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ALGEBRAIC THEORY OF STOCHASTIC AUTOMATA — A CATEGORICAL APPROACH

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Two categories of stochastic (S-) automata $\mathbf{A^S}$ with morphisms — the homomorphisms of S-automata and $\mathfrak{A}^{\mathbf{S}}$ with morphisms — the S-automata are constructed. The special properties of the categories $\mathbf{A^S}$ and $\mathfrak{A}^{\mathbf{S}}$: special objects, special morphisms, limit constructions and certain connections between $\mathbf{A^S}$ and $\mathbf{A^D}$, the category of deterministic automata, are studied It is proved that the functor L from $\mathbf{A^S}$ to $\mathbf{A^D}$ has a left adjoint. The last part of the paper treats certain technically important connections of S-automata, their properties, their relations with certain universal constructions in $\mathbf{A^S}$, the structures of bicategory, of monoidal category and double category which are induced and their categorical interpretation in $\mathbf{A^S}$ and $\mathfrak{A}^{\mathbf{S}}$.

The mathematical theory of automata, related to Logic, Algebra, Analysis and other important fields of Mathematics, has developed rapidly in recent decades.

In particular the algebraic theory of automata, related to some classical algebraic structures (semigroups, groups, vector spaces, etc.) has reached a high level of developement (see [2] and [8]) and completed one stage.

The pure algebraic approach could not however exhaust all the multitude of automata results (mathematical machines) present. Due to this the introduction of the categorical approach in the mathematical theory of automata is a natural process. Effectively the Category Theory gives the basis for a unified building of Mathematics and for a natural unification of different mathematical methods on the subject of research.

From the first papers [3] and [4], devoted to this problem, important results have been achieved up till now. The inner structure of the class of automata and their morphisms, and the study of automata as algebras in a given, arbitrary in general, category, being the two clearly distinct currents in the categorical theory of automata, are closely analyzed in our surveys [14] and [15].

While the problem of deterministic finite automata is solved in general (see [1; 3; 6; 7]), it is not the same with the stochastic automata. Besides the autor's studies (see for ex. [11; 12; 15]), some results are known, which are cited in [1; 3; 6].

The aim of this paper is to construct the categories of stochastic automata, to study the principal inner properties of these categories and to apply the results to the compositions of stochastic automata.

The terminology and the notations are as in [2] and [8] for the mathematical automata, as in [10] for stochastic automata and as in [9] for Categories, with some complements, specified in the text.

1. Stochastic automata — fundamental concepts. Stochastic automata (abbreviated S-automata) have been introduced in 1948 by Shannon and Weaver for the needs of the information theory, as models of information channels. As an independent part of the algebraic theory of mathematical automata, with their own subject and methods, S-automata began to develop in the early 60-ies. They seem to have been "re-invented" in the field of the general theory of automata, as a natural generalization for deterministic automata of different types.

We shall recall some basic definitions of the classical well developed theory of the S-automata, referring to [10].

Let P_1 and P_2 be numerable sets, and $[0, 1]$ — the closed interval of R.

Definition 1. *A mapping* $h: P_1 \times P_2 \to [0, 1]$ *is a stochastic mapping if for each element* $p \in P_1$ *the following condition* $\Sigma_{p' \in P_2} h(p, p') = 1$ *is satisfied.*

Every stochastic mapping can be represented by a $|P_1| \times |P_2|$ — matrix M^h, whose (i, j)-th element is $m^h_{ij} = h(p_i, p'_j)$.

With the aid of the so introduced notion of stochastic mapping and using analogies with deterministic automata, we can give a formal definition of the intuitive notion of a stochastic automata:

Definition 2. *An S-automaton is a quadruple* $A = (X, Q, Y, \{A(y/x)\})$, *where* X, Q, Y *are finite or numerable sets, called inputs (internal) states and outputs respectively, and* $\{A(y/x)\}$ *is the set of* $|X| \times |Y|$ *square matrices of order* $|Q|$ *such that* $a_{ij}(y/x) \geqq 0$ *for all i and j and* $\Sigma_{y \in Y} \Sigma_{j=1}^{|Q|} a_{ij}(y/x) = 1$, *where* $A(y/x) = \| a_{ij}(y/x) \|$.

We shall henceforth use for the stochastic mapping h the notation $h_s: P_1 \to P_2$ such that $h_s(p_1) = (h(p_1, p_{2i}), p_{2i})$, where $p_1 \in P_1$ and $p_{2i} \in P_2$. Obviously h_s is not a mapping but it is determined in the unique way by $h: P_1 \times P_2 \to [0, 1]$ and therefore we shall generally omit the index s.

Now let us consider the quadruple (X, Q, Y, h), where X, Q, Y are the sets mentioned in Def. 2, and $h: X \times Q \times Q \times Y \to [0, 1]$ is the stochastic mapping in the form $h = h_s: P_1 \to P_2$ for $P_1 = X \times Q$ and $P_2 = Q \times Y$. Then the above quadruple defines an S-automaton in an equivalent but not identical with this of Def. 2 way. The verification of the equivalence could be done directly.

One can see by the definition itself that the differences from deterministic automata are due to the imposed conditions on the mapping h. This gives way to use the approach and terminology of [2] and to introduce different types of stochastic automata as well as to construct their classification. Without entering into details, we shall only recall the definitions of the most familiar and special kinds of automata, namely those of Moore and Mealy, keeping the notations of [2] and [10].

Definition 3. *A Mealy-type S-automaton is a quintuple* $(X, Q, Y, \delta, \lambda)$, *where* X, Q, Y *are as in Def. 2, and* $\delta: X \times Q \times Q \to [0, 1]$ *and* $\lambda: X \times Q \times Y \to [0, 1]$ *are stochastic mappings. — A Moore-type S-automaton is a quintuple* (X, Q, Y, δ, β), *where* X, Q, Y, δ *are as above and* $\beta: Q \to Y$ *is a deterministic automata mapping from* Q *to* Y.

The relation of these notions between themselves as well as to the corresponding deterministic automata, is evident.

In practical and theoretical studies in the theory of automata one has often to choose initial inner states depending on certain probabilistic conditions (initial distribution of probabilities). Thus, an **initial stochastic automaton** is the ordered pair (A, π), where A is an S-automaton in the sense of Def. 2, and the mapping $\pi: \{1\}\times Q \to [0, 1]$ is a stochastic one, called **initial distribution**. The mapping π is called **strict initial distribution** when $\pi(1, p)=1$ for an arbitrary value $(1, p)$ of the argument $(p \in Q)$.

Let us denote by beh_A^π the **behaviour** (see [10]) of a stochastic automaton A with initial distribution π, and let A_1 and A_2 be two S-automata with identical sets of inputs and outputs ($X_1=X_2$ and $Y_1=Y_2$). Two initial S-automata (A_1, π_1) and (A_2, π_2) are **equivalent** if $\mathrm{beh}_{A_1}^{\pi_1}=\mathrm{beh}_{A_2}^{\pi_2}$, where $\pi_1: \{1\}\times Q_1 \to [0, 1]$ and $\pi_2: \{1\}\times Q_2 \to [1, 1]$ are initial distributions. If the set of behaviours of the S-automaton A_1 is a subset of the corresponding set of the S-automaton A_2, we can say that the S-automaton A_2 **covers** the S-automaton A_1 and write for this relation $A_2 \geq A_1$.

A number of theoretical and applied problems in the theory of S-automata may be posed. The most important are:

a) among the initial S-automata equivalent in the sense of [10], to a given initial S-automaton to find out an initial S-automaton with minimal number of states;

b) among the S-automata, equivalent (resp. strict equivalent) to a given S-automaton, to find out a minimal S-automaton;

c) for a given S-automaton A to find out an S-automaton with fewer number of states and satisfying the condition $A' \geq A (A' \leq A$ respectively).

While the second problem is always decidable (see [10]), neither the theory, nor the constructive algorithms are advanced enough to solve the first or the third problem.

Moreover, a number of problems for the automata theory cannot be considered as sufficiently solved for the case of S-automata. A special attention has to be paid to the problems of inter-relations of stochastic, deterministic, relational and partially defined (partial) automata; to the problems of minimization, of decomposition and connection of automata and the algebraic structures generated by these connections as composition laws in certain sets of automata.

2. Categories of stochastic automata. Let us denote by $\mathbf{Set_o}$ the universal set (universuum) and let $\mathbf{Set}$ be the corresponding full category of mappings. We write $\mathbf{H_o}$ for the set of the objects (identities) of a given category $\mathbf{H}$, $\mathbf{H^m}$ for the subcategory of the monomorphisms of $\mathbf{H}$, $\mathbf{H^e}$ for the subcategory of the epimorphisms of $\mathbf{H}$ and $\mathbf{H^b}$ for the subcategory of the bimorphisms of $\mathbf{H}$, supposing that $\mathbf{H}$ is a small category [9]. Accordingly to the general use, we denote by $\mathbf{Vct}$ the category of all vector spaces and by $\mathbf{Vct_o}$ the set of all vector spaces over $\mathbf{Set_o}$, identified with the set of the identical homomorphisms of vector spaces (identical linear transformations).

Let $\mathbf{K}$ be a category with two objects: $\mathbf{K_o}=\{e, e'\}$ and one morphism $f: e \to e'$. We call (accordingly with [19] the ordered pair $\mu_A=(K, \{f\})$ a **structural diagram of an automaton** (for short we shall henceforth speak of a skech).

Definition 4. *The functor F: $\mathbf{K} \to \mathbf{Vct}$ is an S-automaton if the following conditions are satisfied for F:*

$$F(e)=V(X\times Q),\ F(e')=V(Y\times Q),\ F(f)=h:\ V(X\times Q) \to V(Y\times Q),$$

where $V(X\times Q), V(Y\times Q) \in \mathbf{Vct_o}$ are free vector spaces with bases $X\times Q$,

*$Y \times Q \in$ **Set**$_0$ respectively and h is a morphism of* **Vct** *such that its restriction on $X \times Q$, i. e. the mapping $h/_{X \times Q}$: $X \times Q \to V(Y \times Q)$ is a stochastic automata mapping. An A-S-morphism is the natural transformation $t: F \dot{\to} F'$, where F and F' are two functors, defining S-automata.*

We identify the image of the category **K** by the functor-automaton F: **K** $\to$ **Vct** with the automaton $A=(X, Q\ Y, h)$, given by the traditional notations and put $F_h(\mathbf{K}) \cong A$. The natural transformation t defines a homomorphism of automata in the sense of [8]. The components of the homomorphism of automata coincide with the components of t.

The above definition of stochastic automaton and the notation of homomorphism from a stochastic automaton to another are identical to the classical one [10]. Let $A=(X, Q, Y, h)$ be an S-automaton defined by an input alphabet X, an output alphabet Y, a set of inner states Q and a stochastic automata mapping h; the ordered quadruple defines in a unique way a functor F: **K** $\to$ **Vct**, for which the conditions of Def. 4 are satisfied. Conversely, each functor of the form of this in Def. 4 defines the sets X, Q, Y having the meaning of input alphabet, inner states and output alphabet respectively, and a stochastic automata mapping, which, put together, define in a unique way an S-automaton. In particular, if h is a mapping with constant value 0 for all values of the argument except one, for which it takes the value 1, we obtain the notion of deterministic auotmaton. It is the same as representing h by a set of values $\{0, 1\}$ instead of by the interval $[0, 1]$. For the definition of such an automaton by a functor it is enough the realization of the skech μ_A to be made in the category **Set**, i. e. by a functor F: **K** $\to$ **Set**. Obviously one can speak of A-S-morphisms if the natural transformations defining them are between functors of the form F: **K** $\to$ **Vct**, and of A-D-morphisms if the corresponding natural transformations are between functors of the form F: **K** $\to$ **Set**.

Starting from Def. 4, one can construct two different categories of stochastic automata over the sets of **Set**$_0$: in the first case the morphisms of the category are the A-S-morphisms and the objects are identified with the identical A-S-morphisms, i. e. with the very S-automata, while in the second case the morphisms are S-automata, and the objects are special classes of stochastic automata.

Let us write **Vct**$^{\mathbf{K}}$ = **Funct(K, Vct)** for the category of functors from **K** to **Vct**, i. e. the category whose objects are the functors from **K** to **Vct** and whose morphisms are the natural transformations between them. The set $\mathbf{A^S}$ of the A-S-morphisms is a subset of **Vct**$^{\mathbf{K}}$.

Proposition 1. $\mathbf{A^S}$ *is a category (with respect to the induced from* **Vct**$^{\mathbf{K}}$ *into* $\mathbf{A^S}$ *composition law), called the category of the A-S-morphisms, whose morphisms are the A-S-morphisms and whose objects — the S-automata over elements of* **Set**$_0$, *identified with the identical A-S-morphisms.*

Let us denote by $\mathbf{A^D}$ the categoy of A-D-morphisms in which the morphisms are A-D-morphisms and the identical A-D-morphisms are the unities identified with the objects of the category [16].

Proposition 2. *For the categories* $\mathbf{A^D}$ *and* $\mathbf{A^S}$ *the inclusions of th following diagram take place:*

$$\mathbf{A^D} \xrightarrow{\;T\;} \mathbf{A^S} \xrightarrow{\;T'\;} \mathbf{Vct^K}, \qquad \mathbf{A^D} \xrightarrow{\;Q\;} \mathbf{Vct^K}$$

where the subcategories contain besides each morphism the related to it objects source and target.

Proof. The inclusion $\mathbf{A^S} \xrightarrow{T'} \mathbf{Vct^K}$ is a consequence of the construction of the A-S-morphisms as natural transformations between functors from $\mathbf{K}$ to $\mathbf{Vct}$; the second part of the statement results from Def. 4. The inclusion $A^{\mathbf{D}} \xrightarrow{T} A^{\mathbf{S}}$ comes from the fact that **Set** and $\mathbf{Set_o}$ could be regarded as a subcategory and a subset of **Vct** and $\mathbf{Vct_o}$ respectively: each set can be regarded as vector space with trivial, i. e. empty operations. The inclusion Q is obviously a result of the composition $T' \circ T$.

Corollary. $\mathbf{A^S}$ *is a category isomorphical to the subcategory of the natural transformations of functors from* $\mathbf{K}$ *to* $\mathbf{Vct}$, *satisfying the condition of stochasticity in Def. 4.*

Let us write $\mathfrak{A}^{\mathbf{S}}$ for the set of S-automata, subset of $\mathbf{Vct_o}$.

Proposition 3. $\mathfrak{A}^{\mathbf{S}}$ *is a category under the induced from* $\mathbf{Vct}$ *in* $\mathfrak{A}^{\mathbf{S}}$ *composition law. The morphisms of* $\mathfrak{A}^{\mathbf{S}}$ *are the S-automata over the elements of* $\mathbf{Set_o}$ *and the objects of* $\mathfrak{A}^{\mathbf{S}}$ *are identified with the S-automata of the form* $A_I=(Y, Q, Y, I)$, *left unity, and* ${}^{I}A=(X, Q, X, I)$, *right unity, for the S-automata-morphism* $A=(X, Q, Y, h)$. *Here we denote by* I *the identical mapping.*

The categorial properties of $\mathfrak{A}^{\mathbf{S}}$ do not present a real interest, particularly from the practical point of view. According to that we shall not treat this problem. Some other composition laws which may be defined in $\mathfrak{A}^{\mathbf{S}}$ allow a direct automata realization (even a technical one) and we shall return to them in 4. The relation existing between the constructions in $\mathfrak{A}^{\mathbf{S}}$ and $\mathbf{A^S}$ has a particular importance. By this relation some non standard constructions in $\mathfrak{A}^{\mathbf{S}}$ are treated as universal constructions in $\mathbf{A^S}$. This problem has its own special importance and will be the subject of a further paper.

3. Properties of the category $\mathbf{A^S}$. Some properties of $\mathbf{A^S}$ are of immediate interest for the theory of S-automata. By and by we shall discuss some special morphisms and objects of $\mathbf{A^S}$, the existence and contents of the limits in $\mathbf{A^S}$ in the automata meaning and the properties of some functors related to the category $\mathbf{A^S}$.

If F and F' are functors-automata and $t\ F \dot{\to} F'$ is a natural transformation between them, we shall denote by A and A' the correspondent automata and by $\bar{t}$: A$\to$A$'$ the correspondent to t A-S-morphism. Two statements concerning the monics and epics in the category $\mathbf{A^S}$ hold:

Proposition 4. *The morphism* $\bar{t}: A \to A' \in \mathbf{A^S}$ *is a monomorphism in* $\mathbf{A^S}$ *iff* $t: F \dot{\to} F' \in \mathbf{Vct^K}$ *is a monomorphism in* $\mathbf{Vct^K}$.

Proof. Let us suppose $\bar{t}$ being a monomorphism in $\mathbf{A^S}$. Then for each pair of A-S-morphisms $\bar{\sigma}_1, \bar{\sigma}_2: A \to A$, $\bar{t}\circ\bar{\sigma}_1=\bar{t}\circ\bar{\sigma}_2$ implies $\sigma_1=\sigma_2$. According to Def. 4 and to the rules for composition of natural transformation one have $\bar{t}\circ\bar{\sigma}_1=\bar{t}\circ\bar{\sigma}_2 \Longrightarrow t\circ\sigma_1=t\circ\sigma_2$ but by $\bar{\sigma}_1=\bar{\sigma}_2$ it follows $\sigma_1=\sigma_2$, and by the two above implications we obtain the result:

$$\tau\circ\sigma_1=\tau\circ\sigma_2 \Longrightarrow \sigma_1=\sigma_2.$$

Conversely, let t be a monomorphism in $\mathbf{Vct^K}$; then for each pair of morphisms σ_1, σ_2: $F \dot{\to} F \in \mathbf{Vct^K}$ such that $t\circ\sigma_1=t\circ\sigma_2$ one have $\sigma_1=\sigma_2$. By this and by Def. 4 one obtains the implication $\bar{t}\circ\bar{\sigma}_1=\bar{t}\circ\bar{\sigma}_2\Rightarrow\bar{\sigma}_1=\bar{\sigma}_2$, i. e. $\bar{t}$ is a monomorphism in $\mathbf{A^S}$.

An automaton $A\in\mathbf{A^S_o}$ is a subautomaton of A' if the ordered pair $(\bar{\sigma}, A)$ is a subobject of A' in $\mathbf{A^S}$.

Let $\bar{\sigma}_1: A_1\to A$ and $\bar{\sigma}_2: A_2\to A$ be two subautomata of A. The intersection $A_3=A_1\cap A_2=(X_3, Q_3, Y_3, h_3)$, where $X_3=X_1\cap X_2$, $Q_3=Q_1=Q_2$, $Y_3=Y_1=Y_2$ and $h_3=h_1\cap h_2$, defines a subautomaton $h_3: A_3\to A$ for which the diagram of Fig. 2 is a universal cartesian square, i. e. defines a fibered product of the pair $(\bar{\sigma}_1, \bar{\sigma}_2)$.

Proposition 5. *The morphism* $\bar{\sigma}: A\to A'\in\mathbf{A^S}$ *is an epimorphism in* $\mathbf{A^S}$ *iff* $\sigma: F\to F'\in\mathbf{Vct^K}$ *is an epimorphism in* $\mathbf{Vct^K}$.

The proof is obtained by analogy to this of Proposition 4.

An automaton $A'\in\mathbf{A^S_o}$ is a factor-automaton of the automaton $A\in\mathbf{A^S_o}$ if the ordered pair $(A', \bar{\sigma})$ is a quotient-object of A in the category $\mathbf{A^S}$. By analogy with the case of subautomata and using the properties of the direct sum of automata in $\mathbf{A^S}$, one can construct a co-cartesian square in $\mathbf{A^S}$, i. e. a fibered sum of two morphisms.

It follows by Proposition 4 and Proposition 5.

Corollary. *The morphism* $\bar{\sigma}: A\to A'\in\mathbf{A^S}$ *is a bimorphism in* $\mathbf{A^S}$ *iff* $\sigma: F\to F'\in\mathbf{Vct^K}$ *is a bimorphism in* $\mathbf{Vct^K}$.

If, by the linear mapping $h\in\mathbf{Vct}$, the bases vectors of the vector space-product, source of h, are transformed in the bases vectors of the vector space — product, target of h, one reaches the case of deterministic automata and simultaneously the well known ([6; 7]) special morphisms in the category $\mathbf{A^D}$ of A-D-morphisms.

The existence of different types of limits in $\mathbf{A^S}$ gives a categorical characterization for the S-automata as well as for the A-S-morphisms. A number of results, similar to some groups of results for D-automata hold.

Theorem 1. *The category* $\mathbf{A^S}$ *is a category with finite products and for each pair of A-S-morphisms with common source and target there exist an equalizer.*

Proof. The existence of finite products in $\mathbf{A^S}$ results from their existence in $\mathbf{Vct}$ and from the fact that $\mathbf{A^S}$ is a saturated subcategory of $\mathbf{Vct^K}$. — Each category $\mathbf{H^P}$, where $\mathbf{P}$ is a small category and $\mathbf{H}$ allows an equalizer for each of the pairs of morphisms with common source and target, also allows an equalizer for each such a pair. $\mathbf{Vct}$ is a category with equalizers and $\mathbf{A^S}$ is full in $\mathbf{Vct^K}$ and with finite products subcategory of $\mathbf{Vct^K}$, $\mathbf{A^S}$ allows an equalizer for each pair of morphisms with common source and target.

Corollary (see [7]). *The category* $\mathbf{A^D}$ *is one with finite products and with equalizers for each pair of morphisms with common source and target.*

The automata meaning of the categorical product of automata in $\mathbf{A^S}$ a parallel simultaneous work of automata with separated inputs (components of the product).

A dual to Theorem 1 is the following:

Theorem 2. *There are in the category* $\mathbf{A^S}$ *an initial object.*

Proof. According to [9] there exists in $\mathbf{Vct}$ an initial object which belongs to the such subcategory of $\mathbf{Vct^K}$, which is isomorphical to the subcategory $\mathbf{A^S}$ of $\mathbf{Vct^K}$. According to the general theory (see [9] and [17]) it is an initial object. If for **Vct** the object is also a final one, i. e. there exists a null object in **Vct**, for $\mathbf{Vct}^{\mathbf{K}}$ and for $\mathbf{A}^{\mathbf{S}}$ this object is however only an initial one.

The initial object of the category $\mathbf{A}^{\mathbf{S}}$ is an S-automaton $I_A=(\emptyset, \emptyset, \emptyset, \emptyset)$ (empty automaton).

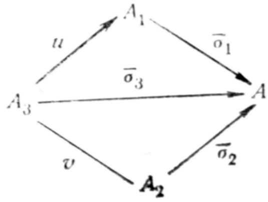

Fig. 2

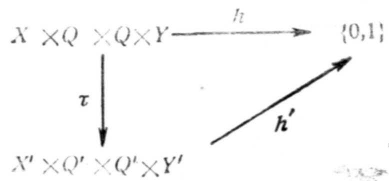

Fig. 3

The relation between the category $\mathbf{A}^{\mathbf{D}}$ of A-D-morphisms and the category $\mathbf{A}^{\mathbf{S}}$ of the A-S-morphisms assumes a clear categorical interpretation by the use of appropriate adjoint functors.

Let the forgetful functor $p^{\mathbf{K}}_{\mathbf{Vct}}$: $\mathbf{Vct}^{\mathbf{K}} \to \mathbf{Set}^{\mathbf{K}}$ be given and let us denote by $\bar{P}^{\mathbf{K}}$: $\mathbf{Set}^{\mathbf{K}} \to \mathbf{Vct}^{\mathbf{K}}$ its left adjoint functor, whose existence and way of construction are stated in [9]. Let also the functor V: $\mathbf{A}^{\mathbf{D}} \to \mathbf{Set}^{\mathbf{K}}$ be given, which is constructed so that if $A=(X, Q, Y, h) \in \mathbf{A}^{\mathbf{D}}_{\mathbf{o}}$, then $V(A)=h{:}X\times Q\to Q\times Y \in \mathbf{Set}^{\mathbf{K}}_{\mathbf{o}}$ and if $\bar{\sigma}$: $A \to A' \in \mathbf{A}^{\mathbf{D}}$, then $V(\bar{\sigma})=\sigma \in \mathbf{Set}^{\mathbf{K}}$ with σ: $(X\times Q \xrightarrow{h} Q\times Y) \dot{\to} (X'\times Q' \xrightarrow{h'} Q'\times Y')$ so that the diagram of Fig. 3 be commutative. Of course, always when the composition of $\bar{\sigma}_1$ and $\bar{\sigma}_2$ is defined in $\mathbf{A}^{\mathbf{D}}$, the following expression holds: $V(\bar{\sigma}_1 \circ \bar{\sigma}_2)=\sigma_1 \circ \sigma_2 = V(\bar{\sigma}_1) \circ V(\bar{\sigma}_2)$.

Let now $A=(X, Q, Y, h)$ be an S-automaton, and $\bar{\sigma}: A \to A'$ be an A-S-morphism. We define $\mathfrak{U}$: $\mathbf{A}^{\mathbf{S}} \to \mathbf{Vct}^{\mathbf{K}}$ as follows:

$$\mathfrak{U}(A)=h:\ V(X\times Q) \to V(Y\times Q) \in \mathbf{Vct}^{\mathbf{K}}_{\mathbf{o}}, \quad \mathfrak{U}(\bar{\sigma})=\sigma \in \mathbf{Vct}^{\mathbf{K}}.$$

Here $V(X\times Q)$, $V(Y\times Q)$ are free vector spaces with bases $X\times Q$, $Y\times Q$ respectively and $\sigma:(V(X\times Y) \xrightarrow{h} V(Y\times Q)) \dot{\to} (V(X'\times Q') \xrightarrow{h'} V(Y'\times Q'))$ is a natural transformation making the diagram of Fig. 4 commutative. The statement $\mathfrak{U}(\bar{\sigma}_1 \circ \bar{\sigma}_2)=\sigma_1 \circ \sigma_2 = \mathfrak{U}(\bar{\sigma}_1) \circ \mathfrak{U}(\bar{\sigma}_2)$, always when the composition of $\bar{\sigma}_1$ and $\bar{\sigma}_2$ is defined in $\mathbf{A}^{\mathbf{S}}$, holds.

By a certain analogy with the ordered pair of adjoint functors $p^{\mathbf{K}}_{\mathbf{Vct}}, \bar{\mathbf{P}}^{\mathbf{K}}$ we can construct the pair of functors $\mathfrak{L}, \mathfrak{G}$ as follows: Let $\mathfrak{L}$: $\mathbf{A}^{\mathbf{S}} \to \mathbf{A}^{\mathbf{D}}$ be defined so that $F_h(K) \cong A=(X, Q, Y, h) \in \mathbf{A}^{\mathbf{S}}_{\mathbf{o}}$, h: $V(X\times Q) \to V(Y\times Q) \in \mathbf{Vct}^{\mathbf{K}}_{\mathbf{o}}$. $\mathfrak{L}(F_h(K))=F_{\bar{h}}(K)$, $\bar{h}$: $V(X\times Q) \to V(Y\times Q) \in \mathbf{Set}_{\mathbf{o}}$, i. e. juxtaposing to each vector space its correspondent set.

Let us in the end consider also the functor $\mathfrak{G}$: $\mathbf{A}^{\mathbf{D}} \to \mathbf{A}^{\mathbf{S}}$, defined so that $\mathfrak{G}(\bar{A})=A$, where $\bar{A}=(X, Q, Y, \bar{h}) \in \mathbf{A}^{\mathbf{D}}_{\mathbf{o}}$, $X, Y, Q \in \mathbf{Set}_{\mathbf{o}}$, and $\bar{h}$: $X\times Q \to Q \times Y \in \mathbf{Set}^{\mathbf{K}}_{\mathbf{o}}$, $A=(X, Q, Y, h) \in A^{\mathbf{S}}_{\mathbf{o}}$, h: $V(X\times Q) \to V(Y\times Q) \in \mathbf{Vct}^{\mathbf{K}}_{\mathbf{o}}$.

As the diagram of Fig. 5 shaws, the functors $\mathfrak{G}, \mathfrak{L}, p^{\mathbf{K}}_{\mathbf{Vct}}, \bar{\mathbf{P}}^{\mathbf{K}}$, V and $\mathfrak{U}$ satisfy the condition $p^{\mathbf{K}}_{\mathbf{Vct}} \circ \mathfrak{U} = V \circ \mathfrak{L}$.

To state the basic result related to the so constructed functors, namely the existence of an adjoint functor, we shall first prove some auxiliary results.

Proposition 6. *The functor* $\mathfrak{L}$: $\mathbf{A^S} \to \mathbf{A^D}$ *is a forgetful functor.*

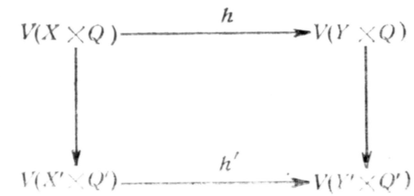

Fig. 4

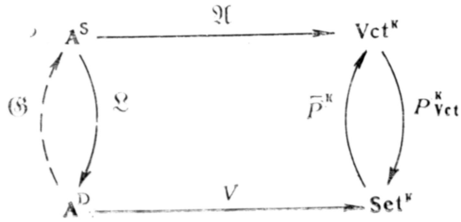

Fig. 5

Proof. According to the way of its construction, the functor $p^{\mathbf{K}}_{\mathbf{Vct}}$ is a forgetful functor and $\mathfrak{L}$, according to its definition, is a restriction of the first over a subcategory of $\mathbf{Vct^K}$, namely $\mathbf{A^S}$. But the restriction of a forgetful functor of some algebraic structure is also a forgetful one.

Corollary. *The functor* $\mathfrak{L}$: $\mathbf{A^S} \to \mathbf{A^D}$ *maintains the projective limits and is a generative one for the category* $\mathbf{A^D}$.

The proof results from the fact that $\mathfrak{L}$ is a restriction of $p^{\mathbf{K}}_{\mathbf{Vct}}$ over $\mathbf{A^S} \subset \mathbf{Vct^K}$, which is a right adjoint for the functor $\bar{\mathbf{P}}^{\mathbf{K}}$, and from the definitions and the general theorems of [5].

Now we can come up to the general result.

Theorem 3. *There is a left adjoint* $\mathfrak{G}$: $\mathbf{A^D} \to \mathbf{A^S}$ *to the functor* $\mathfrak{G}$. *The statement* $\mathfrak{U} \circ \mathfrak{G} = \bar{\mathbf{P}}^{\mathbf{K}} \circ V$ *holds.*

Proof. According to the above construction of $\mathfrak{L}$ and to Prop. 6, $\mathfrak{L}$ maintains the projective limits. By the second part of the Corollary to Proposition 6 it follows that $\mathfrak{L}$ satisfies the condition for the existence (see [9]) of a resolving set. Therefore for the functor $\mathfrak{L}$ all conditions for the holding of the Freyd's theorem for the existence of an adjoint functor [9] are satisfied. The second part of the statement follows immediately from the above constructed diagram of the functors (see Fig. 5).

Theorem 4. *Every S-automaton is a free structure for* $\mathfrak{L}$ *over an object of* $\mathbf{A^D}$.

The proof results from Theorem 3 accordingly with the equivalence of the existence of an adjoint functor and of free structures (see [5] and [9]).

4. Connections of automata in the categories $\mathbf{A^S}$ and $\mathfrak{A}^{\mathbf{S}}$. Let $A=(X, Q, Y, h)$ and $A'=(X', Q', Y', h')$ be two automata. Their composition in automata sense (see [8; 10]) can be done by several nonequivalent ways and described by constructions in the categories $\mathbf{A^S}$ and $\mathfrak{A}^{\mathbf{S}}$. The basic cases of compositions and their categorical representations are the subject of the present paragraph.

Let us consider the parallel connection of the automata A and A' with separated entries (Fig. 6 and [7; 8]). Then the automaton-composition is $A\times A'=(X\times X', Q\times Q', Y\times Y', h\times h')$, where we denote by the symbol "$\times$" the carthesian product of the sets and the linear mappings. The composition law in the set of the S-automata having been written the same way. One obtains the representing matrix for the automaton-composition $A \times A'$ as a direct product of the representing matrices for the automata-components A and A' (see [20]).

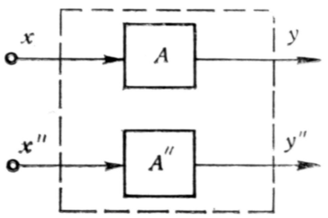

Fig. 6

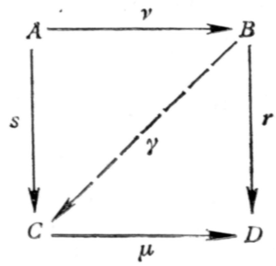

Fig. 7

The automaton $I = (\{1\}, \{1\}, \{1\}, i)$, with regard to the composition law $\times$, is a neutral element (unity) for every S-automaton, i. e. the law $\times$ allows a unique neutral element. One can easily establish that the algebraic structures $(\mathbf{A^S}, \times)$ and $(\mathfrak{A}^{\mathbf{S}}, \times)$ are semigroups with unity I, which are commutative up to isomorphism.

The so constructed composition gives way to the following interesting results.

Theorem 5. *The category* $\mathbf{A^S}$ *of the A-S-morphisms (respectively* $\mathfrak{A}^{\mathbf{S}}$ *of the S-automata) has the structure of a bicategory in the sense of* [17].

Proof. According to the theorem for the homomorphisms every A-S-morphism $f\colon A \to A'$ is representable in the from $f = g \circ h$, where $h\colon A \to A/\tilde{f}$ is an epimophism and $g\colon A/\tilde{f} \to A'$ is a monomorphism. It is obviously true, for $(A/\tilde{f}, h)$ is a factor-object for A in $\mathbf{A^S}$, and $(g, A/\tilde{f})$ is a subobject for A' in $\mathbf{A^S}$; here we have denoted by $\tilde{f}$ the equivalence, canonically generated by the corresponding to f mapping. For each commutative square (Fig. 7), for which $\nu \in \mathbf{A^S_e}$ and $\mu \in \mathbf{A^S_m}$ is diagonalizable, our statement results from the definitions [17].

One can also establish:

Theorem 6. *The ordered triple* $(\mathbf{A^S}, \times, I)$ *is a monoidal category.*

Proof. In fact, every monoid M, regarding as a discrete category, is a monoidal category. But $(\mathbf{A^S_o}, \times)$ is obviously a monoid with regard to the composition law $\times$.

One states analogically that the ordered triple $(\mathfrak{A}^{\mathbf{S}}, \times, I)$ is a monoidal category.

The sets $\mathfrak{A}^{\mathbf{S}}$ and $\mathbf{A^S_o}$, whose elements are the stochastic automata, can obviously be identified; to do this it suffices to juxtapose to the identical A-S-morphism of $\mathbf{A^S_o}$ its correspondent S-automaton. This gives way to the formulation of the following problem: to find out a construction in the category $\mathbf{A^S}$ whose result is an object of $\mathbf{A^S_o}$, identical or isomorphical to the composition of two S-automata in $\mathfrak{A}^{\mathbf{S}}$. Let $\Pi_2 = \Pi_{20} = \{c, c'\}$ be a discrete category and let F: $\Pi_2 \to \mathbf{A^S}$ be a functor from the category Π_2 to the category $\mathbf{A^S}$.

Theorem 7. *If $A, A' \in \mathbf{A^S_o} = \mathfrak{A}^{\mathbf{S}}$ and if $F(\Pi_2) = \{A, A'\}$ then $A \times A' \simeq \lim_{\leftarrow}(F)$ holds.*

Proof. It follows from the construction of Π_2 and from Th. 1 that $\lim_{\leftarrow}(F)$ exists in $\mathbf{A^S}$ and $\lim_{\leftarrow}(F) = A \times A'$; on the other hand, according to the con-

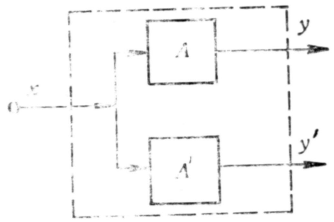

Fig. 8

Fig. 9

struction of the composition law $\times$ in $\mathfrak{A}^{\mathbf{S}}$, $A \times A' = (X \times X', Q \times Q', Y \times Y', h \times h')$. The equality is defined up to isomorphism according to the general properties of the limits [9].

Theorem 7 is evidently valid also for the subcategories $\mathbf{A^D}$ and $\mathfrak{A}^{\mathbf{D}}$ of $\mathbf{A^S}$ and $^{\mathbf{S}}$ respectively.

One can regard the parallel connection of automata with common entry as a special case of the parallel connection of automata with separated entries (Fig. 8). We shall use for this case the following notation $A \otimes A' = (X, Q \times Q', Y \times Y', h \otimes h')$, $X = X'$. In fact the input alphabet for the automaton-composition $A \otimes A'$ is the diagonal of the carthesian product $X \times X'$, i. e. the set $\Delta(X \times X) = \{(x, x);\ x \in X\}$, but the identification $(x, x) \simeq x$ is useful. One obtains the representing matrix for the automaton $A \otimes A'$ as a disjunctive union of the direct product of the matrices for the automata A and A', regarded as with independent entries [18].

Let us now consider the categorical interpretation of the well-known in the theory of automata ([7; 8]) sequential connection of two automata A and A' (see Fig. 9). We shall denote the automaton-composition $A' \circ A = (X, Q \times Q', Y', h' \circ h)$ with $X' = Y$.

Each automaton of the form $E = (M, \{1\}, M, i)$ with i being the identical mapping for M, which is composable with a given automaton with regard to the law $\circ$, is a neutral element (unity) under $\circ$. Of course, an automaton-left unity and an automaton-right unity correspond to each automaton; $(\mathfrak{A}^{\mathbf{S}}, \circ)$ is a category.

Theorem 8. *For the category $(\mathfrak{A}^{\mathbf{S}}, \circ)$ the following statements are valid:*

a) *The category $\mathfrak{A}^{\mathbf{S}}$ is one with finite products;*

b) *The category $\mathfrak{A}^{\mathbf{S}}$ is one with equalizers for each pair of morphisms with common source and target.*

By analogy with the statement of Th. 7, the following problem arises for the category $(\mathfrak{A}^{\mathbf{S}}, \circ)$: to find out a construction in the category $\mathbf{A^S}$, the result of whose applications is an object — S-automaton, which is identical or isomorphical to the sequential connection of two S-automata, i. e. to their composition under the law $\circ$ in $\mathfrak{A}^{\mathbf{S}}$. For that purpose we shall define in the category $\mathbf{A^S}$ a special projection as follows: let $A \times A'$ be a product in $\mathbf{A^S}$ of the two S-automata A and A' such that

$$A \times A' = (X \times X', Q \times Q', Y \times Y', h \times h');$$

the mapping p_V: $\mathbf{A^S_o} \to \mathbf{A^S_o}$ is a *V*-projection is $\mathbf{A^S}$ if

$$p_V(A \times A') = (X, Q \times Q', Y', h' \circ h).$$

The following result gives an essential relation between the categories $(\mathfrak{A}^S, \circ)$ and $\mathbf{A^S}$.

Theorem 9. *If* $A, A' \in \mathbf{A^S_o}$ *and if* $F(\Pi_2) = \{A, A'\}$, *then* $A' \circ A \cong p_V(\lim_{\leftarrow}(F))$ *holds.*

The proof of this statement is analogous to the proof of Th. 7. It is evident that a similar result is also valid for the subcategories $\mathbf{A^D}$ and $(\mathfrak{A}^D, \circ)$ of the categories $\mathbf{A^S}$ and $(\mathfrak{A}^S, \circ)$.

The relation between the composition laws $\times$ (resp. $\otimes$) and o is reflected in the following statement:

Theorem 10. *The oredered triple* $(\mathfrak{A}^S, \times, \circ)$ *is a double categoriy in Ch. Ehresmann's sense* [9]).

Proof. As was stated above, the ordered pairs $(\mathfrak{A}^S, \times)$ and $(\mathfrak{A}^S, \circ)$ are categories (the first is a monoid); therefore what is left to do is to establish the inter-relation between the laws $\times$ and o. For this purpose let us consider the *S*-automata A, B, C, K for which the following expression is valid:

$$(A \circ B) \times (C \circ K) = (A \times C) \circ (B \times K).$$

The expression can be directly verified by means of the schemes for connecting the automata (the immediate calculations are too roomy on accout of which we do not adduce them here).

Corollary. *The ordered triple* $(\mathfrak{A}^D, \times_D, \circ_D)$ *is a double category.*

The proof ensues from the fact that $\mathfrak{A}^D$ is a subset of $\mathfrak{A}^S$, which is closed under the restrictions $\times_D$ and $\circ_D$ on it of the constructed in $\mathfrak{A}^S$ composition laws $\times$ and o.

Without any basical difficulties one can see that the ordered pair $(\mathbf{A^S}, \oplus)$ is a category under the composition law $\oplus$ which defines a direct sum $A \oplus A'$ of two *S*-automata (see for the definitions [7; 8]).

5. Final notes. A series of additional and curious results related to the categories of *S*-automata can be setteld, having in mind a number of standard constructions and methods of the Category Theory (see [9; 17]). But from our point of view they would only be new examples of well-known mathematical substances and therefore they woudn't be of considerable interest. From the point of view of the theory of the *S*-automata we do not see their particular importance because they could only set special cases (of *S*-automata or *A-S*-morphisms) in a categorical form, without being able to further the very theory of the *S*-automata.

But two problems may have solutions in the light of the categories for the *S*-automata and have undeniable importance for the very theory of *S*-automata. They are the problems of decomposition and that of minimization. They will be subjects of some further reports.

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