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ON HAAR BASES IN BESOV SPACES

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Let $-\infty<s<\infty$, $0<p<\infty$, and $0<q<\infty$. The Haar system in a cube Q in the Euclidean n-space R_n is a simultaneous Schauder basis in the Besov spaces $B^s_{p,q}(Q)$ if

$$(s,\ p)\in M=\begin{cases}1<p<\infty, & \frac{1}{p}-1<s<\frac{1}{p}\\ \frac{n}{n+1}<p\leq 1, & n\left(\frac{1}{p}-1\right)<s<1\end{cases}\quad,\quad 0<q<\infty.$$

This assertion cannot be extended to the spaces $B^s_{p,q}(Q)$ if $(s,\ p)\notin \bar{M}$. Similarly for $B^s_{p,q}(R_n)$.

1. Introduction. If Q denotes a cube in the Euclidean n-space R_n, then it is a well-known fact that the Haar system is a Schauder basis in the usual Lebesgue spaces $L_p(Q)$, where $1<p<\infty$. This statement can be extended to the Besov spaces $B^s_{p,q}(Q)$ provided that $1<p<\infty$, $1\leq q<\infty$, and $\frac{1}{p}-1<s<\frac{1}{p}$, cf. [6, 4.9.4], or [5]. By standard arguments (cf. below) one obtains a similar result for $B^s_{p,q}(R_n)$. Recently, the definition of the Besov spaces $B^s_{p,q}(R_n)$ and, by restriction to Q, also of $B^s_{p,q}(Q)$ has been extended in a natural way to $-\infty<s<\infty$, $0<p\leq\infty$, $0<q\leq\infty$. In general, these spaces are only quasi-Banach spaces, however the notion of a Schauder basis can be extended in an obvious way to quasi-Banach spaces. The question arises to find all Besov spaces $B^s_{p,q}(R_n)$ and $B^s_{p,q}(Q)$ for which Haar systems are Schauder bases. Beside some limiting cases, the paper contains a solution of this problem. Since the case $1<p<\infty$ has been treated in [6, 4.9.4], the most interesting case in this paper is $0<p\leq 1$. Roughly speaking, the method developed in [6, 4.9.4], and [5] for $1<p<\infty$ can be extended to $0<p\leq 1$ (however some non-trivial changes are needed). The main results are formulated in the Theorems 1 and 2. All immaterial positive constants are denoted by c, c', etc.

2. Preliminaries. 2.1. Definition of the Besov spaces. Let R_n be the Euclidean n-space. $S(R_n)$ is the usual Schwartz space of all complex-valued infinitely differentiable rapidly descreasing functions on R_n. The dual space $S'(R_n)$ is the set of all tempered distributions. F and F^{-1} denote the Fourier transform and its inverse on $S'(R_n)$, respectively. Let $\{\varphi_j(x)\}_{j=0}^{\infty}\subset S(R_n)$ be a system of functions such that

(i) $\operatorname{supp}\varphi_j\subset\{y\mid 2^{j-1}\leq|y|\leq 2^{j+1}\}$ if $j=1, 2, 3,\ldots$; $\operatorname{supp}\varphi_0\subset\{y\mid |y|\leq 2\}$,

(ii) For any multi-index γ there exists a constant c_γ such that for all $j=0, 1, 2,\ldots$ and all $x\in R_n$ we have $|D^\gamma\varphi_j(x)|\leq c_\gamma 2^{-j|\gamma|}$,

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(iii) For all $x \in R_n$ we have $\sum_{j=0}^{\infty} \varphi_j(x) = 1$.

Definition 1. *If* $-\infty < s < \infty$, $0 < p < \infty$, *and* $0 < q < \infty$, *then*

(1) $$B^s_{p,q}(R_n) = \{ f \mid f \in S'(R_n), \quad \| f \mid B^s_{p,q}(R_n) \| = \left[\sum_{j=0}^{\infty} 2^{sjq} \left(\int_{R_n} | F^{-1}[\varphi_j Ff](x) |^p dx \right)^{q/p} \right]^{1/q} < \infty \right\}.$$

Remark 1. This is the definition of the Besov spaces due to J. Peetre, cf. [3], and the references given there. If $s > 0$, $1 < p < \infty$, and $1 \leq q < \infty$, then $B^s_{p,q}(R_n)$ coincides with the classical Besov spaces. Systematic treatments of these spaces on the basis of the above definition have been given in [3] and [8] (the classical cases, i. e. $1 < p < \infty$ and $1 \leq q < \infty$, may be found in [2] and [6]). The spaces do not depend on the chosen system $\{\varphi_j\}$ (equivalent quasi-norms!). In any case, $B^s_{p,q}(R_n)$ is a quasi-Banach space (Banach space if $1 \leq p < \infty$ and $1 \leq q < \infty$).

Remark 2. In an obvious way the definition can be extended to $p = \infty$ and/or $q = \infty$. One obtains non-separable quasi-Banach spaces which, from the point of view of the existence of Schauder bases, are not of interest. However for the later considerations it is useful to remark that if $s > 0$ then $B^s_{\infty,\infty}(R_n) = \mathcal{C}^s(R_n)$ are the usual Hölder-Zygmund spaces, cf. [8, 2.2.9].

Definition 2. *Let* Q *be a cube in* R_n. *If* $-\infty < s < \infty$, $0 < p < \infty$ *and* $0 < q < \infty$ *then* $B^s_{p,q}(Q)$ *is the restriction of* $B^s_{p,q}(R_n)$ *to* Q. *Furthermore,*

$$\| f \mid B^s_{p,q}(Q) \| = \inf \| g \mid B^s_{p,q}(R_n) \|,$$

where the infimum is taken over all $g \in B^s_{p,q}(R_n)$, *whose restriction to* Q *coincides with* f *(considered as an element of the space* $D'(Q)$ *of all distributions on* Q).

Remark 3. $B^s_{p,q}(Q)$ is a quasi-Banach space (Banach space if $1 \leq p < \infty$ and $1 \leq q < \infty$). If $1 < p < \infty$ and $1 \leq q < \infty$, then it coincides with the classical spaces. The definition can be extended to $p = \infty$ and/or $q = \infty$, cf. Remark 2.

2.2. Some fundamental properties of the Besov spaces. In this subsection, we describe some properties of Besov spaces, proved in other papers, which will be needed later on. If an assertion holds true for $B^s_{p,q}(R_n)$ and for $B^s_{p,q}(Q)$ then we formulate it for $B^s_{p,q}$.

(i) Density. Let $-\infty < s < \infty$, $0 < p < \infty$, and $0 < q < \infty$. $B^s_{p,q}$ is a quasi-Banach space, $S(R_n)$ is dense in $B^s_{p,q}(R_n)$, and $C^\infty(\bar{Q})$ (the set of all complex valued infinitely differentiable functions on the closed cube $\bar{Q}$) is dense in $B^s_{p,q}(Q)$. As far as the spaces on R_n are concerned we refer to [8, 2.1.1] and [7, Chapter 2]. The corresponding assertions for the spaces on Q follow immediately by the restriction process.

(ii) Imbeddings. If $-\infty < \sigma \leq s < \infty$, $0 < p \leq r < \infty$, $0 < q < \infty$, and $\sigma - n/r \leq s - n/p$ then

(2) $$B^s_{p,q} \subset B^\sigma_{r,q}.$$

If $\sigma - n/r < s - n/p$ then $B^{\sigma}_{r,q}$ in (2) can be replaced by $B^{\sigma}_{r,\tilde{q}}$, where $\infty > \tilde{q} > 0$ is an arbitrary number. If $0<p<\infty$, $0<q<\infty$, and $\infty > s - n/p > \sigma > 0$, then

$$B^s_{p,q} \subset \mathcal{C}^\sigma, \tag{3}$$

where $\mathcal{C}^\sigma$ are the Hölder-Zygmund spaces. If $0<p\leq 1$, $0<q<\infty$, and $s>n(p^{-1}-1)$ then

$$B^s_{p,q} \subset L_1. \tag{4}$$

Proofs (and also a precise definition of $\mathcal{C}^s$) may be found in [8], 2.4.1 (as far as the spaces on R_n are concerned; the corresponding assertions for the spaces on Q follow immediately by restriction). Cf. also [6, 2.8.1 and 4.6.1].

(iii) Dual spaces. As mentioned in Remark 2, Definition 1 can be extended to $p=\infty$ and/or $q=\infty$. Furthermore, if $1<p<\infty$ then $1/p+1/p'=1$, and if $0<p\leq 1$ then $p'=\infty$. If $-\infty<s<\infty$, $1\leq p<\infty$, and $0<q<\infty$, then

$$(B^s_{p,q}(R_n))' = B^{-s}_{p',q'}(R_n). \tag{5}$$

If $-\infty<s<\infty$, $0<p<1$, and $0<q<\infty$ then

$$(B^s_{p,q}(R_n))' = B^{-s+n(1/p-1)}_{\infty,q'}(R_n). \tag{6}$$

Proofs (and interpretations) may be found in [8, 2.5].

(iv) Multiplication by characteristic functions. If χ is the characteristic function of a rectangle in R_n, then $f \to \chi f$ yields a bounded mapping from $B^s_{p,q}(R_n)$ into itself if

$$\begin{cases} \text{either} & 1\leq p<\infty, \quad \frac{1}{p}-1<s<\frac{1}{p} \\ \text{or} & \frac{n-1}{n}<p<1, \quad n\left(\frac{1}{p}-1\right)<s<\frac{1}{p} \end{cases} \tag{7}$$

and $0<q<\infty$, cf. [8, 2.6.4, Remark 3]. (Beside limiting cases this assertion cannot be strengthened. In particular (7) is a natural restriction, cf. [8].)

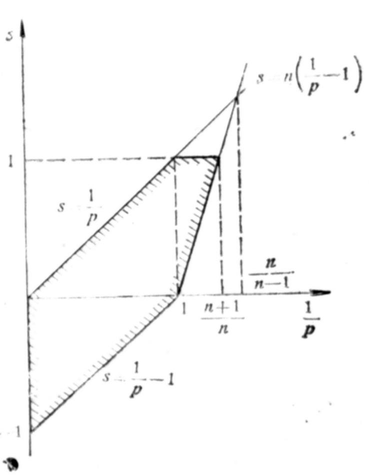

Fig. 1

(v) General multiplications. If $0<p<\infty$, $0<q<\infty$, $-\infty<s<\infty$, and $\varrho>\max(s, n/p-s)$, then there exists a constant c, such that for all $f\in B^s_{p,q}(R_n)$ and all $g\in\mathcal{C}^\varrho(R_n)$

$$\| gf \mid B^s_{p,q}(R_n)\| \leq c \| g \mid \mathcal{C}^\varrho(R_n)\| \, \| f \mid B^s_{p,q}(R_n)\| \tag{8}$$

cf. [8, 2.6.1]. Later on, we need only multiplications with $g\in S(R_n)$ (and in this case, the meaning of gf is clear without further explanations).

vi) Interpolation. If $-\infty<s_0<s_1<\infty$, $0<p<\infty$, $0<q_0<\infty$, $0<q_1<\infty$, $0<q<\infty$, $0<\Theta<1$, and $s=(1-\Theta)s_0+\Theta s_1$ then

$$(B^{s_0}_{p,q_0}(R_n), B^{s_1}_{p,q_1}(R_n))_{\Theta,q} = B^s_{p,q}(R_n), \tag{9}$$

cf. [8, 2.2.10]. Here $(\cdot,\cdot)_{\Theta,q}$ is the real interpolation method. Furthermore, if s_0 and s_1 are real numbers, $0<p_0<\infty$, $0<p_1<\infty$, $0<\Theta<1$, and

$$s=(1-\Theta)s_0+\Theta s_1, \qquad \frac{1}{p}=\frac{1-\Theta}{p_0}+\frac{\Theta}{p_1}, \tag{10}$$

then

$$(B^{s_0}_{p_0,p_0}(R_n),\ B^{s_1}_{p_1,p_1}(R_n))_{\Theta,p}=B^s_{p,p}(R_n), \tag{11}$$

cf. [8, 2.2.10]. (9) and (11) with the cube Q instead of R_n remain true if $p>(n-1)/n$ and if all the other parameters satisfy the above conditions, respectively. In [10], Theorem 2(iii) we proved (9) and (11) for a bounded C^∞ — domain Ω instead of R_n if $p>(n+1)/n$. The proof in [10] works also for cubes Q, and a detailed examination shows that in this case $p>(n-1)/n$ is sufficient.

(vii) Equivalent quasi-norms. If

$$\begin{cases} \text{either} & 1\leq p<\infty, \qquad 0<s<1/p \\ \text{or} & n/(n+1)<p<1, \quad n(1/p-1)<s<1, \end{cases} \tag{12}$$

then

$$\|f\,|\,B^s_{p,p}(R_n)\|^*=\Big(\int_{R_n}|f(x)|^p dx+\int_{R_n\times R_n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy\Big)^{1/p} \tag{13}$$

is an equivalent quasi-norm on $B^s_{p,p}(R_n)$ (norm if $p\geq 1$), cf. [9, Proposition 4]. Obviously, if $p>1$, then we have the well-known classical assertion, cf. e. g. [6, 2.5.1]. The extension to $p\leq 1$ is essentially due to J. Peetre [3, p. 254]. Some modifications of Peetre's proof yield also the following result: If $0<p\leq 1$ and $s>n(1/p-1)$, then there exists a constant c such that for all finite linear combinations $f(x)$ of characteristic functions of rectangles with sides parallel to the axes

$$\|f\,|\,B^s_{p,p}(R_n)\|\leq \tag{14}$$

$$c\Big(\int_{R_n}|f(x)|^p dx+\sum_{j=1}^{n}\int_0^\infty h^{-sp}\int_{R_n}|f(x_1,\ldots,x_{j-1},x_j+h,x_{j+1},\ldots,x_n)-f(x)|^p dx\frac{dh}{h}\Big)^{1/p}.$$

(14) is of interest if $n(1/p-1)<s<1/p$, cf. Fig. 1. In that case, property (iv) and elementary calculations show that both sides of (14) are finite.

2.3. The Haar basis in cubes. If B is a complex quasi-Banach space, then a set $\{b_j\}_{j=1}^\infty\subset B$ is said to be a Schauder basis if each element $b\in B$ can be uniquely represented as $b=\sum_{j=1}^\infty \beta_j b_j$, β_j complex numbers, and the linear operators P_N, acting in B, $P_N b=\sum_{j=1}^N \beta_j b_j$, $N=1, 2, 3,\ldots$, are uniformly bounded. If B is a Banach space then the second part of the definition (the uniform boundedness of P_N) follows from the first part of the definition, cf. [1, p. 88]. In particular, in any case $\beta_j=\beta_j(b)$ is a linear continuous functional, $\beta_j\in B'$. This shows that a quasi-Banach space with a Schauder basis must have a sufficiently rich dual space. (A counterexample is $L_p(R_n)$ with $0<p<1$, because $L'_p=\{0\}$.)

We use the system of Haar functions described in detail in [6, 4.9.4]. Let

$$Q=\{x \mid x\in R_n, \quad 0<x_j<1, \quad j=1,\ldots, n\} \tag{15}$$

be the unit cube. If k is a given natural number then we consider the decomposition $\bar{Q}=\bigcup_{r=1}^{N}\bar{Q}_r$, where the rectangles Q_r are given by

$$Q_r=\Big\{x \mid x\in R_n, \frac{m_j}{2^k}<x_j<\frac{m_j+1}{2^k} \quad \text{if} \quad j=1,\ldots, l; \tag{16}$$

$$\frac{m_j}{2^{k-1}}<x_j<\frac{m_j+1}{2^{k-1}} \quad \text{if} \quad j=l+1,\ldots, n\Big\}.$$

Here l and m_j are appropriate numbers. Obviously, $Q_r\cap Q_s=\emptyset$ if $s\neq r$. If Z symbolizes this decomposition then we introduce the operator P_Z,

$$(P_Z f)(x)=\frac{1}{|Q_r|}\int_{Q_r} f(y)dy \quad \text{if} \quad x\in Q_r \tag{17}$$

(mean value). The set of all these decompositions is enumerated by $Z_1, Z_2, Z_3,\ldots$, in such a way that Z_{j+1} is obtained from Z_j by dividing (bisecting) exactly one of the rectangles Q_r belonging to Z_j. Then it follows that $P_{Z_{j+1}}-P_{Z_j}$ are projections on one-dimensional subspaces, $j=1, 2, 3,\ldots$. We proved in [6, 4.9.4], that the generating elements of the one-dimensional ranges of P_{Z_1} and $P_{Z_{j+1}}-P_{Z_j}$ form a simultaneous Schauder basis in $B^s_{p,q}(Q)$ if $1<p<\infty$, $1\leq q<\infty$, and $1/p-1<s<1/p$ (Haar basis). The representation is given by

$$f=P_{Z_1}f+\sum_{j=1}^{\infty}(P_{Z_{j+1}}-P_{Z_j})f. \tag{18}$$

It is well-known that (18) is also an unique representation (Schauder basis) in $L_p(Q)$ if $1<p<\infty$. Finally we add a technical remark: If f is smooth, then (17) and (18) have an immediate meaning. If the projections P_{Z_j} are uniformly bounded in a given space $B^s_{p,q}(Q)$ then the density of $C^\infty(\bar{Q})$ in $B^s_{p,q}(Q)$ (cf. 2.2.(i)) and the usual completion argument shows that each element $f\in B^s_{p,q}(Q)$ can be represented by (18), where $P_{Z_j}f$ is defined via this limiting process. The statements below must be understood in this sense.

3. Special properties of the Besov spaces. 3.1 Equivalent quasi-norms.

Proposition 1. *Let Q be the cube described in (15) and let s and p be given by (12). Then the following three quasi-norms*

$$\|f \mid B^s_{p,p}(Q)\|^{(1)}=\Big(\int_Q |f(x)|^p dx+\int_{Q\times Q}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy\Big)^{1/p}, \tag{19}$$

$$\|f \mid B^s_{p,p}(Q)\|^{(2)}=\int_Q |f(x)|\,dx+\Big(\int_{Q\times Q}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy\Big)^{1/p} \tag{20}$$

(21) $$\|f \mid B^s_{p,p}(Q)\|^{(3)} = \left|\int_Q f(x)dx\right| + \left(\int_{Q\times Q} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{1/p},$$

are equivalent quasi-norms in $B^s_{p,p}(Q)$.

Proof. Step 1. Obviously, Definition 2 and (13) prove that

(22) $$\|f \mid B^s_{p,p}(Q)\|^{(1)} \leq c \|f \mid B^s_{p,p}(Q)\|.$$

On the other hand, let $f \in B^s_{p,p}(Q)$ and $f(x)=0$ if $x \in Q$ and $|x| \geqq 1/2$. (The general multiplication properties described in 2.2(v) show that it is sufficient to prove the reversion of (22) for functions of that type.) The special multiplication properties described in 2.2 (iv) yield that S_0,

(23) $$(S_0 f)(x) = \begin{cases} f(x) & \text{if} \quad x \in Q, \\ 0 & \text{if} \quad x \in R_n - Q, \end{cases}$$

is an extension from $B^s_{p,p}(Q)$ into $B^s_{p,p}(R_n)$. Then it follows that S_1,

(24) $$(S_1 f)(x) = f(|x_1|, |x_2|, \ldots, |x_n|),$$

is also an extension. Here we assume that $(S_1 f)(x)=0$ if $|x| \geqq {}^1/_2$.) Hence, by (13),

(25) $$\|f \mid B^s_{p,p}(Q)\| \leq \|S_1 f \mid B^s_{p,p}(R_n)\|^* \leq c \|f \mid B^s_{p,p}(Q)\|^{(1)}.$$

This proves that (19) is an equivalent quasi-norm in $B^s_{p,p}(Q)$.

Step 2. By (4) (if $p<1$) and Hölder's inequality (if $p>1$) we have

(26) $$\|f \mid B^s_{p,p}(Q)\|^{(3)} \leq \|f \mid B^s_{p,p}(Q)\|^{(2)} \leq c \|f \mid B^s_{p,p}(Q)\|^{(1)}.$$

Hence, we must show that there exist a constant c such that for all $f \in B^s_{p,p}(Q)$

(27) $$\|f \mid L_p(Q)\| \leq c \|f \mid B^s_{p,p}(Q)\|^{(3)}.$$

Let us assume that there does not exist a constant c with (27). Then we find a sequence $\{f_j\} \subset B^s_{p,p}(Q)$ with

(28) $$\|f_j \mid L_p(Q)\| = 1 \quad \text{and} \quad \|f_j \mid B^s_{p,p}(Q)\|^{(3)} \to 0 \quad \text{when} \quad j \to \infty.$$

In particular, by (19), $\{f_j\}$ is a bounded set in $B^s_{p,p}(Q)$. If $1<p<\infty$ then it is well-known that $\{f_j\}$ is pre-compact in $L_p(Q)$ (compact imbedding). Hence, in that case, without loss of generality, $f_j \to f$ in $L_p(Q)$. By (19) and (28), $f_j \to f$ in $B^s_{p,p}(Q)$. Hence,

(29) $$\int_Q f(x)dx = 0 \quad \text{and} \quad \int_{Q\times Q} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy = 0.$$

Consequently, $f(x) \equiv 0$. This is a contradiction, since $\|f \mid L_p(Q)\| = 1$. If $p \leq 1$, then (2) and (4) show that $\{f_j\}$ is pre-compact in $L_1(Q)$. By Hölder's inequality, $\{f_j\}$ is also pre-compact in $L_p(Q)$. Then we can apply the above arguments in this case, too. The proof is complete.

Corollary 1. *S_0 in (23) is an extension operator from $B^s_{p,p}(Q)$ into $B^s_{p,p}(R_n)$. Consequently,*

$$\|S_0 f \mid B^s_{p,p}(R_n)\| \sim \|f \mid B^s_{p,p}(Q)\|.$$

On the other hand if $d(x)$ denotes the distance of $x\in Q$ from the boundary of Q, then

$$\| S_0 f \mid B^s_{p,p}(R_n) \|^* \sim \left(\int_Q d^{-sp}(x) \mid f(x) \mid^p dx + \int_{Q\times Q} \frac{\mid f(x)-f(y) \mid^p}{\mid x-y \mid^{n+sp}} dx dy \right)^{1/p},$$

cf. (13). If one uses the equivalent quasi-norm (19) then one obtains the following fractional Hardy inequality: *If p and s satisfy (12), then there exists a constant c such that for all $f\in B^s_{p,p}(Q)$*

$$\left(\int_Q d^{-sp} \mid f(x) \mid^p dx \right)^{1/p} \leq c \left(\int_Q \mid f(x) \mid^p dx + \int_{Q\times Q} \frac{\mid f(x)-f(y) \mid^p}{\mid x-y \mid^{n+sp}} dx dy \right)^{1/p}. \tag{30}$$

Inequalities of type (30) are known if $1<p<\infty$, cf. [6, 3.2.6], and the references given there (essentially, assertions of this type are due to P. Grisvard). An extension to values $p\leq 1$ and a direct proof may be found in [9], formula (72) (bounded C^∞ — domains instead of cubes).

As usual we set

$$(\Delta^1_{h,j} f)(x) = f(x_1, \ldots, x_{j-1}, x_j+h, x_{j+1}, \ldots, x_n) - f(x), \quad \Delta^l_{h,j} = \Delta^1_{h,j}\Delta^{l-1}_{h,j}$$

if $h>0$ and $(\Delta^1_h f)(x) = f(x+h) - f(x)$, $\Delta^l_h = \Delta^1_h \Delta^{l-1}_h$ if $h\in R_n$. Here $l=2, 3, \ldots$

Proposition 2. *Let $0<p\leq 1$, $0<q<\infty$, and $0<s<\infty$. If l is a natural number with $l>s$, then there exists a constant c such that for all $\in S(R_n)$,*

$$\sum_{j=1}^n \left(\int_0^1 h^{-sq} \| \Delta^l_{h,j} f \mid L_p(R_n) \|^q \frac{dh}{h} \right)^{1/q} + \| f \mid L_p(R_n) \| \leq c \| f \mid B^s_{p,q}(R_n) \|. \tag{31}$$

Proof. Step 1. First we prove an inequality. If $f\in S(R_n)$ and if $\{\varphi_j\}$ is a system as it has been described at the beginning of 2.1, then

$$f = \sum_{j=0}^\infty F^{-1}\varphi_j F f \quad \text{(convergence in } S(R_n)). \tag{32}$$

Furthermore, let $\{\psi_j\}_{j=0}^\infty$ be a second system with the properties (i) and (ii) in 2.1 (with ψ instead of φ) and $\psi_j(x)=1$ if $x\in \operatorname{supp}\varphi_j$; $j=0, 1, 2, \ldots$. It is not hard to see that there exist couples of systems $\{\varphi_j\}$, $\{\psi_j\}$ having these properties. Furthermore, we may assume that $\psi_j(x)=\psi(2^{-j}x)$ if $j=1, 2, \ldots$ We have

$$\| \Delta^l_{h,1} F^{-1}\varphi_j F f \mid L_p(R_n) \| \leq c \| F^{-1}\varphi_j F f \mid L_p(R_n) \| \tag{33}$$

and $$\| \Delta^l_{h,1} F^{-1}\varphi_j F f \mid L_p(R_n) \| = \| F^{-1}(e^{ih\xi_1}-1)^l \psi_j F F^{-1}\varphi_j F f \mid L_p(R_n) \|.$$

Let $j=1, 2, \ldots$. Applying formula (7) in [7, p. 57], then it follows that

$$\begin{aligned}(34)\quad \| \Delta^l_{h,1} F^{-1}\varphi_j F f \mid L_p(R_n) \| &\leq c 2^{jn\left(\frac{1}{p}-1\right)} \| F^{-1}(e^{ih\xi_1}-1)^l \psi_j \mid L_p(R_n) \| \, \| F^{-1}\varphi_j F f \mid L_p(R_n) \| \\ &\leq c' 2^{jn\left(\frac{1}{p}-1\right)} \| \Delta^l_{h,1} [F^{-1}\psi(2^{-j}\cdot)] \mid L_p(R_n) \| \, \| F^{-1}\varphi_j F f \mid L_p(R_n) \|.\end{aligned}$$

Here c and c' are independent of j. Because

$$\Delta^l_{h,1}[F^{-1}\psi(2^{-j}\cdot)](x)=2^{jn}(\Delta^l_{2^jh,1}F^{-1}\psi)(2^jx)$$

we have

$$\|\Delta^l_{h,1}[F^{-1}\psi(2^{-j}\cdot)]\mid L_p(R_n)\|\leq c2^{jn-jn/p}\|\Delta^l_{2^jh,1}F^{-1}\psi\mid L_p(R_n)\|\leq c'2^{jn(1-1/p)}(2^jh)^l,$$

where c and c' are independent of j and h. Putting this estimate in (34) and using (33) we have

$$\|\Delta^l_{h,1}[F^{-1}\varphi_jFf]\mid L_p(R_n)\|\leq c\min(1,(2^jh)^l)\|F^{-1}\varphi_jFf\mid L_p(R_n)\|. \tag{35}$$

Here $j=0, 1, 2,\ldots$, and c is independent of j and h. Obviously, $\Delta^l_{h,1}$ can be replaced by $\Delta^l_{h,k}$, where $k=2,\ldots, n$.

Step 2. We prove (31). It is sufficient to consider the term with $\Delta^l_{h,1}$. Let $h\sim 2^{-k}$, where $k=1, 2, 3,\ldots$. (32) and (35) yield

$$\|\Delta^l_{h,1}f\mid L_p(R_n)\|^p\leq c\sum_{j=0}^{k}2^{(j-k)lp}\|F^{-1}\varphi_jFf\mid L_p(R_n)\|^p+c\sum_{j=k+1}^{\infty}\|F^{-1}\varphi_jFf\mid L_p(R_n)\|^p.$$

Consequently,

$$\int_0^1 h^{-sq}\|\Delta^l_{h,1}f\mid L_p(R_n)\|^q\frac{dh}{h}\leq c\sum_{k=0}^{\infty}2^{skq}\sup_{2^{-k-1}\leq h\leq 2^{-k}}\|\Delta^l_{h,1}f\mid L_p(R_n)\|^q$$

$$\leq c'\sum_{k=0}^{\infty}2^{skq}\Big(\sum_{j=0}^{k}2^{(j-k)lp}\|F^{-1}\varphi_jFf\mid L_p(R_n)\|^p\Big)^{q/p} \tag{36}$$

$$+c'\sum_{k=0}^{\infty}2^{skq}\Big(\sum_{j=k+1}^{\infty}\|F^{-1}\varphi_jFf\mid L_p(R_n)\|^p\Big)^{q/p}.$$

If ε is an arbitrary positive number, then the right-hand side of (36) can be estimated from above by

$$c\sum_{k=0}^{\infty}\sum_{j=0}^{k}2^{(j-k)(l-s-\varepsilon)q}2^{jsq}\|F^{-1}\varphi_jFf\mid L_p(R_n)\|^q$$

$$+c\sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty}2^{(k-j)(s-\varepsilon)q}2^{jsq}\|F^{-1}\varphi_jFf\mid L_p(R_n)\|^q.$$

If ε is chosen in an appropriate way then $0<\varepsilon<s<s+\varepsilon<l$ and the last estimate show that

$$\int_0^1 h^{-sq}\|\Delta^l_{h,1}f\mid L_p(R_n)\|^q\frac{dh}{h}\leq c\sum_{j=0}^{\infty}2^{jsq}\|F^{-1}\varphi_jFf\mid L_p(R_n)\|^q\leq c'\|f\mid B^s_{p,q}(R_n)\|^q.$$

Similarly for $\|f\mid L_p(R_n)\|$. The proof is complete.

Remark 4. Immaterial modifications of the above proof show that under the hypotheses of Proposition 2

$$\Big(\int_{|h|\leq 1}|h|^{-sq}\|\Delta^l_hf\mid L_p(R_n)\|^q\frac{dh}{|h|^n}\Big)^{1/q}+\|f\mid L_p(R_n)\|\leq c\|f\mid B^s_{p,p}(R_n)\|. \tag{37}$$

Remark 5. Let $0<p\leq 1$, $0<q<\infty$, and $n\left(\frac{1}{p}-1\right)<s<l$. Then (4) yields that one can add the term $\|f|L_1(R_n)\|$ on the left-hand side of (31). Hence, a fundamental sequence in the quasi-norm of the left-hand side of (31) with the added term $\|f|L_1(R_n)\|$ is both a fundamental sequence in the space of all measurable functions and in $S'(R_n)$. This shows that (31) can be extended by continuity to all $f\in B^s_{p,q}(R_n)$, provided that $s>n(1/p-1)$. This argument fails if $s\leq n(1/p-1)$. This follows also from the fact that $\delta\in B^s_{p,q}(R_n)$ if $0<p<\infty$, $0<q<\infty$, and $s<n(1/p-1)$.

Proposition 3. *Let* $(n-1)/n<p<1$ *and* $\max(1, n(p^{-1}-1))<s<\min(2, 1/p)$, (*cf. the Figure*). *Then* $\|f|B^s_{p,p}(R_n)\|$ *and*

$$(38)\qquad \|f|L_p(R_n)\|+\sum_{j=1}^{n}\left(\int_0^\infty h^{-sp}\|\Delta^1_{h,j}f|L_p(R_n)\|^p\frac{dh}{h}\right)^{1/p}=\|f|B^s_{p,p}(R_n)\|^+$$

are equivalent quasi-norms on the set of all finite linear combinations $f(x)$ of characteristic functions of rectangles with sides parallel to the axes.

Proof. In this proof, $f(x)$ denotes always a finite linear combination of characteristic functions of rectangles with sides parallel to the axes. 2.2 (iv) and elementary calculations show that

$$(39)\qquad \|f|B^s_{p,p}(R_n)\|<\infty \quad\text{and}\quad \|f|B^s_{p,p}(R_n)\|^+<\infty.$$

Using $\Delta^1_{2h,1}=\Delta^2_{h,1}+2\Delta^1_{h,1}$ then we have

$$\int_0^\infty h^{-sp}\|\Delta^1_{h,1}f|L_p(R_n)\|^p\frac{dh}{h}=2^{-sp}\int_0^\infty h^{-sp}\|\Delta^1_{2h,1}f|L_p(R_n)\|^p\frac{dh}{h}$$

$$\leq 2^{-sp}\int_0^\infty h^{-sp}\|\Delta^2_{h,1}f|L_p(R_n)\|^p\frac{dh}{h}+2^{-sp+p}\int_0^\infty h^{-sp}\|\Delta^1_{h,1}f|L_p(R_n)\|^p\frac{dh}{h}.$$

Because $s>1$ and all terms in the last estimate are finite, we have

$$\int_0^\infty h^{-sp}\|\Delta^1_{h,1}f|L_p(R_n)\|^p\frac{dh}{h}\leq c\int_0^\infty\|\Delta^2_{h,1}f|L_p(R_n)\|^p h^{-sp}\frac{dh}{h}.$$

Similarly for $\Delta^1_{h,k}$ with $k=2,\ldots,n$. Because $s<2$ and $s>n(p^{-1}-1)$, Proposition 2, Remark 5, and (39) prove that $\|f|B^s_{p,p}(R_n)\|^+\leq c\|f|B^s_{p,p}(R_n)\|$. This estimate and (14) prove the proposition.

Corollary 2. *If* $(n-1)/n<p<1$ *and* $\max(1, n(p^{-1}-1))<s<p^{-1}$ *then any finite linear combination of characteristic functions of rectangles with sides parallel to the axes belongs to* $B^s_{p,p}(R_n)$, *however the set of all these functions is not dense in* $B^s_{p,p}(R_n)$.

The first assertion follows from 2.2 (iv). In order to prove the non-density we may assume that $s<2$. This follows from (2). (This additional assumption is automatically satisfied if $n\geq 2$.) Then Proposition 3 is applicable. If we assume that the finite linear combinations of characteristic functions of rectangles with sides parallel to the axes are dense in $B^s_{p,p}(R_n)$ then it follows that $\|f|B^s_{p,p}(R_n)\|^+$ is an equivalent quasi-norm of $B^s_{p,p}(R_n)$. However, if

$\|f \mid B^s_{p,p}(R_n)\|^+ < \infty$ and $f \in S(R_n)$ then it is not hard to see that $f(x) \equiv 0$, (here we use $s > 1$). This is a contradiction.

3.2. Direct products. Let N be the lattice of all points k in R_n such that $k = (k_1, \ldots, k_n)$, k_j are integers. Let $Q^{(k)} = \{x \mid k_j < x_j < k_j + 1$, where $j = 1, \ldots, n\}$. The characteristic function of $Q^{(k)}$ is denoted by χ_k. Obviously, $R_n = \bigcup_{k \in N} \bar{Q}^{(k)}$.

Proposition 4. *If p and s satisfy (12), then $\left(\sum_{k \in N} \|\chi_k f \mid B^s_{p,p}(R_n)\|^p\right)^{1/p}$ and $\left(\sum_{k \in N} \|f \mid B^s_{p,p}(Q^{(k)})\|^p\right)^{1/p}$ are equivalent quasi-norms in $B^s_{p,p}(R_n)$. In other words*

$$B^s_{p,p}(R_n) = \left(\sum_{k \in N} \oplus B^s_{p,p}(Q^{(k)})\right)_p. \tag{40}$$

Proof. (13) and the fact that S_0 from formula (23) is an extension operator from $B^s_{p,p}(Q)$ into $B^s_{p,p}(R_n)$ prove that

$$\|f \mid B^s_{p,p}(R_n)\|^p \leq c \sum_{k \in N} \|\chi_k f \mid B^s_{p,p}(R_n)\|^p \leq c' \sum_{k \in N} \|f \mid B^s_{p,p}(Q^{(k)})\|^p.$$

The reverse inequality is an easy consequence of (19).

4. Haar bases in Besov spaces. 4.1. Haar bases in $B^s_{p,q}(Q)$. Q has the meaning of (15) (unit cube in R_n). The system of Haar functions in Q is that one described in 2.3.

Theorem 1. (i). *The Haar functions in Q are a simultaneous Schauder basis in $B^s_{p,q}(Q)$ if*

$$\begin{cases} \text{either} & 1 < p < \infty, \quad 1/p - 1 < s < 1/p, \quad 0 < q < \infty \\ \text{or} & n/(n+1) < p \leq 1, \quad n(1/p - 1) < s < 1, \quad 0 < q < \infty. \end{cases} \tag{41}$$

(ii) *The assertion in (i) cannot be extended to $B^s_{p,q}(Q)$ if*

$$\begin{cases} \text{either} & 1 < p < \infty, & s \notin [1/p - 1, 1/p], & 0 < q < \infty \\ \text{or} & n/(n+1) \leq p \leq 1, & s \notin [n(1/p - 1), 1], & 0 < q < \infty \\ \text{or} & 0 < p < n/(n+1), & -\infty < s < \infty, & 0 < q < \infty. \end{cases} \tag{42}$$

Proof. Step 1. If $1 < p < \infty$, $p^{-1} - 1 < s < 1/p$, and $1 < q < \infty$ then part (i) has been proved in [6], 4.9.4. Let $n/(n+1) < p \leq 1$, $n(p^{-1} - 1) < s < 1$ and $0 < q < \infty$. If we assume that $f(x) \equiv 0$ is represented by Haar functions in $B^s_{p,q}(Q)$ then (2) shows that $f(x) \equiv 0$ is also represented by Haar functions in $B^{\tilde{s}}_{\tilde{p},\tilde{p}}(Q)$, where $\tilde{p}$ and $\tilde{s}$ are appropriate numbers with $1 < \tilde{p} < \infty$ and $0 < \tilde{s} < 1/\tilde{p}$. Since the Haar functions are a Schauder basis in $B^{\tilde{s}}_{\tilde{p},\tilde{p}}(Q)$ we obtain that the above representation of $f(x) \equiv 0$ must be the trivial representation. This proves the uniqueness of the representation. Hence, we must show that (18) is a representation in the above-mentioned sense. Using the interpolation formula (9), then it follows that we may assume, without loss of generality $p = q$.

Step 2. We prove that (18) is a representation in $B^s_{p,p}(Q)$ if $n/(n+1)<p\leq 1$ and $n(p^{-1}-1)<s<1$. First we show that there exists a constant c such that for all $j=1, 2, 3, \ldots$ and all $f\in C^\infty(\bar{Q})$,

$$\|P_{Z_j} f \mid B^s_{p,p}(Q)\| \leq c \|f \mid B^s_{p,p}(Q)\| . \tag{43}$$

We extend the proof given in [6, 4.9.4] (or in [5]) to the case under consideration here. We use the equivalent quasi-norm (20). Obviously,

$$\|P_{Z_j} f \mid L_1(Q)\| \leq \|f \mid L_1(Q)\| \leq c \|f \mid B^s_{p,p}(Q)\|. \tag{44}$$

Furthermore, if Q_r has the meaning of 2.3 with respect to the fixed decomposition Z_j, then we have

$$\int_{Q\times Q} \frac{|P_{Z_j}f(x)-P_{Z_j}f(y)|^p}{|x-y|^{n+sp}}\,dxdy \leq \sum_{l=1}^{N} \sum_m{}' \int_{Q_l\times Q_m} \ldots + \sum_{l=1}^{N} \sum_m{}'' \int_{Q_l\times Q_m} \ldots \tag{45}$$

In the first sum we put all couples (l, m) with $\bar{Q}_l\cap\bar{Q}_m\neq\emptyset$ and $l\neq m$. The second sum contains the rest. In order to estimate the first sum we fix l and assume, without loss of generality,

$$\int_{\tilde{Q}_l} f(x)dx=0 \quad \text{with} \quad \tilde{Q}_l=\bigcup Q_m, \tag{46}$$

where the union is taken over all m such that $\bar{Q}_m\cap\bar{Q}_l\neq\emptyset$. We have

$$\sum_m{}' \int_{Q_l\times Q_m} \ldots \leq c\sum_m{}'\Big[|Q_l|^{-p}\Big(\int_{Q_l}|f(x)|\,dx\Big)^p+|Q_m|^{-p}\Big(\int_{Q_m}|f(x)|\,dx\Big)^p\Big]\int_{Q_l\times Q_m}\frac{dxdy}{|x-y|^{n+sp}} \tag{47}$$
$$\leq c'|\tilde{Q}_l|^{-p+1-sp/n}\Big(\int_{\tilde{Q}_l}|f(x)|dx\Big)^p.$$

The equivalence of the quasi-norms (20) and (21), a simple homogeneity argument and (46) yield

$$\int_{\tilde{Q}_l}|f(x)|\,dx\leq c\,|\tilde{Q}_l|^{\frac{s}{n}+1-\frac{1}{p}}\Big(\int_{\tilde{Q}_l\times\tilde{Q}_l}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy\Big)^{1/p}. \tag{48}$$

Putting (48) in (47) we have

$$\sum_m{}' \int_{Q_l\times Q_m} \ldots \leq c\int_{\tilde{Q}_l\times\tilde{Q}_l}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy, \tag{49}$$

where c is independent of j (and l). Hence,

$$\sum_{l=1}^{N}\sum_m{}'\int_{Q_l\times Q_m}\ldots\leq c\|f\mid B^s_{p,p}(Q)\|^p. \tag{50}$$

In order to estimate the second sum in (45) we first note that

$$A_{l,m}=\Big|\frac{1}{|Q_l|}\int_{Q_l}f(x)dx-\frac{1}{|Q_m|}\int_{Q_m}f(x)dx\Big|\leq\frac{1}{|Q_l|}\int_{Q_l}|f(x)-c_m|\,dx, \tag{51}$$

where $c_m = |Q_m|^{-1} \int_{Q_m} f(x)dx$. The equivalence of the quasi-norms (19) and (20) and a homogeneity argument yield

$$A_{l,m} \leq c\,|Q_l|^{s/n-1/p}\Big(\int_{Q_l\times Q_l} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy\Big)^{1/p} + c\,|Q_l|^{-1/p}\Big(\int_{Q_l}|f(x)-c_m|^p dx\Big)^{1/p}. \tag{52}$$

Furthermore, if $x\in Q_l$ is fixed then

$$|f(x)-c_m| \leq \frac{1}{|Q_m|}\int_{Q_m}|f(x)-f(y)|\,dy.$$

Applying again the above inequality we have

$$|f(x)-c_m| \leq c\,|Q_m|^{s/n-1/p}\Big(\int_{Q_m\times Q_m}\frac{|f(z)-f(y)|^p}{|z-y|^{n+sp}}\,dzdy\Big)^{1/p} + c\,|Q_m|^{-1/p}\Big(\int_{Q_m}|f(x)-f(y)|^p dy\Big)^{1/p}.$$

Integration over $x\in Q_l$ yields

$$|Q_l|^{-1/p}\Big(\int_{Q_l}|f(x)-c_m|^p dx\Big)^{1/p} \leq c\,|Q_m|^{s/n-1/p}\Big(\int_{Q_m\times Q_m}\frac{|f(z)-f(y)|^p}{|z-y|^{n+sp}}\,dzdy\Big)^{1/p} + c\,|Q_l|^{-1/p}\,|Q_m|^{-1/p}\Big(\int_{Q_m\times Q_l}|f(x)-f(y)|^p dxdy\Big)^{1/p}.$$

Putting this estimate in (52) we have

$$A_{l,m} \leq c\,|Q_l|^{s/n-1/p}\Big(\int_{Q_l\times Q_l}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy + \int_{Q_m\times Q_m}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy\Big)^{1/p} + c\,|Q_l|^{-1/p}\,|Q_m|^{-1/p}\Big(\int_{Q_l\times Q_m}|f(x)-f(y)|^p dxdy\Big)^{1/p}. \tag{53}$$

Now, we are able to estimate the second sum in (45). If $b_{l,m}$ denotes the distance of the rectangles Q_l and Q_m then we have

$$\sum_{l=1}^{N}\sum_{m}{}'' \int_{Q_l\times Q_m} \ldots \leq c \sum_{l,m}\frac{A_{l,m}^p}{b_{l,m}^{n+sp}}\,|Q_l|\,|Q_m|. \tag{54}$$

It is easy to see that for fixed l

$$\sum_{m}{}''\frac{1}{b_{l,m}^{n+sp}} \leq c\,|Q_l|^{-1-sp/n}. \tag{55}$$

Similarly for fixed m. Putting (53) in (54) and using (55) then we have

$$\sum_{l=1}^{N}\sum_{m}{}''\int_{Q_l\times Q_m}\ldots \leq c\int_{Q\times Q}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy. \tag{56}$$

(44), (50), and (56) prove (43).

Step 3. Again we assume that $n/(n+1)<p\leqq 1$ and $n(p^{-1}-1)<s<1$. If f is a finite linear combination of Haar functions then $f(x)=P_{Z_j}f$, provided that $j\geqq j_0$ is sufficiently large. Hence, functions of that type are represented in $B^s_{p,p}(Q)$ by (18). Furthermore, any smooth function can be approximated in $B^s_{p,p}(Q)$ by finite linear combinations of Haar functions. The proof of this fact is the same as in [6, 4.9.4] (Step 3 of the proof of Theorem 1). (43), 2.2 (i), and the just-mentioned facts prove that each function $f\in B^s_{p,p}(Q)$ can be represented by (18). The proof of (i) is complete.

Step 4. We prove (ii). Let $0<p<\infty$, $0<q<\infty$ and $s>1/p$. In that case the Haar functions do not belong to $B^s_{p,q}(Q)$ (with exception of $f(x)=1$). Let us assume that all Haar functions are elements of $B^s_{p,q}(Q)$. Then it follows from the multiplication properties described in 2.2 (v) that $\psi(x')\varkappa(x_n)\chi(x_n)\in B^s_{p,q}(R_n)$, where $x'=(x_1,\ldots,x_{n-1})$, $\psi(x')\in S(R_{n-1})$, $\varkappa(x_n)\in S(R_1)$ and $\chi(x_n)$ is a characteristic function of an appropriate interval. In [8], 2.6.4, Remark 1 (and Substep 4.1 of the proof of the theorem in 2.6.4 in [8]) we proved that this is not possible. (For details we refer to [8], 2.6.4.) Next we assume $1<p<\infty$ and $s<p^{-1}-1$ and $0<q<\infty$. The operators P_{Z_j} from (17), (18) are formally self-adjoint, i. e. if $\varphi\in C_0^\infty(Q)$ and $\psi\in C_0^\infty(Q)$ then

$$\int_Q (P_{Z_j}\varphi)(x)\psi(x)dx=\int_Q \varphi(x)(P_{Z_j}\psi(x))dx. \tag{57}$$

In an obvious way this relation can be extended to $\varphi\in S(R_n)$, where we assume that $\psi\in C_0^\infty(Q)$ is fixed. If we assume that P_{Z_j} is a bounded operator from $B^s_{p,q}(Q)$ into itself with $1<p<\infty$, $0<q<\infty$ and $s<p^{-1}-1$, then (5), the multiplication properties in 2.2 (v) and the usual duality argument show that $P_{Z_j}\psi$ belongs to $B^{-s}_{p',q'}(R_n)$. However $-s>1/p'$ and we obtain a contradiction to the above case. Next we assume that $0<p\leqq 1$, $0<q<\infty$, and $s<n(p^{-1}-1)$. We use the same duality argument and (6) and (3). We have

$$P_{Z_j}\psi\in B^{-s+n(1/p-1)}_{\infty,q'}(R_n)\subset\mathcal{C}^\sigma(R_n), \tag{58}$$

where $0<\sigma<-s+n(1/p-1)$. Since any function belonging to $\mathcal{C}^\sigma(R_n)$ is continuous, (58) yields a contradiction. It remains the case (cf. the Figure) $(n-1)/n\leqq p\leqq 1$, $0<q<\infty$, $\max(1, n(p^{-1}-1))\leqq s\leqq 1/p$, where $s>1$. The interpolation formulas (9) and (11) show that we may assume $(n-1)/n<p=q<1$, $\max(1, n(p^{-1}-1))<s<1/p$ (in particular, the interpolation formulas (9) and (11) prove that the maximal extension of the shaded region in the Figure, characterizing the spaces $B^s_{p,q}(Q)$, $0<q<\infty$, for which the Haar functions are a simultaneous Schauder basis, must be convex). Now Corollary 2 shows that the finite linear combinations of the Haar functions are not dense in $B^s_{p,p}(Q)$. Hence, they are not a Schauder basis. The proof is complete.

4.2. Haar bases in $B^s_{p,q}(R_n)$. If $Q^{(k)}$ has the meaning of 3.2, then we denote the corresponding Haar functions in the sense of 2.3 (with respect to $Q^{(k)}$, instead of Q) by $H_j(Q^{(k)})$ (generating function of the range of $P_{Z_{j+1}}-P_{Z_j}$, resp. P_{Z_0}, with respect to $Q^{(k)}$ instead of Q). Here $j=1, 2, \ldots$. The set of all these functions is ordered by $\{H_l(R_n)\}_{l=1}^\infty$, where $H_l(R_n)=H_j(Q^{(k)})$, $l=l(j,k)$, in such a way that $l(j_1, k_1) > l(j_2, k_2)$ if either $k_1 \neq k_2$ or $k_1 = k_2$ and $j_1 > j_2$. $H_l(R_n)$ are denoted as the Haar functions in R_n.

Theorem 2. (i) *The Haar functions in R_n are a simultaneous Schauder basis in $B^s_{p,q}(R_n)$ if (41) is satisfied.*

(ii) *The assertion in (i) cannot be extended to $B^s_{p,q}(R_n)$ if (42) is satisfied.*

Proof. Part (i) with $p=q$ follows from Theorem 1 (i), Proposition 4, and well-known standard arguments. The general case $(p \neq q)$ is a consequence of (9). The proof of part (ii) is the same as in Theorem 1.

4.3. Remarks.

Remark 6. The Theorems 1 and 2 have a final character beside limiting cases. Probably, the corresponding systems of Haar functions are not Schauder bases for $B^s_{p,q}$ if (s, p) belongs to the boundary of the shaded region in the Figure and $0 < q < \infty$. In some cases one can prove this conjecture, i. e. if $1 \leqq p < \infty$, $s = 1/p$, and $0 < q < \infty$. In this case, the argument at the beginning of Step 4 of the proof of Theorem 1 is applicable, cf. [8, 2.6.3], formula (18). By duality one obtains the case $1 < p < \infty$, $s = p^{-1} - 1$, and $1 < q < \infty$.

Remark 7. We considered in this paper only Haar functions. One could try to use spline functions of higher order for the construction of Schauder basis. We refer to S. Ropela [4]. There are reasons that the shaded region in the Figure, characterizing simultaneous Schauder bases, can be extended (in dependence of the order of the spline functions) if one uses higher spline functions. Negative results in this sense of the part (ii) of the two theorems can be obtained on the basis of [8, 2.6.4].

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