta}$ satisfy Hölder's condition with respect to t (exponent $h' \in (0, 1]$) and that the characteristic form $\Sigma_{\alpha,\beta=1}^{n} a_{\alpha\beta}(X, t) \lambda_\alpha \lambda_\beta$ is positively definite in D_*.

We shall examine the following integrals

$$U(X, t) = \int_0^t \int_{S_\tau} \Gamma(X, t;\ Q, \tau) \varphi(Q, \tau) dQ d\tau \tag{2}$$

(the potential of single layer with respect to equation (1));

$$V(X, t) = \int_0^t \int_{\Omega_\tau} \Gamma(X, t;\ Y, \tau) \varrho(Y, \tau) dY d\tau \tag{3}$$

(the potential of spatial charge with respect to (1));

$$J(X, t) = \int_{\Omega_0} \Gamma(X, t;\ Y, 0) f(Y) dY \tag{4}$$

(the Fourier-Poisson integral with respect to (1)).

Here Γ is the fundamental solution of equation (1), constructed by W. Pogorzelski [10; 11, p. 379].

2. Regular continuity of the potential of single layer. We assume that satisfies Lapunov's conditions [11, p. 231], one of which is of the form

$$(N_{P_t}, N_{Q_\tau}) \leq C(|PQ|^{\varkappa_1} + |t-\tau|^{\varkappa_2}), \tag{5}$$

where $C>0$, $\varkappa_i \in (0,1]$, $i=1,2$, and (N_{P_t}, N_{Q_τ}) is the angle formed by the inward normals to σ_T at its points $P_t=(P,t)$ and $Q_\tau=(Q,\tau)$.

Further we assume that the orientation of σ_T with respect to equation (1) is time-like (i. e. that the tangent plane to σ_T is nowhere perpendicular to the t-axis).

We also assume that

$$|\varphi(Q,\tau)| \leq M_\varphi \tau^{-\mu_\varphi}, \quad ((Q,\tau)\in\sigma_T) \tag{6}$$

holds for $M_\varphi>0$, $\mu_\varphi\in[0,1)$.

A. Piskorek obtained some estimates of the potential $U(X,t)$ and its first-order derivatives $U_{x_\alpha}(X,t)$ [8, theorem 1].

We shall prove that $U(X,t)$ and $U_{x_\alpha}(x,t)$ satisfy Hölder's conditions.

Theorem 1. *Under the aforegoing assumptions on* σ_T *and* φ, *the inequalities*

$$|U(X,t)-U(X_1,t_1)| \leq \text{const}\, M_\varphi t^{-(\mu_\varphi+\mu_0-1)}(|XX_1|^{h_0}+(t_1-t)^{h_0/2}), \tag{7}$$

$$|U_{x_\alpha}(X,t)| \leq \text{const}\, M_\varphi |XP|^{-(2-2\mu)} t^{-(\mu_\varphi+\mu-1)}, \tag{8}$$

$$|U_{x_\alpha}(X,t)-U_{x_\alpha}(X_1,t_1)| \tag{9}$$
$$\leq \text{const}\, M_\varphi |XP|^{-(2-2\mu_*+h_*)} t^{-(\mu_\varphi+\mu_*-1)}(|XX_1|^{h_*}+(t_1-t)^{h_*/2})$$

are valid, where const *is a positive constant independent of* φ; (X,t), (X_1,t_1) $\in D_T$ *with* $t\leq t_1$; P *and* P_1 *are two points on* S_t *and* S_{t_1} *respectively, such that* $|XP|=\inf\{|XQ|: Q\in S_t\}$ *and* $|X_1P_1|=\inf\{|X_1Q|: Q\in S_{t_1}\}$ (*we assume that* $|XP|\leq|X_1P_1|$), *and the exponents* $\mu_0,\ldots,h_*$ *satisfy*

$$(1+h_0)/2<\mu_0<1,\ 0<h_0<1,\ 1/2<\mu<1,\ (1+h_*)/2<\mu_*<1,\ 0<h_*<1. \tag{10}$$

Inequality (8) is analogous to inequality (16) in [8] but is sharper.

To prove theorem 1 we shall need the following.

Lemma 1. *If* φ *and* σ_T *satisfy the assumptions of theorem 1, then for* $\vartheta\in[0,1)$ *and* $\alpha=1,2,\ldots,n$ *the estimate*

$$H(X,t) := \int_0^t \int_{S_\tau} |XQ|^{-\vartheta} |\Gamma_{x_\alpha}(X,t;Q,\tau)|\,|\varphi(Q,\tau)|\,dQ\,d\tau \tag{11}$$
$$\leq \text{const}\, M_\varphi |XP|^{-(2-2\mu+\vartheta)} t^{-(\mu_\varphi+\mu-1)}$$

is valid, where $(1+\vartheta)/2<\mu<1$.

Proof of lemma 1. Let δ be the Lapunov constant for σ_T and let $\delta'\leq\delta/3$ be a sufficiently small positive constant. The proof, being similar but easier in the case $\delta'\leq t\leq T$, will be given only for $0<t<\delta'$.

We shall base in our reasoning on some concepts of paper [8]. Accordingly, let $\bar{P}$ denote a point of S_τ such that $|X\bar{P}|=\inf\{(XQ|: Q\in S_\tau\}$, let $W(\bar{P},\delta')$ be an n-dimensional cylinder of radius δ' and of axis $n_{\bar{P}}$ (the normal to S_τ at $\bar{P}$) and let, finally, Σ_τ^0 be the part of S_τ placed inside $W(\bar{P},\delta')$. We can write

$$(12)\qquad H(X,t)=\int_0^t\int_{\Sigma_\tau^0}|XQ|^{-\vartheta}|\Gamma_{x_\alpha}(X,t;Q,\tau)||\varphi(Q,\tau)|\,dQd\tau$$

$$+\int_0^t\int_{S_\tau-\Sigma_\tau^0}|XQ|^{-\vartheta}|\Gamma_{x_\alpha}(X,t;Q,\tau)||\varphi(Q,\tau)|\,dQd\tau=H_1(X,t)+H_2(X,t).$$

Evidently, the integral $H_2(X,t)$ is bounded.

In the evaluation of $H_1(X,t)$ we use the estimate

$$(13)\qquad |\Gamma_{x_\alpha}(X,t;Q,\tau)|\leqq\text{const}\,(t-\tau)^{-(n+1)/2}\exp[-c|XQ|^2/(t-\tau)],$$

$c>0$, proved in [4, p. 24], and the inequality

$$(14)\qquad |XP|\leqq|X\widetilde{P}|+|P\widetilde{P}|\leqq\text{const}\,(|X\widetilde{P}|+t-\tau),$$

$Q\in\Sigma_\tau^0$, where $\widetilde{P}$ denotes the intersection of S_τ and the orthogonal projection o the axis n_P (normal to S_t at P) on the plane $t=\tau$.

We obtain

$$(15)\qquad H_1(X,t)$$

$$\leqq\text{const}\,M_\varphi|XP|^{-(2-2\mu)}\{\int_0^t\tau^{-\mu_\varphi}(t-\tau)^{-(n+1)/2}\int_{\Sigma_\tau^0}|X\widetilde{P}|^{2-2\mu}|XQ|^{-\vartheta}\exp\Big[-\frac{c|XQ|^2}{t-\tau}\Big]dQd\tau$$

$$+\int_0^t\tau^{-\mu_\varphi}(t-\tau)^{-(n+1)/2+2-2\mu}\int_{\Sigma_\tau^0}|XQ|^{-\vartheta}\exp\Big[-\frac{c|XQ|^2}{t-\tau}\Big]dQd\tau=\overset{*}{H}_1(X,t)+\overset{**}{H}_1(X,t),$$

where μ is arbitrary in $(0,1)$.

In order to estimate $\overset{**}{H}_1(X,t)$ above let us note that

$$(16)\qquad \chi_2|XQ'|\geqq|XQ|\geqq\chi_1|XQ'|\geqq\chi_1|\bar{P}Q'|$$

$(\chi_1,\chi_2>0)$ holds, where Q' is the orthogonal projection of $Q\in\Sigma_\tau^0$ on the plane Π_τ tangent to S_τ at $\bar{P}$.

Basing on (15), introducing the polar coordinates system in Π_τ (with the pole at $\bar{P}$) and making the substitution

$$(17)\qquad |\bar{P}Q'|=z((t-\tau)/c\chi_1)^{1/2},$$

we have

$$(18)\qquad \overset{**}{H}_1(X,t)\leqq\text{const}\,M_\varphi|XP|^{-(2-2\tilde{\mu})}\int_0^\infty z^{n-2-\vartheta}e^{-z^2}dz\int_0^t\tau^{-\mu_\varphi}(t-\tau)^{1-2\tilde{\mu}-\vartheta/2}d\tau$$

$$\leqq\text{const}\,M_\varphi|XP|^{-(2-2\tilde{\mu})}t^{-(\mu_\varphi+2\tilde{\mu}+\vartheta/2-2)}$$

for $1/2<\tilde{\mu}<1-\vartheta/4$.

In a similar way we arrive at the inequalities

$$(19)\qquad \overset{*}{H}_1(X,t)\leqq\text{const}\,M_\varphi|X\widetilde{P}|^{-(2-2\bar{\mu})}\int_0^t\tau^{-\mu_\varphi}(t-\tau)^{-(1+\vartheta/2)}|X\widetilde{P}|^{2-2\bar{\mu}}\exp\Big[-\frac{c_0|X\widetilde{P}|^2}{t-\tau}\Big]d\tau$$

$$\leqq\text{const}\,M_\varphi|XP|^{-(2-2\bar{\mu})}t^{-(\varphi+\mu+\vartheta/2-1)}$$

or $c_0>0$, $1/2<\bar{\mu}<1-\vartheta/2$.

Relations (12), (15), (18) and (19) yield (11), which completes the proof of lemma 1.

Proof of theorem 1. Inequality (8) follows directly from lemma 1. We shall prove inequality (9) (the assertion of (7) is similar but much easier), confining ourselves to the harder case, when $|XX_1|>0$. We shall also assume in the proof that the inequalities $|XX_1|<\delta'/2$, $(t_1-t)^{1/2}<\delta'$ and $t_1\leq 2t$ are valid because, if they were not, (9) would result from (8).

Let $K(X, 2|XX_1|)$ be an n-dimensional ball with centre X and radius $2|XX_1|$, placed in the plane $t=\tau$. We can write

$$(20)\qquad |U_{x_\alpha}(X,t)-U_{x_\alpha}(X_1,t_1)|\leq\int_0^{t_1}\int_{S_\tau^K}|\Gamma_{x_\alpha}(X_1,t_1;Q,\tau)|\,|\varphi(Q,\tau)|\,dQd\tau$$

$$+\int_0^{t}\int_{S_\tau^K}|\Gamma_{x_\alpha}(X,t;Q,\tau)|\,|\varphi(Q,\tau)|\,dQd\tau+\int_t^{t_1}\int_{S_\tau-S_\tau^K}|\Gamma_{x_\alpha}(X_1,t_1;Q,\tau)|\,|\varphi(Q,\tau)|\,dQd\tau$$

$$+\int_0^{t}\int_{S_\tau-S_\tau^K}|\Gamma_{x_\alpha}(X_1,t_1;Q,\tau)-\Gamma_{x_\alpha}(X,t;Q,\tau)|\,|\varphi(Q,\tau)|\,dQd\tau=I_1+\cdots+I_4$$

with $S_\tau^K=S_\tau\cap K$ and $S_\tau-S_\tau^K=S_\tau\setminus K$.

From the inequalities

$$(21)\qquad |XQ|\leq 2|XX_1|,\quad |X_1Q|\leq 3|XX_1|,$$

$Q\in S_\tau^K$, and from lemma 1 it follows that

$$(22)\qquad I_i\leq\text{const}\,M_\varphi|XP|^{-(2-2\mu_*+h_*)}t^{-(\mu_\varphi+\mu_*-1)}(|XX_1|^{h_*}+(t_1-t)^{h_*/2})$$

$i=1,2$, where $(1+h_*)/2<\mu_*<1$.

For the integral I_3 in (20), by basing on (13) and on applying an inequality analogical to (14) we can write

$$I_3\leq\text{const}\,M_\varphi|XP|^{-(2-2\mu_*)}\int_t^{t_1}\tau^{-\mu_\varphi}(t_1-\tau)^{-(n+1)/2}d\tau$$

$$\times\Big\{\int_{S_\tau^*}|X_1\widetilde{P}_1|^{2-2\mu_*}\exp\Big[-\frac{c_0|X_1\widetilde{P}_1|^2}{t_1-\tau}\Big]\exp\Big[-\frac{c_0|P_1Q_1'|^2}{t_1-\tau}\Big]dQ$$

$$+\int_{S_\tau^*}(t_1-\tau)^{2-2\mu_*}\exp\Big[-\frac{c_0|\overline{P}_1Q'|^2}{t_1-\tau}\Big]dQ+\int_{S_\tau^{**}}\exp\Big[-\frac{c|X_1Q|^2}{t_1-\tau}\Big]dQ\Big\},$$

where $\overline{P}_1\in S_\tau$ and $|X_1P_1|=\inf\{|X_1Q|:Q\in S_\tau\}$; $\widetilde{P}_1$ is the point of intersection of S_τ and the axis n_{P_1} normal to S_{t_1} at P_1 ($|X_1P_1|=\inf\{|X_1Q|:Q\in S_{t_1}\}$, Q_1' is the orthogonal projection of $Q\in S_\tau$ on the plane Π_1 tangent to S_τ at $\overline{P}_1$, and S_τ^* and S_τ^{**} are defined by $S_\tau^*:=(S_\tau-S_\tau^K)\cap W_1$ and $S_\tau^{**}:=(S_\tau-S_\tau^K)/W_1$ respectively, with W_1 denoting an $(n-1)$-dimensional cylinder of axis $n_{\overline{P}_1}$ and radius δ'.

Hence, by an argument similar to that in the prove of (18) and (19) we obtain

$$(23)\qquad I_3\leq\text{const}\,M_\varphi|XP|^{-(2-2\mu_*)}t^{-(\mu_\varphi+\mu_*-1)}(t_1-t)^{h_*/2},$$

where $(1+h_*)/2<\mu_*<1$.

Proceeding to the examination of integral I_4 in (20), we consider the following cases:

$$\text{(a)}\quad (t_1-t)^{1/2} \leqq 2|XX_1|, \qquad \text{(b)}\quad (t_1-t)^{1/2} > 2|XX_1|.$$

In case (a) we use the definition of $\Gamma(X,t;Q,\tau)$ (see [11 p. 379]) and we break the integral considered into two integrals by

$$I_4 = \overset{*}{I}_4 + \overline{I}_4, \tag{24}$$

where $\overset{*}{I}_4$ and $\overline{I}_4$ denote the integrals generated by the functions
$e_1 := [\omega_{x_\alpha}^{Q,\tau}(X_1,t_1;Q,\tau)-\omega_{x_\alpha}^{Q,\tau}(X,t;Q,\tau)]$, $e_2 := [\overline{\omega}_{x_\alpha}(X_1,t_1;Q,\tau)-\overline{\omega}_{x_\alpha}(X,t;Q,\tau)]$ respectively, with $\omega^{M,\zeta}(Z,\xi;M,\zeta)$ denoting the well known quasi-solution of equation (1) and $\overline{\omega}(Z,\xi;M,\zeta)$ being given by the formula

$$\overline{\omega}(z,\xi;Q,\tau) = \int_\tau^\xi \int_{\Omega_*} \omega^{M,\zeta}(Z,\xi;M,\zeta)\,\Phi(M,\zeta;Q,\tau)\,dM d\zeta, \tag{25}$$

Φ is the solution of an integral equation (see [11, p. 382]).

For the integral $\overset{*}{I}_4$ above, by applying to e_1 the mean-value theorem and by using the inequality

$$2|XX_1| \leqq (t_1-t)^{1/2} \leqq |XQ|, \tag{26}$$

$Q \in S_\tau - S_\tau^K$, we can write

$$\overset{*}{I}_4 \leqq \text{const}\, M_\varphi \Big\{ |XX_1|^{h_*} \int_0^{t_1}\int_{S_\tau} \tau^{-\mu_\varphi}(t_1-\tau)^{-(n+1)/2}|X_1Q|^{-h_*}\exp\Big[-\frac{c_0|X_1Q|^2}{t_1-\tau}\Big] dQd\tau$$
$$+(t_1-t)^{h_*/2}\int_0^t\int_{S_\tau}\tau^{-\mu_\varphi}(t_*-\tau)^{-(n+1)/2}|XQ|^{-h_*}\exp\Big[-\frac{c_0|XQ|^2}{t-\tau}\Big] dQd\tau\Big\},$$

where $t<t_*<t_1$ and $h_* \in (0,1)$.

Hence (see the assertion of (18) and (19) above) we obtain

$$\overset{*}{I}_4 \leqq \text{const}\, M_\varphi |XP|^{-(2-2\mu_*+h_*)} t^{-(\mu_\varphi+\mu_*-1)}(|XX_1|^{h_*}+(t_1-t)^{h_*/2}), \tag{27}$$

where μ_* is a parameter chosen as in (22).

Now, we shall consider the integral $\overline{I}_4$ in (24).

To this end we make the decomposition

$$e_2 = [\overline{\omega}_{x_\alpha}(X_1,t;Q,\tau)-\overline{\omega}_{x_\alpha}(X,t;Q,\tau)]+[\overline{\omega}_{x_\alpha}(X_1,t_1;Q,\tau)-\overline{\omega}_{x_\alpha}(X_1,t;Q,\tau)] = e_2^1+e_2^1.$$

In order to examine the expression e_2^1 above we consider again the ball $K(X,2|XX_1|)$ and we break the domain Ω_* into two sets $\Omega_*\cap K$ and $\Omega_*|K$. Hence we can write

$$e_2^1 \leqq \int_\tau^t \int_{\Omega_*\cap K} (|\omega_{x_\alpha}^{M,\zeta}(X_1,t;M,\zeta)|+|\omega_{x_\alpha}^{M,\zeta}(X,t;M,\zeta)|)\Phi(M,\zeta;Q,\tau)\,dMd\zeta \tag{28}$$
$$+\int_\tau^t\int_{\Omega_*-K}|\omega_{x_\alpha}^{M,\zeta}(X_1,t;M,\zeta)-\omega_{x_\alpha}^{M,\zeta}(X,t;M,\zeta)|\,|\Phi(M,\zeta;Q,\tau)|dMd\zeta = \Xi_1+\Xi_2.$$

The member Ξ_1 in (28) is estimated by using the inequalities $|XM| \leq 2|XX_1|$, $|X_1M| \leq 3|XX_1|$, $M \in \Omega_* \cap K$, and with the help of [4, formula (4.15) on p. 16 and lemma 3 on p. 15] we get

$$\Xi_1 \leq \text{const}\, |XX_1|^{h_*} (t-\tau)^{-(n+1+h_*-h_1)/2} \exp[-\tilde{c} r^2/(t-\tau)], \tag{29}$$

$\tilde{c} > 0$, $h_* \in (0, 1)$, $h_1 = \min(h, 2h')$, where $r = \min(|XQ|, |X_1Q|)$.

In the evaluation of Ξ_2 we apply the mean value theorem, the result of this evaluation being identical with the estimate of Ξ_1 above.

Hence also the expression e_2^1 satisfies an inequality analogical to (29).

By a similar argument, on introducing the ball $\tilde{K}(X_1, (t_1-t)^{1/2})$ and on subsequent breaking the integrals in e_2^2, we arrive at the following inequality

$$\begin{aligned} e_2^2 \leq \text{const}\, (t_1-t)^{h_*/2} \{ (t_1-\tau)^{-\alpha} \exp[-c_* |X_1Q|^2/(t_1-\tau)] \\ + (t_*-\tau)^{-\alpha} \exp[-c_* |X_1Q|^2/(t_*-\tau)] + (t-\tau)^{-\alpha} \exp[-c_* |X_1Q|^2/(t-\tau)] \}, \end{aligned} \tag{30}$$

where $t < t_* < t_1$, $h_* \in (0, 1)$ and $\alpha = (n+1+h_*-h_1)/2$.

From the foregoing results, it follows that integral I_4 (see (24)) satisfies

$$|I_4| \leq \text{const}\, M_\varphi |XP|^{-(2-2\mu_*+h_*-h_1)} t^{-(\mu_\varphi+\mu_*-1)} (|XX_1|^{h_*} + (t_1-t)^{h_*/2}) \tag{31}$$

with $(1+h_*-h_1)/2 < \mu_* < 1$.

Hence (see (27) and (31)) we can assert that in the case (a), the integral I_4 in (20) satisfies an inequality analogous to (27).

In the case *(b)* ($(t_1-t) > 4|XX_1|^2$) the reasoning does not differ essentially from that presented above and is based on the consideration of the ball $\tilde{K}(X_1, (t_1-t)^{1/2})$, on breaking the domain $S_\tau - S_\tau^K$ (see (20)) into the sets $S_\tau^{\tilde{K}} \setminus S_\tau^K$ and $S_\tau \setminus S_\tau^{\tilde{K}}$ and on subsequent use of lemma 1.

On joining all the results obtained above we arrive at the thesis (9).

3. The oblique derivatives of the potential of single layer. Define on σ_T a vector field $\{l_P\}$ attaching to each point $(P, t) \in \sigma_T$ a vector l_P placed in the plane $t = \text{const}$. For $(x, t) \notin \sigma_T$ we have

$$\frac{d}{dl_P} U(X, t) = \int_0^t \int_{S_\tau} \frac{d}{dl_P} \Gamma(X, t; Q, \tau) \varphi(Q, \tau) dQ d\tau. \tag{32}$$

The following theorem is valid.

Theorem 2. *If φ is continuous on $\sigma_T \setminus S_0$, satisfies inequality* (6) *and the inequality*

$$|\varphi(Q, \tau) - \varphi(Q_1, \tau)| \leq k_\varphi \tau^{-\mu_\varphi} |QQ_1|^{h_\varphi}, \tag{33}$$

where $k_\varphi > 0$, $h_\varphi \in (0, 1]$, then the oblique derivative $\frac{d}{dl_P} U(X, t)$ satisfies

$$\lim_{X \to P} \frac{d}{dl_P} U(X, t) = -\frac{(2\sqrt{\pi})^n \cos(n_P, l_P)}{2\omega(P, t)\sqrt{\det |a^{\alpha\beta}(P, t)|}} \varphi(P, t) + \frac{d}{dl_P} U(P, t) \tag{34}$$

with

$$\omega(P, t) = \sum_{\alpha, \beta=1}^{n} a_{\alpha\beta}(P, t) \cos(n_P, x_\alpha) \cos(n_P, x_\beta) \tag{35}$$

(n_P *is the inward normal to* S_t *at* P), *the singular integral* $\frac{d}{dl_P} U(P, t)$ *being understood in the sense of Cauchy's principal value* (*c. f.* [2, 195]).

Proof. Let us consider at (P, t) a local rectangular coordinates system $P\zeta_1 \dots P\zeta_n$ with $P\zeta_n = n_P$ and the axes $P\zeta_1, \dots, P\zeta_{n-1}$ placed in the plane Π_t tangent to S_t at P.

We can write

$$\frac{\partial U}{\partial x_a}(X, t) = \sum_{i=1}^{n-1} \frac{\partial U}{\partial \zeta_i}(X, t) \cos(\zeta_i, x_a) + \frac{\partial U}{\partial \zeta_n}(X, t) \cos(n_P, x_a)$$

$a = 1, 2, \dots, n$, and using the definition of the transversal derivative $\frac{dU}{dT_P}(X, t)$ [11], we have

$$\frac{\partial}{\partial \zeta_n} U(X, t) = \frac{1}{\omega(X, t)} \Big\{ \frac{d}{dT_P} U(X, t)$$

$$- \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(X, t) \sum_{i=1}^{n-1} \frac{\partial U}{d\zeta_i}(X, t) \cos(\zeta_i, x_\alpha) \cos(n_P, x_\beta) \Big\}.$$

Hence, by basing on the formula

$$\frac{d}{dl_P} U(X, t) = \sum_{a=1}^{n} \frac{\partial}{\partial x_a} U(X, t) \cos(l_P, x_a)$$

and on applying [7, theorem 1] and [2, theorem 3] we arrive at thesis (39). (The said theorems were proved for $\mu_\varphi = 0$. Their extension on the case $0 < \mu_\varphi < 1$ is straightforward).

Theorem 3. *Under the assumptions of the preceding theorem, the following inequality*

$$(36) \qquad \left| \frac{d}{dl_P} U(P, t) \right| \leq (C_1 M_\varphi + C_2 k_\varphi) t^{-\mu_\varphi}$$

is valid with C_1 *and* C_2 *being positive constants independent of* φ. *Moreover, if the vector field* $\{l_P\}$ *satisfies*

$$(37) \qquad (l_P, l_{P_1}) \leq k_l (|PP_1|^{h_l} + |t_1 - t|^{h_l}),$$

$(P, t) \in S_t$, $(P_1, t_1) \in S_{t_1}$, $k_l > 0$, $h_l \in (0, 1]$, *then*

$$(38) \quad \left| \frac{d}{dl_{P_1}} U(P_1, t_1) - \frac{d}{dl_P} U(P, t) \right| \leq (C_3 M_\varphi + C_4 k_\varphi) t^{-\mu_\varphi} (|PP_1|^{\tilde{h}_\varphi} + |t_1 - t|^{\theta\varkappa_*/3}),$$

$t \leq t_1$, holds true where

$$(39) \quad \tilde{h}_\varphi = \min(h_\varphi, \theta' \min(h, 2h', \varkappa, h_l)), \; \varkappa_* = \min(h, 2h', 3\varkappa/2, 3h_\varphi/2, 3h_l/2),$$

$\varkappa = \min(\varkappa_1, \varkappa_2)$, θ and θ' are arbitrary constants in $(0, 1)$, and C_3 and C_4 are positive constants independent of φ.

The proof of theorem 3 rests upon two lemmas.

Lemma 2. *If* $\{l_P\}$ *and* φ *satisfy the assumptions of theorem 3 with* $\mu_\varphi = 0$, *then* (36), (37) *hold true* (*with* $\mu_\varphi = 0$).

Proof of lemma 2 is analogous to that of [2, theorem 3], with the necessary modifications due to the replacement of the tangential derivative $\frac{dU}{ds_P}(P, t)$ (see [2, p. 195]) by the oblique derivative $\frac{d}{dl_P}U(P, t)$.

Lemma 3. *If $\{l_P\}$ and φ satisfy the assumptions of theorem 3, then for each point $(P, t)\in \sigma_T\setminus S_0$ the inequalities*

$$\left|\frac{d}{dl_P}U^{(0,\, t_0)}(P, t)\right|\leq (C_1M_\varphi+C_2k_\varphi)\, t^{-\mu_\varphi}, \tag{40}$$

$$\left|\frac{d}{dl_{P_1}}U^{(0,\, t_0)}(P_1, t_1)-\frac{d}{dl_P}U^{(0,\, t_0)}(P, t)\right|\leq (C_3M_\varphi+C_4k_\varphi)t^{-\mu_\varphi}(|PP_1|^{\widetilde{h}_\varphi}+|t_1-t|^{\theta\varkappa_*/3} \tag{41}$$

hold, where $t_0=t/2$, $\frac{d}{dl_X}U^{\alpha,\beta}(X, \xi)$, (with $X=P_1, P$; $\xi=t_1, t$), denotes integral (32) *with the integration being performed over $\sigma_\beta\setminus\sigma_\alpha$, and the remaining symbols are understood analogously as in* (36)—(39).

Proof of lemma 3 is similar to that of lemma 2 above, with the appropriate modifications caused by the unboundedness of φ.

Proof of theorem 3. The validity of theorem 3 follows immediately from the decomposition

$$\frac{d}{dl_{P_1}}U(P_1, t_1)-\frac{d}{dl_P}U(P, t)=\left[\frac{d}{dl_{P_1}}U^{(t_0,\, t)}(P_1, t_1)-\frac{d}{dl_P}U^{(t_0,\, t)}(P, t)\right]$$

$$+\left[\frac{d}{dl_{P_1}}U^{(0,\, t_0)}(P_1, t_1)-\frac{d}{dl_P}U^{(0,\, t_0)}(P, t)\right],$$

$t_0=t/2$, and from lemmas 2 and 3.

4. The potential of spatial charge and the Fourier-Poisson integral. In this section will be assumed that $\varkappa_1=1$ (c. f. condition (5)).

Let us consider the potential of spatial charge (3) assuming that ϱ is integrable in each closed subdomain of D_T and that

$$|\varrho(Y, \tau)|\leq M_\varrho\, \tau^{-\mu_\varrho}\, |YQ_Y|^{-p} \tag{42}$$

holds, where $M_\varrho>0$, $\mu_\varrho\in[0, 1)$, $p\in[0, 1)$ and Q_Y is a point on the surface S_τ such that $|YQ_Y|=\inf\{|YQ|: Q\in S_\tau\}$.

Theorem 4. *Under the aforegoing assumptions on ϱ and σ_T, the inequalities*

$$|V(X_1, t_1)-V(X, t)|\leq \text{const}\, M_\varrho t^{-\mu_\varrho}(|XX_1|+|t_1-t|^{h_\varrho/2}), \tag{43}$$

$$|V_{x_\alpha}(X_1, t_1)-V_{x_\alpha}(X, t)|\leq \text{const}\, M_\varrho t^{-\mu_\varrho}(|XX_1|^{\widetilde{h}_\varrho}+|t_1-t|^{h^*_\varrho/2}), \tag{43'}$$

$0<h_\varrho<1$, $0<\widetilde{h}_\varrho<1-p$, $0<h^*_\varrho<1-\max(p, \mu_\varrho)$, *are valid, where* $V_{x_\alpha}(Z, \xi)=\frac{\partial}{\partial x_\alpha}V(z, \xi)$ $(z=X_1, X;\ \xi=t_1, t)$, *and (X, t) and (X_1, t_1) are arbitrary points of $\bar{D}_T/\Omega_0$ such that* $0<t\leq t_1\leq T$.

Proof of theorem 4 is similar to that of theorem 1 with some parts of the reasoning being based on the concepts of [8].

We shall end this paper with the following theorem concerning the Fourier-Poisson integral (4).

Theorem 5. *If the function $f(Y)$ is defined and continuous in Ω_0 and satisfies*

$$|f(Y)| \leq M_f |YQ_Y|^{-q},$$

$M_f > 0$, $q \in [0, 1)$, *where Q_Y is a point of S_0 such that $|YQ_Y| = \inf\{|XQ| : Q \in S_0\}$, then integral (4) and its first-order derivatives satisfy*

$$|J(X_1, t_1) - J(X, t)| \leq \text{const}\, M_f t^{-\mu_f} (|XX_1| + |t_1 - t|^{h_f}),$$

$$|J_{x_\alpha}(X_1, t_1) - J_{x_\alpha}(X, t)| \leq \text{const}\, M_f t^{-\mu_f} (|XX_1|^{\tilde{h}_f} + |t_1 - t|^{h_f/2}),$$

where $(1+q)/2 < \mu_f < 1$, *and* h_f *and* $\tilde{h}_f$ *are in* $(0, 1-q)$.

Proof of theorem 5 is similar to that of theorem 4, but much easier.

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