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LINEAR OPERATORS DEFINED IN SPACES OF COMPLEX FUNCTIONS OF MANY VARIABLES AND COMMUTING WITH THE OPERATORS OF INTEGRATION

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An integral-differential representation is found for operators defined on topological spaces of analytical functions of many complex variables and commuting with the operators of integration.

In this paper we denote by $\bar{A}_0$ the space of the functions $f(z)$, $z=(z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$, analytical at the origin. The operators $I_k = \bar{A}_0 \to \bar{A}_0$ defined by the equalities $I_k f(z) = \int_0^{z_k} f(z_1, z_2, \ldots, \tau_k, \ldots, z_m)\, d\tau_k$, $k=1, 2, \ldots, m$, where $|z|$ is small enough and the integration is accomplished over the segment $[0, z_k]$ are called operators of integration (with respect to the variable z_k).

Some analytical characteristics of classes of linear operators in a closed form, acting from different subspaces of $\bar{A}_0$ in $\bar{A}_0$ and commuting with the operators I_k, $k=1, 2, \ldots, m$ are obtained in this paper.

Analogical problems and their applications are treated in the case of one variable (see for instance [1—13]).

1. Linear operators defined in the space of the polynomials and commuting with the operators of integration. Let S_m denote the subset of $\bar{A}_0$ composed by the polynomials $P(z) = \sum_{|k| \leq n} a_k z^k$ $(z=(z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$, $k=(k_1, k_2, \ldots, k_m)$, $a_k \in \mathbb{C})$, where n is an arbitrary non-negative integer.

Let H denote an arbitrary linear subset ot A_0, closed with respect to the operators I_k, $k=1, 2, \ldots, m$, i. e. such that: $f(z) \in H \Longrightarrow I_k f(z) \in H$, $k=1,2, \ldots, m$.

Let further $I(S_m; H)$ denote the class of the linear operators $L: S_m \to H$, commuting with the operators I_k, $k=1, 2, \ldots, m$.

In this paragraph some analytical characteristics of the class $I(S_m; H)$ will be given. Furthermore, these characteristics will be established for classes of linear continuous operators defined in more general topological subspaces of A_0. For these purposes we shall establish three lemmas.

L e m m a 1.1. *If $f(z) \in \bar{A}_0$ and $n=(n_1, n_2, \ldots, n_m)$ is an arbitrary m-dimensional multiindex with non zero components, for the operator $I=I_1 I_2 \ldots I_m$ the formula*

$$I^n f(z) = \int_0^z \frac{(z-\tau)^{n-1}}{(n-1)!} f(\tau)\, d\tau \tag{1.1}$$

holds, where $\tau=(\tau_1, \tau_2, \ldots, \tau_m)$ is a vector of integration, and $\int_0^z = \int_0^{z_1}\int_0^{z_2} \cdots \int_0^{z_m}$

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denotes a consecutive integration with respect to the variables τ_k, with boundaries from 0 *to* z_k, $k=m, m-1, \ldots, 1$, *respectively.*

The proof of this lemma can be carried out by a multiple application of the well known formula of Cauchy [14] and accomplishment of the necessary number of integrations under the sign of the integral.

Note. One can easily conclude that the representation (1.1) holds also in the cases when the multiindex n has zero components as soon as the following conditions are satisfied: 1°. The zero degree of each operator I_k is the identity; 2°. The integral of the right-hand side of (1.1) means multiple integration from 0 to z_k, but for values of k only, for which $n_k\neq 0$; 3°. For $n_s=0$ the factors $(z_{n_s}-\tau_s)^{n_s-1}d\tau_s/(n_s-1)!$ in the integrand are omitted; 4°. The function $f(\tau)$ under the integral is replaced by the function obtained from $f(\tau)$, when the the integration variables τ_s for which $n_s=0$ are substituted by the parameters z_s.

Lemma 1.2. *Let* $L\in I(S_m;H)$ *and* $(s_1, s_2, \ldots, s_p)$, $p=1, 2, \ldots, m$ *be a p-class variation of the numbers* $1, 2, \ldots, m$ *such that* $s_1<s_2<\cdots<s_p$. *Let* $n=(n_{s_1}, n_{s_2}, \ldots, n_{s_p})$ *be an arbitrary p-dimensional multiindex with non zero components. Then the representation*

$$(1.2)\qquad L\left(\prod_{k=1}^{p} z_{s_k}^{n_{s_k}}\right)=\int_0^{z_{s_1}}\ldots\int_0^{z_{s_p}}\varphi(z_1, \ldots, \tau_{s_1}, \ldots, \tau_{s_p}, \ldots, z_m)$$

$$\frac{\partial^p}{\partial z_{s_1}\ldots\partial z_{s_p}}\prod_{k=1}^{p}(z_{s_k}-\tau_{s_k})^{n_{s_k}}d\tau_{s_1}, \ldots, d\tau_{s_p},$$

holds, where $\varphi(z)=L1\in H$.

Proof. The equality $I_{s_1}^{n_{s_1}}I_{s_2}^{n_{s_2}}\ldots I_{s_p}^{n_{s_p}}1=\prod_{k=1}^{p}z_{s_k}^{n_{s_k}}/n_{s_k}!$ can be verified directly. Using it and taking into account that the linear operator L commutes with the operators I_k, $k=1, 2, \ldots, m$, we obtain:

$$(1.3)\qquad L\left(\prod_{k=1}^{p} z_{s_k}^{n_{s_k}}\right)=\left[\prod_{k=1}^{p}(n_{s_k}!)\right]I_{s_1}^{n_{s_1}}I_{s_2}^{n_{s_2}}\ldots I_{s_p}^{n_{s_p}}\varphi(z),$$

where $\varphi(z)=L1\in H$.

Applying Lemma 1.1, from (1.3) we obtain the equality

$$L\left(\prod_{k=1}^{p} z_{s_k}^{n_{s_k}}\right)=\left(\prod_{k=1}^{p} n_{s_k}\right)\int_0^{z_{s_1}}\int_0^{z_{s_2}}\ldots\int_0^{z_{s_p}}\left[\prod_{k=1}^{p}(z_{s_k}-\tau_{s_k})^{n_{s_k}-1}\right]\varphi(z_1, \ldots,$$

$$\tau_{s_1}, \ldots, \tau_{s_p}, \ldots, z_m)\, d\tau_{s_1}\ldots d\tau_{s_p},$$

which evidently can be written in the form (1.2).

Lemma 1.3. *Each operator* L *of the class* $I(S_m;H)$ *acts on functions of the kind*

$$(1.4)\qquad q(z)=\prod_{k=1}^{p} z_{s_k}^{n_{s_k}},\quad n_{s_k}>0,$$

according to the formula

$$(1.5)\qquad Lq(z)=\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z q(z-\tau)\varphi(\tau)d\tau$$

($|z|$ — small enough), where $z=(z_1, z_2,\ldots, z_m)$, $\tau=(\tau_1, \tau_2,\ldots,\tau_m)$ and $\int_0^z=\int_0^{z_1}\int_0^{z_2}\ldots\int_0^{z_m}$ means consecutive m-multiple integration (p, s_k and n has values given in Lemma 1.2).

Proof. Let $r_1, r_2,\ldots, r_{m-p}$ denote those of indexes $1, 2,\ldots, m$ which are not included in the multiindex $(s_1\ s_2,\ldots, s_p)$ and let $z^*=(z_{r_1}, z_{r_2},\ldots, z_{r_{m-p}})$.

Accomplishing the necessary derivations and integrations under the sign of the integral, for the right-hand side of (1.5) we obtain

$$\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z q(z-\tau)\varphi(\tau)\,d\tau$$

$$=\frac{\partial^p}{\partial z_{s_1}\ldots\partial z_{s_p}}\int_0^{z_{s_1}}\ldots\int_0^{z_{s_p}} q(z-\tau)\frac{\partial^{m-p}}{\partial z_{r_1}\ldots\partial z_{r_{m-p}}}\left[\int_0^{z^*}\varphi(\tau)d\tau_{r_1}\ldots d\tau_{r_{m-p}}\right]d\tau_{s_1}\ldots d\tau_{s_p}.$$

From here, taking into account that

$$\frac{\partial^{m-p}}{\partial z_{r_1}\partial z_{r_2}\ldots\partial z_{r_{m-p}}}\int_0^{z^*}\varphi(\tau)d\tau_{r_1}\ldots d\tau_{r_{m-p}}=\frac{\partial^{m-p}}{\partial z_{r_1}\partial z_{r_2}\ldots\partial z_{r_{m-p}}}\int_0^{z_{r_1}}d\tau_{r_1}\ldots\int_0^{z_{r_{m-p}}}\varphi(\tau)d\tau_{r_{m-p}}$$

$$=\frac{\partial^{m-p-1}}{\partial z_{r_1}\partial z_{r_2}\ldots\partial z_{r_{m-p-1}}}\int_0^{z_{r_1}}d\tau_{r_1}\ldots\int_0^{z_{r_{m-p-1}}}\varphi(\tau_1,\ldots, z_{r_{m-p}},\ldots,\tau_m)d\tau_{r_{m-p-1}}$$

$$=\cdots=\varphi(z_1, z_2,\ldots,\tau_{s_1},\tau_{s_2},\ldots,\tau_{s_p},\ldots, z_m)$$

(the arguments of φ, under which the derivation was accomplished are replaced by the fixed values $z_{r_1}, z_{r_2}\ldots z_{r_{m-p}}$ and the other are integration variables), we obtain the equality

$$\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z q(z-\tau)\varphi(\tau)d\tau$$

$$=\frac{\partial^p}{\partial z_{s_1}\partial z_{s_2}\ldots\partial z_{s_p}}\int_0^{z_{s_1}}d\tau_{s_1}\ldots\int_0^{z_{s_p}}q(z-\tau)\varphi(z_1,\ldots,\tau_{s_1},\ldots,\tau_{s_p},\ldots, z_m)d\tau_{s_p}.$$

Accomplishing the derivations of the respective integrals depending on parameters (containing the parameter in the integration boundaries as well as in the subintegral function), taking into account that $q(z-\tau)$ is zero in the points of the hyperplanes $\tau_{r_k}=z_{r_k}$, $k=1, 2,\ldots, m-p$ we obtain the equality

$$\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z q(z-\tau)\varphi(\tau)d\tau$$

$$=\int_0^{z_{s_1}}\ldots\int_0^{z_{s_p}}\varphi(z_1,\ldots,\tau_{s_1},\ldots,\tau_{s_p},\ldots, z_m)\frac{\partial^p}{\partial z_{s_1}\partial z_{s_2}\ldots\partial z_{s_p}}\prod_{k=1}^{p}(z_{s_k}-\tau_{s_k})^{n_{s_k}}d\tau_{s_1}\ldots d\tau_{s_p}$$

which, according to Lemma 1.2 implies (1.5). Thus, the lemma is proved.

Theorem 1.1. *Each operator* $L \in I(S_m; H)$ *admits a representation of the form*

$$LP(z) = \frac{\partial^m}{\partial z_1 \partial z_2 \ldots \partial z_m} \int_0^z P(z-\tau)\varphi(\tau)d\tau \tag{1.6}$$

($|z|$ *— small enough), where* $\varphi(z) = L1$ *is a single-valued function from the set* H.

Proof. Each polynomial $P(z) \in S_m$ can be represented in the form

$$P(z) = P(0) = \sum_{p=1}^{m} \sum_{s_1<s_2<\ldots<s_p} \sum_{|n_s|\leq n} a_{n_{s_1} n_{s_2} \ldots n_{s_p}} \prod_{k=1}^{p} z_{s_k}^{n_{s_k}}, \tag{1.7}$$

where the multiindex $n_s = (n_{s_1}, n_{s_2}, \ldots, n_{s_p})$ has nonzero components, n is a natural number and $(s_1, s_2, \ldots, s_p)$ describes p-class variations without inversions of the numbers $1, 2, \ldots, m$.

Since the operator L is linear, from (1.6), the equality

$$LP(z) = P(0)\varphi(z) + \sum_{p=1}^{m} \sum_{s_1<s_2<\ldots<s_p} \sum_{|n_s|\leq n} a_{n_{s_1} n_{s_2} \ldots n_{s_p}} Lq(z) \tag{1.8}$$

follows, where $\varphi(z) = L1$ and $q(z)$ is the function (1.4).

From (1.8), applying lemma 1.3 and representing $\varphi(z)$ under the form

$$\varphi(z) = \frac{\partial^m}{\partial z_1 \partial z_2 \ldots \partial z_m} \int_0^z \varphi(\tau)\, d\tau,$$

we obtain immediately the representation (1.6).

Theorem 1.2. *If* H *is a linear subset of* A_0*, invariant with respect to the operators* I_k, $k = 1, 2, \ldots, m$ *and closed with respect to multiplication with independent variables* z_k, $k = 1, 2, \ldots, m$*, then each operator defined in* S_m *by an equality of the form* (1.6)*, where* $\varphi(z)$ *in an arbitrary function of the set* H*, is an operator of the class* $I(S_m; H)$.

Proof. First of all we have to prove that the right-hand side of (1.6) is a function of H, for any function $\varphi(z) \in H$ and for every polinomial $P(z) \in S_m$. For this purpose, it is enough to develop the polynomial $P(z-\tau)$ according to the formula of Taylor in the neighbourhood of the point z and to accomplish the derivation $\partial^m/\partial z_1 \ldots \partial z_m$. Thus, according to the properties of the set H, the right-hand side of (1.6) can be represented as a linear combination of elements of H, hence it is an element of H (H is a linear subset of $\bar{A}_0$).

Hence, under the hypotheses of the theorem, each operator L, defined in S_m by the equality (1.6), acts from S_m into H.

On the other hand, one can easily verify that L is a linear operator, hence theorem 1.2 would be proved if we establish that the considered operator $L: S_m \to H$ defined by the equality (1.6), commutes with every operator I_k, $k = 1, 2, \ldots, m$, i. e. the equalities

$$LI_kP(z) = I_kLP(z), \quad \forall P(z) \in S_m, \quad k = 1, 2, \ldots, m \tag{1.9}$$

hold (of course for $|z|$ — small enough).

Let us fix k and $P(z)$ in (1.9) and $g_1(z) = LI_kP(z)$, $g_2(z) = I_kLP(z)$.

Using the representation (1.6) of the operator L and the definition of the operator I_k, we can easily see that the equalities

$$\partial g_1(z)/\partial z_k = \partial g_2(z)/\partial z_k \tag{1.10}$$

hold, and

$$g_1(z_1,\ldots,z_{k-1},0,z_{k+1},\ldots,z_m)=g_2(z_1,\ldots,z_{k-1},0,z_{k+1},\ldots,z_m)=0 \tag{1.11}$$

for every point $(z_1,\ldots,z_{k-1},0,z_{k+1},\ldots,z_m)$ as soon as its distance from the origin is small enough.

The equality (1.10) indicates, that the difference $g_1(z)-g_2(z)$ is a constant with respect to z_k, i. e. the equality $g_1(z)-g_2(z)=c(z_1,z_2,\ldots,z_{k-1},z_{k+1},\ldots,z_m)$ holds ($|z|$ — small enough).

From the last equality taking $z=(z_1,z_2,\ldots,z_{k-1},0,z_{k+1},\ldots,z_m)$ according to (1.11), we can conclude that $c(z_1,z_2,\ldots,z_{k-1},\ldots,z_m)=0$ for every $z'=(z_1,z_2,\ldots,\widehat{z_k},\ldots,z_m)$, for which $|z|$ is small enough, i. e. $g_1(z)=g_2(z)$ for such $|z|$, hence (1.9) holds. Thus, theorem 1.2 is proved.

Theorem 1.3. *If H is an arbitrary linear subset of $\bar{A}_0$, then an operator $L:S_m\to H$ is of the class $I(S_m;H)$ if it admits a representation of the form* (1.6)

$$LP(z)=\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z P(z-\tau)\varphi(\tau)d\tau$$

($|z|$ — small enough), where $\varphi(z)$ is a function from H.

This theorem follows immediately from theorem 1.1 and the proof of theorem 1.2.

This shows that the representation (1.6) is an analytical charecteristic of the class $I(S_m;H)$.

From formula (1.6) it appears that the character of the map of the space S_m by an operator $L\in I(S_m;H)$ is determined by the function $\varphi(z)=L1$. For instance, if $L1\in S_m$, the operator L acts from S_m into S_m; if $L1\in E(G)$, where $E(G)$ is the class of the analytical functions in the domain G containing the origin, the operator L acts from S_m into $E(G)$; if $L1$ is an entire function ($L1\in \bar{A}_\infty$), L acts from S_m into $\bar{A}_\infty$, etc.

2. Linear oparators defined in arbitrary subsets of $\bar{A}_0$ and commuting with the operators of integration. The results obtained in 1 can be applied for finding the general type of linear continuous operators commuting with the operators of integration and acting in considerably wider subspaces of the space $\bar{A}_0$.

To do that we shall first of all topologize the space $\bar{A}_0$ introducing into it the natural topology h_0, according to which, a sequence $\{f_n(z)\}$ is convergent to a function $f(z)\in\bar{A}_0$ iff there exists a polycircle K with a centre in the origin, such that:

1) the functions $f_n(z)$, $n=1,2,\ldots$ are analytical in K, and

2) the sequence $\{f_n(z)\}$ converges uniformly in K to the function $f(z)$.

Without entering into details which do not differ essentially from their respective ones in the case of analytical functions of one argument (see [15]—[18]), we shall point out that the mentioned topology can be introduced considering the space $\bar{A}_0$ as the inductive limit of Banach's spaces:

$$\bar{A}_0 = \lim_{0<R<\infty} \text{ind}\, \widetilde{A}_R,$$

where $\widetilde{A}_R$ is the space of functions $f(z)$, analytical in the open polycircle K_R with centre the origin, radius R and continuous in the closure $\bar{K}_R$, normalized with the norm $\|f(z)\| = \sup_{z \in \bar{K}_R} |f(z)|$.

The same topology in $\bar{A}_0$ can be introduced considering the inductive limit of Fréchet spaces $(A_R; h_R)$, where $A_R = \{f(z);\ f(z)$ is analytical in the polycircle $K_R\}$, and h_R is the topology of uniform convergence on every compact contained in the polycircle K_R.

Now let H_i, $i=1,2$ be arbitrary linear subspaces of $\bar{A}_0$ with respective topologies h_i, $i=1,2$. We shall assume that the subspace S_m is sequentially dense in the topological space $(H_1; h_1)$. Besides, we shall suppose that the topologies h_i, $i=1,2$ majorize the respective topologies, which are induced on the spaces H_i by the natural topology h_0 in $\bar{A}_0$. Finally, we shall assume that the spaces H_i are closed with respect to the operators of integration I_k, $k=1,2,\ldots,m$, which are continuous in the topological space $(H_1; h_1)$.

The described general conception is realized in a number of particular cases. The simplest ones are:

a) $H_1 \equiv H_2 \equiv \bar{A}_0$ with topologies h_1 and h_2, identical to the topologies induced in H_i, $i=1,2$ by h_0;

b) $H_1 \equiv A_{R_1}$; $H_2 \equiv A_{R_2}$; h_i, $i=1,2$ are the topologies of uniform convergence on every compact contained in the polycircle with centre the origin and radius R_i, $i=1,2$.

Let us denote by $I(H_1, H_2)$ the class of the linear operators acting from H_1 to H_2 and commuting with the operators of integration I_k, $k=1,2,\ldots,m$

Theorem 2.1. *Each continuous operator of the class $I(H_1; H_2$ admits a representation of the form*

$$Ly(z(= \frac{\partial^m}{\partial z_1 \partial z_2 \ldots \partial z_m} \int_0^z y(z-\tau)\varphi(\tau)d\tau \tag{2.1}$$

($|z|$ — small enough), where $\varphi(z) = L1$ is a function from H_2.

Proof. Let $y(z)$ be arbitrary fixed function from the space H_1 and $P_1(z)$ $P_2(z), \ldots, P_n(z), \ldots$ is a sequence of polynomials convergent to $y(z)$ according to the topology h_1.

Taking into account that the operator $L: H_1 \to H_2$ is a continuous map of Hausdorff's space H_1 in Hausdorff's space H_2 (from which follows its sequential continuity) we conclude that the sequence

$$LP_1(z),\ LP_2(z), \ldots,\ LP_n(z), \ldots \tag{2.2}$$

is convergent to the function $Ly(z)$ according to the topology h_2.

On the other hand the topology h_2 majorizes the topology induced in H_2 by the natural topology h_0 therefore (2.2) is convergent to $Ly(z)$ according to the topology h_0 in the space $\bar{A}_0$, i. e. (2.2) is uniformly convergent to the function $Ly(z)$ in a certain circle $|z| < r_1$.

Taking into account that the restriction of the operator L on S_m is an operator from the class $I(\delta_m; \bar{A}_0)$, according to theorem 1.1 we obtain

$$LP_n(z)=\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z P_n(z-\tau)\varphi(\tau)d\tau,\quad n=1,2,\ldots \tag{2.3}$$

However, the right-hand side of (2.3) tends uniformly in a certain circle $|z|<r_2$ to the function

$$\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z y(z-\tau)\varphi(\tau)d\tau. \tag{2.4}$$

This follows from the fact that after the substitution $z-\tau=\bar{\tau}$ the derivation $\partial^m/\partial z_1\partial z_2\ldots\partial z_m$ can be transferred to the function $\varphi(z)$ and then the right-hand side of (2.3) will be represented as a finite sum of integral expressions in which $P_n(z)$ participates as a subintegral factor. In this representation, because of the continuity of the operators I_k we can accomplish the limit operation $P_n(z)\Rightarrow y(z)$ and then again transform the expression in the form (2.4).

Thus, there exists a circle $|z|<r_3$ in which the left-hand side of (2.3) tends uniformly to $Ly(z)$ and its right-hand side tends uniformly to the function (2.4).

In this case, because of the theorem of identity, the equality (2.1) holds in the whole circle, where the function $y(z)$ is analytical. Theorem 2.1 is proved.

Theorem 2.2. *Each operator L defined in the space H_1 by an equality of the type*

$$Ly(z)=\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z y(z-\tau)\varphi(\tau)d\tau, \tag{2.5}$$

where $\varphi(z)$ is an arbitrary function from H_2, is a continuous operator from the class $I(H_1;H_2)$.

Proof. We have to show only that each operator L defined by (2.5) commutes with the operators of integration $I_k y(z)=\int_0^{z_k}y(z_1, z_2,\ldots, \tau_k,\ldots, z_m)d\tau_k$. Indeed the statement that the operators of the type (2.5) act from H_1 to H_2 and are continuous in H_1, follows immediately from the assumptions about the spaces H_1 and H_2 and the continuity of the operators I_k in the space $(H_1;h_1)$.

Let us consider an arbitrary operator L of the type (2.5). Its restriction in the subspace S_m of the space H_1 obviously satisfies the requirements of theorem (1.1) and therefore belongs to the class $I(S_m;\bar{A}_0)$, i. e. for any polynomial $P(z)\in S_m$ the commutation $I_kLP(z)=LI_kP(z), k=1, 2,\ldots, m$ holds. However, from it follows the equality $I_kLy(z)=LI_k y(z)$, $k=1,2,\ldots, m$, $y(z)\in H_1$, since by assumption the space S_m is sequentially dense in H_1 according to the topology h_1 and the operators $L, I_1, I_2,\ldots, I_m$ are continuous in $(H_1;h_1)$. Theorem 2.2 is proved.

The proved theorems 2.1 and 2.2 obviously allow us to state that the analytical representation (2.1) is a characteristic property (or a general form) of the continuous operators of the class $I(H_1;H_2)$, i. e. the following theorem holds.

Theorem 2.3. *A continuous opepator $L:H_1\to H_2$ commutes with the operators of integration $I_k y(z)=\int_0^{z_k} y(z_1,\ldots, \tau_k,\ldots, z_m)d\tau_k$ ($|z|$ — small enough), $k=1,\ldots, m$, if the operator L can be represented in the form*

$$L\,y(z)=\frac{\partial^m}{\partial z_1\partial z_2\ldots\partial z_m}\int_0^z y(z-\tau)\varphi(\tau)d\tau,$$

where $\varphi(z)$ is an arbitrary function of the space H_2.

In conclusion we shall note that the given proofs of theorems 2.1 and 2.2 allow us to omit some of the assumptions; this will make the exposition easier.

So for instance:

1°) In theorem 2.1 the assumption that the space $(H_1; h_1)$ should be such that the operators of integration are continuous can be omitted;

2°) The same assumption can be omitted in the particular case of theorem 2.2 where the space H_2 coincides with $\bar{A}_0$.

Some applications of the obtained representations will be a subject of a paper to follow.

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