Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

DISTANCE CONDITIONS FOR THE ZEROS OF PEAKING COSINE POLYNOMIALS

G. BLEIMANN, E. L. STARK

The purpose of this paper is to establish estimates for

$$z_n := \min\{x_0(n)>0;\ t_n(x_0(n))=0\}, \text{ where } t_n=1+2\Sigma_{k=1}^{n}\varrho_{k,n}\cos kx,$$

by applying certain inequalities (of Bernstein-type) for the C-norms of trigonometric polynomials and their derivatives.

1. Introduction. In the constructive theory of (2π-periodic) functions, approximation by means of singular convolution integrals plays a fundamental role; See e. g. [5]. In this connection, the description of the underlying kernels (more precisely: approximate identities), particularly a characterization of their graphical behaviour, is of importance. One aspect, the distribution of zeros, leads to the investigation of distance conditions for the first zero of peaking cosine polynomials; compare e. g. [6, p. 95], [17, p. 38, 43].

One peculiarity of these approximate identities (for a complete definition see [5, p. 31] but also [4]) which are represented by a normalized, even, trigonometric polynomal of degree n, i. e.

$$t_n(x)=1+2\sum_{k=1}^{n}\varrho_{k,n}\cos kx,\quad n\in\mathbb{N}, \tag{1.1}$$

is the *peaking property*, in particular at the origin (mod 2π), i. e. $t_n(0)>t_n(x)$ for all $x\in(-\pi,\pi]$ such that (with the C-norm)

$$\|t_n\| := \max\{|t_n(x)|;\ -\pi<x\leqq\pi\}=t_n(0)>0. \tag{1.2}$$

With respect to the fine structure of these (even) kernels some information on the distance of the first (positive) zero from the origin (i. e. the peaking point), thus of

$$z_n := \min\{x_0(n)>0;\ t_n(x_0(n))=0\} \tag{1.3}$$

is indeed of interest.

The purpose of this paper is to establish estimates for the z_n's by applying certain inequalities (of Bernstein-type) for the C-norms of trigonometric polynomials and their derivatives. For certain sharper estimates (concerning constants), a theorem of M. Riesz and an adequate geometric construction, respectively, are used.

It should be noticed that the procedures described below may also be applied with obvious modifications in order to derive distance conditions for the zeros of cosine polynomials which are located in the neighbourhood of any positive (relative) maximum of the polynomial.

2. A review of some Bernstein-type inequalities. Let $n \in \mathbb{N}$ and a_0, a_k, $b_k \in \mathbb{R}$, $1 \leq k \leq n$; then the following classes of trigonometric polynomials are needed:

$$T_n := \{t_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)\},$$

$$T_n^+ := \{t_n \in T_n;\ t_n(x) \geq 0\},\quad NT_n^+ := \{t_n \in T_n^+;\ a_0 = 1\}.$$

For these polynomials the following general theorems have far-reaching consequences.

Theorem (S. N. Bernstein [1]). *If* $t_n \in T_n$, *then*

$$\|t_n^{(r)}\| \leq n^r \|t_n\|,\quad r \in \mathbb{N}. \tag{2.1}$$

In view of the extremal function $t_n(x) = \sin nx$, the constant 1 on the right-hand side of (2.1) is optimal. For different proofs and further literature see e. g. [15, footnote p. 356], [2], [11, p. 39], [7, p. 91, 228], [14, p. 90, 287, Ex. 82].

Theorem (L. Fejér [10]). *If* $t_n \in NT_n^+$, *then*

$$\|t_n\| \leq n+1. \tag{2.2}$$

Again the inequality is sharp; this is due to Fejér's kernel (which indeed is a peaking kernel), i. e.

$$F_n(x) = \frac{1}{n+1}\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2 = 1 + 2\sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)\cos kx \geq 0,\quad F_n(0) = n+1;$$

see also [14, p. 83, Ex. 50], [5, p. 84, Probl. 1.6.4].

Theorem (M. Riesz [15]). *If* $t_n \in T_n$ *and* $h < \pi/n$, *then*

$$\|t_n'\| \leq n\,\|t_n(x+h) - t_n(x-h)\|/2\sin nh. \tag{2.3}$$

(In the original paper [15, p. 365] inequality (2.3) appears for $h = \pi/2n$ only; compare the literature following inequality (2.6).)

Corollary. *If* $t_n \in T_n^+$, *then*

$$\|t_n^{(r)}\| \leq n^r \|t_n\|/2,\quad r \in \mathbb{N}. \tag{2.4}$$

Proof. Setting $h = \pi/2n$ in (2.3) yields

$$\|t_n'\| \leq n\,\|t_n(x+\pi/2n) - t_n(x-\pi/2n)\|/2 \leq n\,\|t_n(x+\pi/2n)\|/2 = n\,\|t_n\|/2$$

in view of the positivity. Again there exists an extremal function, namely

$$t_n(x) = \frac{1}{2} + \frac{1}{2}\cos nx = \cos^2(nx/2) \geq 0. \quad \square \tag{2.5}$$

The improvement of the constant in the Bernstein inequality (2.1) to the desirable factor 2^{-r}, as a consequence of the positivity of $t_n \in T_n^+$, is limited to the first derivative ($r = 1$), as also follows by (2.5).

For the sake of completeness the general M. Riesz — S. N. Bernstein inequality has to be added. It is given by the following

Theorem. *Let* $t_n \in T_n$. *If* $\Delta_h^r t_n(x) = \Sigma_{k=0}^r (-1)^k \binom{r}{k} t_n(x-[r-2k]h)$, $r \in \mathbb{N}$ *and* $h < \pi/n$ *or* $h \leq \pi/2n$, *respectively, then*

$$\|t_n^{(r)}\| \leq (n/2 \sin nh)^r \|\Delta_h^r t_n\| \leq n^r \|t_n\|, \tag{2.6}$$

respectively; the inequalities being sharp.

For proofs, extremal functions, and the literature we refer, in particular, to [19, p. 213 (17), p. 214 (18)]; moreover, see also [12, p. 259 (3. 5. 28)] and the literature cited there as well as [3; 13; 18]. It should be noticed that — strange to say — the name of M. Riesz is not mentioned in any of these sources!

Theorem (E. v. Egerváry — O. Szász [9]). *If* $t_n \in NT_n^+$, *then*

$$\|t_n'\| \leq \sqrt{\frac{n+1}{2}\binom{n+2}{3}} = \frac{n}{2}(n+1)\sqrt{\frac{n+2}{3n}}. \tag{2.7}$$

For an alternative proof (of this extraordinary, yet rather unnoticed inequality) [16, p. 278]; inequality (2.7) is again sharp in view of an extremal function [9, p. 652]. It is obviously better than the inequality $\|t_n'\| \leq n(n+1)/2$ which follows from (2.4), $r=1$, together with (2.2); however (2.4) only assumes $t_n \in T_n^+$.

3. Distance conditions via Bernstein-type inequalities. The inequalities of Sec. 2 now easily provide estimates for the first zero z_n of cosine polynomials as defined by (1.3). Here, Π_n denotes the class of all polynomials satisfying (1.1) and (1.2); Π_n^+ is used if $t_n(x) \leq 0$.

Proposition. (i) *If* $t_n \in \Pi_n$, *there holds the global distance condition*

$$z_n \geq \sqrt{2}/n; \tag{3.1}$$

(ii) if $t_n \in \Pi_n^+$, *then globally*

$$z_n \geq 2/n; \tag{3.2}$$

(iii) *if* $t_n \in \Pi_n^+$, *one has the individual estimate*

$$z_n \geq \frac{2\sqrt{3}}{n} \frac{t_n(0)}{n+1} \sqrt{\frac{n}{n+2}}. \tag{3.3}$$

Proof. (i) For z_n of (1.3) a 3-term Taylor formula for $x=0$ gives

$$0 = t_n(z_n) = t_n(0) + z_n t_n'(0) + z_n^2 t_n''(x_0)/2, \quad 0 < x_0 < z_n.$$

So in view of (1.2) and $t_n'(0)=0$,

$$t_n''(x_0) = -2\|t_n\|/z_n^2. \tag{3.4}$$

By Bernstein's inequality (2.1) for $r=2$ there results $|t_n''(x_0)| = 2\|t_n\|/z_n^2 \leq \|t_n''\| \leq n^2\|t_n\|$, yielding (3.1).

(ii) The mean value theorem when applied to t_n reads

$$t_n'(x_0) = (t_n(z_n) - t_n(0))/(z_n - 0) = -\|t_n\|/z_n, \quad 0 < x_0 < z_n. \tag{3.5}$$

The M. Riesz inequality (2.4) for $r=1$ then gives $|t_n'(x_0)| = \|t_n\|/z_n \leq \|t_n'\| \leq n\|t_n\|/2$, proving (3.2). (For these two proofs t_n needs not be normalized!)

(iii) From (3.5) it is inferred that $t_n'(x_0) = -t_n(0)/z_n$, $0 < x_0 < z_n$; applying the Egerváry-Szász inequality (2.7) gives

$$t_n(0)/z_n \leq \| t'_n \| \leq n(n+1)((n+2)/3n)^{1/2}/2$$

from which there follows (3.3). □

Remark. In case (i) the mean value theorem would give merely $z_n \geq 1/n$; concerning case (ii), a 3-term Taylor formula produces the same result; as to case (iii), the Taylor formula would result in an estimate worse than (3.3)!

4. Distance conditions via a theorem of M. Riesz. An improvement of the constants which appear in (3.1), (3.2) will be provided by the following "curve fitting"

Theorem (M. Riesz [15, p. 363 f]). *Let $t_n \in T_n$ satisfy $\|t_n\| \leq M$ together with $t_n(\xi) = M$. If $t_n(x) \neq M \cos n(x-\xi)$, then for all $x \in (\xi - \pi/n, \xi + \pi/n)$, $x \neq \xi$, it holds that*

$$t_n(x) > M \cos n(x-\xi). \tag{4.1}$$

An almost immediate consequence is

Proposition. (i) *If $t_n \in \Pi_n$, then in general*

$$z_n \geqq \pi/2n; \tag{4.2}$$

(ii) if $t_n \in \Pi_n^+$, *then the constant is doubled according to*

$$z_n \geq \pi/n. \tag{4.3}$$

Proof. (i) $t_n \in \Pi_n$ implies $M = \|t_n\| = t_n(0) > 0$ and $\xi = 0$ in (4.1), i. e. $t_n(x) > t_n(0) \cos nx$, $-\pi/n < x < \pi/n$; moreover, the right-hand side is positive (merely) for $x \in (-\pi/2n, \pi/2n)$. This proves (4.2).

(ii) Let $t_n \in \Pi_n^+$, thus $M^+ := \|t_n\| = t_n(0) > 0$, $\xi = 0$. Define

$$q_n(x) := t_n(x) - M^+/2, \tag{4.4}$$

then $q_n \in T_n$ and $\|q_n\| = q_n(0) = M^+/2 > 0$. Now, (4.1) when applied to (4.4) yields $t_n(x) - M^+/2 > (M^+/2) \cos nx, (-\pi/n, \pi/n)$ or, equivalently, $t_n(x) > (M^+/2)(1 + \cos nx)$, $(-\pi/n, \pi/n)$ where the right-hand side is positive for all x as indicated; this proves (4.3). □

5. Distance conditions via an osculating parabola. Whereas the Bernstein-type inequalities give at once rough estimates for the smallest zero z_n of $t_n \in \Pi_n$ or Π_n^+, respectively, which, in turn, are partly improved by Proposition (4.2), (4.3), another more refined geometrical approach will bring forth approximations having constants that are better in some cases, too.

To this end, the function

$$p_n(x) := \frac{1}{2} t''_n(0) x^2 + t_n(0) \tag{5.1}$$

involving $t_n(x)$ of (1.1) and satisfying $p_n^{(i)}(0) = t_n^{(i)}(0)$, $i = 1, 2, 3$, is introduced. Thus (5.1) is the osculating parabola to t_n, i. e., both p_n and t_n exhibit the same curvature at the origin. Then, considering the positive zero of (5.1), it is read off that $p_n(y_n) = 0 \Leftrightarrow y_n = \sqrt{-2t_n(0)/t''_n(0)}$. This enables one to establish the following

Proposition. *For $t_n \in \Pi_n$ satisfying the additional condition $\varrho_{k,n} \geq 0$, $1 \leq k \leq n$, it holds that*

$$z_n \geq \sqrt{-2t_n(0)/t''_n(0)} = \sqrt{(1 + 2 \sum_{k=1}^{n} \varrho_{k,n}) / \sum_{k=1}^{n} k^2 \varrho_{k,n}}\,. \tag{5.2}$$

Proof. It remains to show that $z_n \geq y_n$. Considering (3.4) for the left-hand side below, then

$$z_n^2 = t_n(0) / \sum_{k=1}^{n} k^2 \varrho_{k,n} \cos kx_0 \geq t_n(0) / \sum_{k=1}^{n} k^2 \varrho_{k,n} = y_n^2, \quad 0 < x_0 < z_n,$$

if and only if $\sum_{k=1}^{n} k^2 \varrho_{k,n} \cos kx_0 \leq k^2 \varrho_{k,n}$, $0 < x_0 < z_n$; but this is true in view of $\varrho_{k,n} \geq 0$, $1 \leq k \leq n$ (which, for many concrete applications, is not a very restrictive condition). □

Kernel Type	Dirichlet Π_n	Fejér Π_n^+	Rogosinski Π_n	Fejér-Korovkin Π_n^+	de la Vallée Poussin Π_n^+
$\varrho_{k,n}$	1	(2.4)	$\cos\frac{k\pi}{2n+1}$	(6.1)	(6.2)
$t_n(0)$	$2n+1$ $\simeq 2n$	$n+1$ $\simeq n$	$1/\sin\frac{\pi/2}{2n+1}$ $\simeq (4/\pi)n$	$\frac{2}{n+2}\cot^2\frac{\pi/2}{n+2}$ $=(8/\pi^2)n$	$\frac{(n!)^2 2^{2n}}{(2n)!}$ $\simeq \sqrt{\pi}\sqrt{n}$
z_n	$\frac{2\pi}{2n+1}$ $\simeq \pi/n$ 3.1416	$\frac{2\pi}{n+1}$ $\simeq 2\pi/n$ 6.2832	$\frac{3\pi}{2n+1}$ $\simeq (3\pi/2)/n$ 4.7124	$\frac{3\pi}{n+2}$ $\simeq 3\pi/n$ 9.4225	π multiplicity $2n$
(3.1)	$\sqrt{2}/n$ 1.4142	./.	$\sqrt{2}/n$ 1.4142	./.	./.
(3.2)	./.	$2/n$ 2.0000	./.	$2/n$ 2.0000	$2/n$ ./.
(3.3)	./.	$\frac{2\sqrt{3}}{n}\sqrt{\frac{n}{n+2}}$ $\simeq 2\sqrt{3}/n$ 3.4641	./.	. . . $\simeq\frac{16\sqrt{3}}{\pi^2}\frac{1}{n}$ 2.8079	. . . $\simeq\frac{2\sqrt{3\pi}}{n^{3/2}}$ ./.
(4.2)	$(\pi/2)/n$ 1.5708	./.	$(\pi/2)/n$ 1.5708	./.	./.
(4.3)	./.	π/n 3.1416	./.	π/n 3.1416	π/n ./.
(5.2)	$\frac{\sqrt{6}}{n}\sqrt{\frac{n}{n+2}}$ $\simeq\sqrt{6}/n$ 2.4496	$\frac{2\sqrt{3}}{n}\sqrt{\frac{n}{n+2}}$ $\simeq 2\sqrt{3}/n$ 3.4641	. . . $\simeq\sqrt{\frac{2\pi^2}{\pi^2-8}}\frac{1}{n}$ 3.2493	. . . $\simeq\frac{2\pi}{\sqrt{\pi^2-6}}\frac{1}{n}$ 3.1941	. . . $\simeq 2/\sqrt{n}$ ./.

A similar method using a fourth degree parabola may be applied to $t_n \in \Pi_n^+$ where z_n is then at least of multiplicity 2; hewever, the calculations involved become more intricate.

6. Check-out. We conclude with a comparison of the accuracy of the different distance conditions by means of selected polynomials, in particular of well-known kernels of approximation theory, the zeros of which happen to be known explicitly. These results are collected in the Table (see p. 95).

Remarks. The global estimates (3.1), (3.2), (4.2), (4.3) as well as the distance conditions (3.3), (5.2) depending upon the individual polynomial give the exact order of the first zero with varying constants; the interpretation of their sharpness is obvious. However, a striking "counter"-example is the kernel (Example 5) of de La Vallée Poussin revealing certain restrictions of the methods of Sec. 3/5. — For the definition of the kernels of the Table, in particular, for their closed representations, enabling the exact determination of the corresponding z_n's see [5, p. 517 f.] —

$$\varrho_{k,n}(\boldsymbol{K})=\left(1-\frac{k}{n+2}\right)\cos\frac{k\pi}{n+2}+\frac{1}{n+2}\cot\frac{\pi}{n+2}\sin\frac{k\pi}{n+2};$$

$$\varrho_{k,n}(\boldsymbol{V})=(n!)^2/(n-k)!\,(n+k)!.$$

Acknowledgement. The authors would like to thank Dipl. — Math. G. Wilmes for a profound discussion concerning the material of Sec. 2, in particular in connection with (2.4) as well as Professor P. L. Butzer for supporting the preparation of this note and for a critical reading of the manuscript. — The contribution of the first named author was supported by a DFG grant under Bu 166/27 which is gratefully noted.

REFERENCES

1. S. N. Bernstein. Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné. *Mém. Cl. Sci. Acad. Roy. Belg.*, (2), **4**, 1912, 1—104.
2. S. N. Bernstein. Démonstration nouvelle d'une inégalité relative aux polynômes trigonométriques. *Atti Accad. Zincei Naz., Cl. Sci. Fis. Mat. Natur.* 5, 1927, 558—561.
3. С. Н. Бернштейн. Распространение неравенства С. Б. Стечкина на целые функции конечной степени. *Доклады АН СССР*, **60**, 1948, 1487—1490.
4. G. Bleimann, E. L. Stark. The fine structure of periodic approximate identities. In: Proceedings of the Colloquium on Fourier-Analysis and Approximation Theory, Budapest, 16—21 Aug., 1976 (in print).
5. P. L. Butzer, R. J. Nessel. Fourier Analysis and Approximation. I. One-Dimensional Theory. Basel, 1971.
6. P. L. Butzer, R. J. Nessel, K. Scherer. Trigonometric convolution operators with kernels having alternating signs and their degree of convergence. *Jber. Deutsch. Math.-Verein.*, **70**, 1967, 86—99.
7. E. W. Cheney. Introduction to Approximation Theory. New York, 1966.
8. E. W. Cheney, Th. J. Rivlin. Some polynomial approximation operators. *Math. Z.*, **145**, 1975, 33—42.
9. E. V. Egerváry, O. Szász. Einige Extremalprobleme im Bereich der trigonometrischen Polynome. *Math. Z.*, **27**, 1928, 641—652.
10. L. Fejér. Sur les polynômes harmoniques quelconques. *C. R. Acad. Sci. Paris*, **157**, 1913. 506—509.

11. G. G. Lorentz. Approximation of Functions. New York, 1966.
12. D. S. Mitrinović. Analytic Inequalities. Berlin-Heidelberg-New York, 1970.
13. С. М. Никольский. Обобщение одного неравенства С. Н. Бернштейна. *Доклады АН СССР*, **60**, 1948, 1507—1510.
14. G. Pólya, G. Szegö. Aufgaben und Lehrsätze aus der Analysis, II. Berlin-Heidelberg-New York, 1971.
15. M. Riesz. Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome. *Jber. Deutsch. Math.-Verein.* **23**, 1914, 354—368.
16. W. W. Rogosinski. On non-negative polynomials. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **3—4**, 1960/61, 253—280.
17. E. L. Stark. Über trigonometrische singuläre Faltungsintegrale mit Kernen endlicher Oszillation. Dissertation, RWTH Aachen, 1970.
18. С. Б. Стечкин. Обобщение некоторых неравенств С. Н. Бернштейна. *Доклады АН СССР* **60**, 1948, 1511—1514.
19. А. Ф. Тиман. Теория приближения функций действительного переменного. Москва, 1960. (A. F. Timan. Theory of approximation of functions of a real variable. Oxford. 1963).

Lehrstuhl A für Mathematik
Rheinisch-Westfälische
Technische Hochschule Aachen
Templergraben 55, 5100 Aachen, BRD

Received 31. 12. 1977