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LINEARLY VARYING BANACH-VALUED MEASURES

ELIAS FLYTZANIS

We generalize the notion of a positive measure being preserved by a transformation. We allow the measure to take values in a Banach space and to vary linearly under transformations of the measurable space. For the main results we assume that the Banach space has the R—N property and the measure is of bounded variation. For the cases considered we show that there exists an equivalent finite positive invariant measure and we identify the resulting R—N derivatives.

1. We consider a σ-algebra Σ of subsets of a set S, a separable complex Banach space B and a set function $M:\Sigma\to B$ which is a B-measure in the sense that $\Sigma_1^n M(E_i)\to M(E)$ for every sequence of pairwise disjoint sets $\{E_i: i=1, 2, 3, \dots\}\subset\Sigma$ with $F=\cup E_i$. A set $E\in\Sigma$ is said to be M-null if $M(F)$ is the zero vector for every $F\subset E$, $F\in\Sigma$. We consider also point transformations $h:S\to S$ defined M — a. e., which are measurable having a measurable inverse and are nonsingular in the sense that E is M-null iff $h^{-1}(E)$ is M-null. Generalizing the notion of positive finite invariant measures (p. f. i. m.) and also that of vector-valued eigenfunctions of measure preserving transformations [2] we examine solutions M, h, T, to the equation

$$Mh^{-1}=TM, \tag{1}$$

where M, h, are as above, $T:B\to B$ is a bounded linear operator and Mh^{-1} is defined by $Mh^{-1}(E)=M(h^{-1}(E))$. In the rest of this section we solve equation (1) in finite dimmensions and in 2 we extend this solution to some cases in infinite dimmensions. The results obtained in the sequel parallel those obtained in [2]. The present setting however is more general and also it seems more natural. Certainly the generalization of equation (1) to locally convex spaces would present special interest since the particular case of operator valued measures can be used in the study of symmetry properties of the spectral decomposition of operators.

First we give some preliminary notions. If M, h are as above we consider the spaces $L_\infty(M)$ of complex measurable functions M-ess. bd. and ca(M) of countably additive positive finite measures absolutely continuous with respect to M in the sense that if E is M-null then $\mu(E)=0$ for $\mu\in$ ca(M). We note that a positive finite measure (p. f. m.) m is said to be equivalent to M if $\sup\{\|M(F)\|: F\subset E_i\}\to 0$ iff $m(E_i)\to 0$ where E_i, $i=1, 2, \dots$, is a sequence in Σ. The point transformation h induces in $L_\infty(M)$ an invertible isometry V by $Vf=f(h(\,.\,))$ whose dual $V^*:\text{ca}(M)\to\text{ca}(M)$ is given by $V^*\mu=\mu h^{-1}$. The eigenvalues of h are defined as the eigenvalues of V. If now c is an eigenvalue of V^* then clearly $|c|=1$ and the eigenvalue equation $\mu h^{-1}=c\mu$ implies:

(i) The total variation $|\mu|$ is a p. f. m. invariant under h.

(ii) $d\mu = f(\,.\,)d|\mu|$ where $f \in L_\infty(\mu)$ and $cf(h(\,.\,)) = f(\,.\,)$, i. e. $f(h^{-1}(\,.\,)) = cf(\,.\,)$.

In fact the remark above gives the solutions of equation (1) if B is one-dimmensional and generalizes directly to finite dimmensions as follows.

Theorem 1. *The solutions M, h, T of* (1) *for which M has finite dimensional range are of the form $dM = X(\,.\,)dm$, where*

(i) *m is a p. f. m. invariant under h;*

(ii) *$X(\,.\,) \in L_\infty(m)$ satisfies the equation $X(h^{-1}(\,.\,)) = TX(\,.\,)$. In particular we have $X(\,.\,) = \Sigma_1^n f_i(\,.\,)x_i$ where $Tx_i = c_i x_i$, $f_i(h^{-1}(\,.\,)) = c_i f_i(\,.\,)$ with $f_i \in L_\infty(m)$, $|c_i| = 1$.*

Proof. Assuming for convenience that B is spanned by the range of B it follows from the invertibility of h and the finite dimmesionality of B that T is invertible. From the boundedness of the range of B [1] it follows that vectors in the range have bounded orbits under both T, T^{-1} and then from finite dimmensionality that T, T^{-1} are totally bounded in the sense that the norms $\{\|T^n\| : n = 0, \pm 1, \ldots\}$ are uniformly bounded. Renorming, if necessary, the space by the equivalent norm $\|x\|' = \sup\{\|T^n x\| : n = 0, \pm 1, \ldots\}$ we can assume w. l. o. g. that T is an isometry. In finite dimmensions the total variation $|M|$ is a p. f. m. and it is invariant under h by equation (1) and by T being an isometry. If we write $dM = X(\,.\,)d|M|$ we clearly have $X(h^{-1}(\,.\,)) = TX(\,.\,)$. Also T being an isometry in finite dimmensions it has a complete set of (independent) eigenvectors $\{x_i : i = 1, 2, \ldots, n\}$ and writing $X(\,.\,) = \Sigma f_i(\,.\,)x_i$ we have clearly $f_i(h^{-1}(\,.\,)) = c_i f_i(\,.\,)$ if $Tx_i = c_i x_i$.

If h is not assumed invertible then the above theorem still holds if M is assumed to have the property that its restriction to the h-invariant sub-σ-algebra Σ_0 of Σ has the same span as M itself, where $\Sigma_0 = \bigcap_{-\infty}^{\infty} h^{-i}(\Sigma)$. We note that for Σ_0 we have $\Sigma_0 = h^{-1}(\Sigma_0) = h(\Sigma_0)$ and the function $X(\,.\,)$ occuring in the theorem is Σ_0-measurable so that $X(h^{-1}(\,.\,))$ is well defined.

2. That Theorem 1 does not hold in infinite dimmensions can be seen by the following example.

Example. (S, Σ, m) is a positive finite measure space and $h : S \to S$ in invertible m-nonsingular transformation for which the R—N derivative dmh^{-1}/dm is ess. bd. so that the operator $T : L_p(m) \to L_p(m)$ defined by $Tf(\,.\,) = f(h(\,.\,))$ is a bounded linear operator. Also we define the set function $M : \Sigma \to L_p(m)$ by $M(E) = \chi_E(\,.\,)$, where χ_E is the characteristic function of $E \in \Sigma$. M is a B-valued measure if $1 \leq v < \infty$ and satisfies (1) by $M(h^{-1}E) = \chi h^{-1}E = T\chi_E = TM(E)$. In particular if m does not have an equivalent p. f. m. invariant under h then neither does M. We note, however, that if T is an isometry, then m is invariant under h and is such a measure. In fact we have the following result:

Theorem 2. *M, h, T, is a solution of* (1) *and T is an isometry. Then there exists a p. f. m. equivalent to M and invariant under h.*

Proof. By [1] there exists a p. f. m. m equivalent to M. In order to show that there exists one such, invariant under h it suffices to show that lim inf $(mh^{-i}(E)) = 0$ implies $m(E) = 0$. Indeed assume lim $(mh^{-k}(E)) = 0$ for some sequence $k \to +\infty$. If $m(E) > 0$ then by the equivalence $m \sim M$ we have that E is not a M-null set and there exists $F \subset E$ for which $M(F)$ is not the zero vector. We consider now the sequence $h^{-k}F$. By assumption we have $m(h^{-k}F) \to 0$ and the equivalence $m \sim M$ implies $M(h^{-k}F) = TM(F) \to 0$ which is a contradiction because T is an isometry.

It is seen from the proof that the theorem holds more generally if we assume that the zero vector is not in the closure of any orbit of T which in the case of the example is in fact a sufficient and necessary condition for the existence of an equivalent invariant measure.

Considering again the example above but with m assumed invariant we note that part (ii) of the theorem is not generally valid in infinite dimmensions even with T assumed to be an isometry. Noting also that for $p=1$ the space $B=L_1(m)$ does not have the R—N property [5] while for $1<p<\infty$ the measure $M:\Sigma\to L_p(m)$ is not of bounded variation it is the case that Theorem 1 holds if B is assumed to have the R—N property, M to be of bounded variation and $T:B\to B$ to be an isometry. Using the techniques of [3] we will prove the theorem for a larger class of operators T.

Theorem 3. *B is a separable complex Banach space having the* R—N *property and $T:B\to B$ has the property that for collection $\{x^*\}\subset B^*$ dense in the B-topology of B^* the orbits under T^* are bounded. Then the solution M, h, T of* (1) *for which M has bounded variation is of the form*

$$dM=X(\,.\,)dm,$$

where

(i) *m is a p. f. m. equivalent to M and invariant under h;*

(ii) *$X:S\to B$ is m-integrable and satisfies $X(h^{-1}(\,.\,))=TX(\,.\,)$.*

Remark. Part (ii) follows directly from (i) and equation (1), so we need prove only (i). We note also that the equation in (ii) has been solved in [3] without the assumption of R—N property. In particular it implies that T, T^* have the same eigenvalues, all of norm 1, they are also eigenvalues of h and the eigenvectors span B in the norm topology and B^* in the B-topology. Also we have that $X(\,.\,)$ is measurable with respect to the sub-σ-algebra of Σ generated by the eigenfunctions of h. In particular, if we denote by $L^e_\infty(m)$ the subspace of $L_\infty(m)$ spanned by the eigenfunctions of h, we have that X is uniquely determined by the map $K:L^e_\infty(m)\to B$ defined by $Kf=fdM$.

Proof (i). Denoting by $|M|$ the total variation of M we have that the operator $K:L_\infty(M)\to B$ defined by $Kf=\int fdM$ is a bounded linear operator and also compact by [4], assuming the R—N property. We also have the adjoint operator $K^*:B^*\to \mathrm{ca}(M)\subset L^*_\infty(M)$ defined by $K^*x^*=x^*M\in\mathrm{ca}(M)$. K^* is compact by the compactness of K and also injective if we assume, as we do w. l. o. g., that B is spanned by the range of M. We have clearly $KT^*=V^*K^*$, where V^* is the invertible isometry induced in ca (M) by $V^*\mu=\mu h^{-1}$. Let now $\{x^*\}$ be a collection spanning B^* in the B-topology and whose elements have bounded orbits under T^*. Then $\mu=K^*x^*\in\mathrm{ca}(M)$ has conditionally compact orbit in the norm topology under the isometry V^* by the compactness of K^* and the equation above. It follows that the subspace spanned by the orbit of μ is also spanned by the eigenvectors of V^*. Since $\{x^*\}$ spans B^* in the B topology it follows that their image under K^* span the range of K^* which by above is also spanned by a countable collection $\{\mu_i\}$ of eigenvectors of V^*. We have $\mu_i h^{-1}=c_i\mu_i$ and from 1 it follows that the total variation $|\mu_i|$ is a p. f. m. invariant under h. Setting $m=\Sigma|\mu_i|/2^i\|\mu_i\|$, we have that m also has this property. Next we show that $m(E)=0$, iff E is M-null, which implies the equivalence $m \sim M$ under the assumption of bounded variation for M. Indeed, if $m(E)=0$ then $|\mu_i|(E)=0$ which implies $\mu_i(F)=0$ for each $F \subset E$ and all $\mu_i \in \{\mu_i\}$. Since $\{\mu_i\}$ spans the range of K^*, it follows that for each $F \subset E$ we have $x^*(M(F))=0$ for all $x^* \in B^*$ and hence $M(F)=0$. This implies that E is an M-null set. The other direction follows from the fact that each $\mu_i \in ca(M)$.

From the proof above we also have:

Corollary. *If B, M are as in Theorem 3, h does not have any eigenvalues different from* 1 *and M, h, T give a solution of* (1) *then either T is the identity or M has an infinite dimmensional range and all the orbits of T^* are unbounded.*

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School of Technology
University of Thessaloniki
Thessaloniki

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