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ON THE CAUCHY PROBLEM FOR A CLASS OF QUASILINEAR DEGENERATE PARABOLIC AND ULTRAPARABOLIC EQUATIONS

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The local existence of classical solutions is studied for the Cauchy problem for a class quasilinear degenerate parabolic and ultraparabolic equations of second order.

In the present paper are proved the existence and uniqueness of classical solutions for small values of the variable t for equations of the form

$$a^{ij}(x, t, u)u_{ij}+a^{i}(x, t, u)u_{i}-u_{t}-a(x, t, u)u=f(x, t) \tag{1}$$

with $a^{ij}(x, t, s)\xi_i\xi_j\geqq 0$, $\xi=(\xi_1, \ldots, \xi_n)\in R^n$ (where subscripts are used to denote differentiation and summation convention is accepted) in cylindrical regions of the form $G=\Omega\times(0, T)$, where Ω is a bounded region with piece-wise smooth boundary in R^n. The study of the case $\Omega=R^n$ as well as the study of some boundary value problems for ultraparabolic equations are based on the results thus obtained. The necessary a priori estimates are corollaries of those obtained in [1; 2] (of which works the present paper is a continuation) on the assumption that

$$a(x, t, s)\geqq a_0>0, \tag{2}$$

where a_0 is large positive constant. This assumption is omitted in the present work because of the particular form of the equation (1), the existence of solution for small values of t only being proved with the aid of substitution of the form $u=e^{\lambda t}v$ (leading to inequality of the kind (2) in the linear case for appropriate λ).

In what follows, where there is no need to emphasize the special role of some of the variables (for instance t in (1)), we shall put for brevity $(x, t)=y$, i. e. $y=(y_1, \ldots, y_m)\in R^m$, where m is the number of independent variables, and we can rewrite the lefthand side of (1) in the form

$$M(u)\equiv b^{kl}(y, u)u_{kl}+b^{k}(y, u)u_{k}-b(y, u)u=f(y), \tag{3}$$

$k, l=1, \ldots, n+1$, where the meaning of the coefficients is obvious.

As in [1] the following notations are accepted. If the coefficients of (3) are defined for $(y, s)\in\bar{G}\times[-M, M]$ $(M>0)$ and the boundary ∂G is piece-wise smooth, let $\partial G=S_3\cup S$, where

$$S_3=\{y\in\partial G: b^{kl}(y, s)\nu_k(y)\nu_l(y)>0,\ s\in[-M, M]\}$$

$$S=\{y\in\partial G: b^{kl}(y, s)\nu_k(y)\nu_l(y)=0,\ s\in[-M, M]\}$$

and if $\beta(y, s)=(b^{k}(y, s)-b^{kl}_{l}(y, s))\nu_k(y)$ let $S=S_1\cup S_2$, where

(4) $$S_1=\{y\in S:\beta(y,s)\leqq 0,\quad s\in[-M,M]\},$$

(5) $$S_2=\{y\in S:\beta(y,s)>0,\ s\in[-M,M]\}$$

(here $\nu_k(y)$ denotes the k-th component of the unit exterior normal at a point $y\in\partial G$).

If $f(y)$ and $g(y,s)$ are differentiable functions with domains in R^m and R^{m+1}, respectively, and $\alpha=(\alpha_1,\ldots,\alpha_m)$ is an ordered m-tuple of nonnegative integers, let by definition

$$D^\alpha f(y)=\partial^{|\alpha|}f(y_1,\ldots,y_m)/\partial y_1^{\alpha_1}\ldots\partial y_m^{\alpha_m},$$

$$D^{\alpha+p}g(y,s)=\partial^{|\alpha|+p}g(y_1,\ldots,y_m,s)/\partial y_1^{\alpha_1}\ldots\partial y_m^{\alpha_m}\partial s^p,$$

where $|\alpha|=\Sigma_{\mu=1}^m\alpha_\mu$. If k is a nonnegative integer and if f is k times continuously differentiable in $\bar G$, let by definition

(6) $$\|f\|_{C^0(\bar G)}=\max_{\bar G}|f(y)|,$$

(7) $$\|f\|^2_{C^k(\bar G)}=\Sigma_{|\alpha|\leqq k}\|D^\alpha f\|^2_{C^0(\bar G)},$$

(8) $$|D^k f|=\max_{|\alpha|=k}\|D^\alpha f\|_{C^0(\bar G)}.$$

The following notations shall be used too [1].

(9) $$M_r=\max_{\bar G\times[-M,M]}\{|D^{\alpha+p}b^{kl}|,\quad |D^{\alpha+p}b^k|,\quad |D^{\alpha+p}b|\},$$

$|\alpha|+p=r$, $r=0, 1, 2, 3$, and

$$C_1(u)=M_1(1+|D^1u|),$$

$$C_2(u)=M_2(1+|D^1u|+|D^1u|^2)+M_1|D^2u|,$$

$$C_3(u)=M_3(1+|D^1u|+|D^1u|^2+|D^1u|^3)+M_2|D^2u|(1+|D^1u|)+M_1|D^3u|.$$

The following propositions are preparatory for the proof of the main result.

Proposition 1. *Let the coefficients of the operator*

(10) $$M_\varepsilon(v;u)\equiv\varepsilon\Delta u+b^{kl}(y,v)u_{kl}+b^k(y,v)u_k-b(y,v)u$$

with $b^{kl}(y,s)\xi_k\xi_l\geqq 0$, $b(y,s)\geqq b_0>0$ are defined and four times continuously differentiable in $R^m\times[-M,M]$, the inequalities (9) *hold with $G=R^m$ and outside some bounded region ω, the operator $M_\varepsilon(v;u)$ has the form*

(11) $$M_\varepsilon(v;u)=(\mu+\varepsilon)\Delta u-b_0u,$$

where $1/4\leqq\mu+\varepsilon\leqq 1/3$, $0<\varepsilon$, Δ is the Laplace operator. Let Q_R be the cube $Q_R=\{y\in R^m: |y_k|<R,\ k=1,\ldots,m\}$, where $R>0$ is such that $Q_R\supset\bar\omega$. If $v\in C^4(\bar Q_R)$ and $h\in C^4(\bar Q_R)$ with supp $h\subset\omega$, *then every solution of the boundary value problem*

(12) $$M_\varepsilon(v;u)=h(y)$$

in Q_R and

(13) $$u/_{\partial Q_R}=0$$

is of the class $C^5(\overline{Q}_R)$. *Furthermore if* $b_0>1$ *and* R *is sufficiently large, the inequality*

(14) $$\Sigma_{|\alpha|\leqq 2}\max_{\partial Q_R}|D^\alpha u|^2\leqq b_0^{-1}\|h\|^2_{C^0(\overline{Q}_R)}$$

holds.

Proof. As it is well known ([3, p. 235]), the boundary value problem (12), (13) has an unique solution u, which is at least of the class C^5 in Q_R and on the smooth parts of the boundary. We shall prove that actually u can be extended to a function $\tilde{\tilde{u}}\in C^5(Q_{3R})$. This may be done by "extension by symmetry" as in [4]. To this end let us consider in the region

$$P_1=\{y\in R^m: -R<y_1<3R,\ |y_k|<R,\quad k=2,\dots,m\}$$

the operator

(15) $$\tilde{L}(v;\ u)\equiv\varepsilon\Delta u+\tilde{b}^{kl}(y)u_{kl}+\tilde{b}^k(y)u_k-\tilde{b}(y)u,$$

whose coefficients for fixed $v\in C^4(\overline{Q}_R)$ and $y=(y_1, y')$, $y'=(y_2,\dots,y_m)$, are defined by

(16) $$\tilde{b}^{kl}(y_1, y')=\begin{cases}b^{kl}(y, v(y)) & \text{for}\quad |y_1|<R,\\ (-1)^{\sigma_1}b^{kl}(2R-y_1, y', v(2R-y_1, y')) & \text{for}\quad R\leqq y_1<3R,\end{cases}$$

where $\sigma_1=0$ for $k\neq 1$, $l\neq 1$ and for $k=1$, $l=1$ and $\sigma_1=1$ otherwise;

$$\tilde{b}^{k}(y_1, y')=\begin{cases}b^{k}(y, v(y)) & \text{for}\quad |y_1|<R,\\ (-1)^{\sigma_2}b^{k}(2R-y_1, y', v(2R-y_1, y')) & \text{for}\quad R\leqq y_1<3R,\end{cases}$$

where $\sigma_2=0$ for $k\neq 1$ and $\sigma_2=1$ otherwise;

$$\tilde{b}(y_1, y')=\begin{cases}b(y, v(y)) & \text{for}\quad |y_1|<R\\ b(2R-y_1, y', v(2R-y_1, y')) & \text{for}\quad R\leqq y_1<3R\end{cases}$$

and let

$$\tilde{h}(y_1, y')=\begin{cases}h(y) & \text{for}\quad |y_1|<R,\\ -h(2R-y_1, y') & \text{for}\quad R\leqq y_1<3R.\end{cases}$$

Because of (11) and supp $h\subset\overline{\omega}$, the extensions defined above will have in P_1 the same smoothness as the coefficients of (10) have in Q_R. Furthermore, the operator (15) is an elliptic operator, since for $R\leqq y_1<3R$ and $\xi^*=(\xi_1^*,\dots,\xi_m^*)=(-\xi_1,\dots,\xi_m)$

$$\tilde{b}^{kl}(y_1, y')\xi_k\xi_l=b^{kl}(2R-y_1, y', v(2R-y_1, y'))\xi_k^*\xi_l^*\geqq\varepsilon|\xi^*|^2=\varepsilon|\xi|^2.$$

Let $\tilde{u}$ be the solution of the boundary value problem

(17) $$\tilde{L}(v;\ u)=\tilde{h}$$

in P_1 and

(18) $$u|_{\partial P_1}=0.$$

Obviously $\tilde{u}\in C^5(P_1)$ and on the smooth parts of ∂P_1. From (16) it follows that $\tilde{b}^{kl}(y_1, y')=(-1)^{\sigma_1}\tilde{b}^{kl}(2R-y_1, y')$ for $(y_1, y')\in P_1$ and analogous equalities hold for the remaining coefficients. This allows the computation of the value of the operator (15) for the function $\tilde{u}(2R-y_1, y')$, whence

(19) $$\tilde{L}(v;\ \tilde{u}(2R-y_1, y'))=-\tilde{L}(v;\ u(y_1, y')).$$

Let

(20) $$\tilde{w}(y_1, y')=\tilde{u}(y_1, y')+\tilde{u}(2R-y_1, y').$$

Then (18) and (19) imply $\tilde{L}(v;\ \tilde{w})=0$ in P_1 and $\tilde{w}|_{\partial P_1}=0$. Now the uniqueness theorem for elliptic equations implies that $\tilde{w}\equiv 0$. From (20) it follows that $\tilde{u}|_{y_1=R}=0$, i. e. that u and $\tilde{u}$ are the solutions of the same boundary value problem in Q_R and by the uniqueness theorem $u(y_1, y')=\tilde{u}(y_1, y')$ for $-R<y_1\leq R$. Successive "symmetries" with respect to the walls $y_1=-R$, $|y_k|=R$, $k=2, \ldots, m$ allow the construction of a function $\tilde{\tilde{u}}\in C^5(Q_{3R})$ such that $\tilde{\tilde{u}}(y)=u(y)$ for $y\in Q_R$. When R is large (11) implies that for instance in the region $Q_{R+1}\setminus Q_{R-1}$, i. e. in a neighbourhood of ∂Q_R, the function $\tilde{\tilde{u}}$ satisfies the equation

(21) $$L'(u)\equiv(\mu+\varepsilon)\Delta\tilde{\tilde{u}}-b_0u=0.$$

According to the wellknown Schauder estimates [5, § 5.5], the inequality

(22) $$\|u\|_{C_{2+\gamma}(\Sigma_\varrho)}\leq C(1-\varrho/r)^{-(2+\gamma)^2}(|L'u|_{C_\gamma(\Sigma_r)}+|u|_{C_0(\Sigma_r)}),$$

holds, where $0<\gamma<1$, $\Sigma_\varrho=\{y\in R^m: |y|<\varrho\}$, $0<\varrho<r\leq r_0$ for every $u\in C_{2+\gamma}(\Sigma_r)$ and the constant C depends on γ, m, μ, b_0 and r_0 is determined from μ and b_0 only. The norms above are the Hölder's, defined as in [5]. Now (22) with $r=r_0$, $\varrho=r_0/2$ and (21) imply

$$\Sigma_{|\alpha|\leq 2}\max_{\Sigma_\varrho}|D^\alpha\tilde{\tilde{u}}|\leq C'2^9(2/r_0)^3(\|L'\tilde{\tilde{u}}_{C_\gamma(\Sigma_{r_0})}+|\tilde{\tilde{u}}|_{C_0(\Sigma_{r_0})})\leq C''\,|\tilde{\tilde{u}}\|_{C_0(\Sigma_{r_0})},$$

where C'' depends on γ, m, μ and b_0 only. From the above inequality applied in vicinity of every boundary point of Q_R and the way u is extended to $\tilde{\tilde{u}}$ it follows that

$$\Sigma_{|\alpha|\leq 2}\max_{\partial Q_R}|D^\alpha u|=\Sigma_{|\alpha|\leq 2}\max_{\partial Q_R}|D^\alpha\tilde{\tilde{u}}|\leq C''\|\tilde{\tilde{u}}|_{C_0(Q_{R+1}\setminus Q_{R-1})}=C''\|u|_{C_0(Q_R\setminus Q_{R-1})}.$$

Now lemma 1 from [1], applied to the boundary value problem (12), (13) implies $|u(y)|\leq b_0^{-1}\|h\|_{C^0(\bar{Q}_R)}$ and the considerations in [2], preceding lemma 2 imply that for R sufficiently large the inequality (14) will hold.

P r o p o s i t i o n 2. *Let the hypotheses of proposition 1 be satisfied and $b_0>1+(2m+1)C_1(v)+m'C_2(v)$, where m' depends on the dimension m only. Then $\Sigma_{|\alpha|=1}(D^\alpha u)^2\leq b_0^{-1}\|h\|^2_{C^1(\bar{Q}_R)}$.*

Proof. [2, lemma 2].

Proposition 3. *Let the hypotheses of proposition* 1 *be satisfied and* $b_0>1+(4m+2)C_1(v)+m''C_2(v)$, *where* $m''>m'$ *depends on* m *only. Then*

$$\Sigma_{|\alpha|\leq 2}(D^{\alpha}u)^2\leq b_0^{-1}\,\|h\|^2_{C^2(\bar{Q}_R)}. \tag{23}$$

Proof. From [1, lemma 3, (69)] follows the inequality

$$\Sigma_{|\alpha|\leq 2}(D^{\alpha}u)^2\leq \max\{b_0^{-1}\,\|h\|_{C^2(\bar{Q}_R)},\quad \max_{\partial Q_R}\Sigma_{|\alpha|\leq 2}(D^{\alpha}u)^2\}.$$

Now (23) follows from (14).

Proposition 4. *Let the hypotheses of proposition* 1 *be satisfied and* $b_0>2+m_1C_1(v)+m_2C_2(v)$, *where* $m_1>4m+2$ *and* $m_2>m''$ *are constants depending on* m *only. If* $R_u=\Sigma_{|\alpha|=3}(D^{\alpha}u)^2$ *then*

$$R_u\leq b_0^{-2}m_2^2M_1^2\max(\Sigma_{|\alpha|\leq 2}(D^{\alpha}u)^2)\max R_v+M', \tag{24}$$

where M' *does not depend on* u, v *and* ε.

Proof. From the interior Schauder estimates [3] in the form

$$\|u\|_{C_{3+\gamma}(\Sigma_\varrho)}\leq C'''(\|L'u\|_{C_{1+\gamma}(\Sigma_r)}+\|u\|_{C_0(\Sigma_r)}),$$

as above it follows that there exists a constant p, which does not depend on v and ε, such that $\max\partial_{Q_R} R_u\leq p$. Now (24) follows from [1, lemma 4, (112)].

Theorem 1. *Let* $\Omega\subset R^n$ *be a region with a piece-wise smooth boundary and* $G=\Omega\times(0,T)$. *For* $G'\supset\bar{G}$ let in $\bar{G}'\times[-M,M]$ *be defined and two times continuously differentiable the coefficients of the operator*

$$M(u)\equiv a^{ij}(x,t,u)u_{ij}+a^i(x,t,u)u_i-u_t-a(x,t,u)u, \tag{25}$$

theirs second derivatives being Lipschitz continuous and let $a^{ij}(x,t,s)\xi_i\xi_j\geq 0$ $(\xi\in R^n)$ *for* $(x,t)\in\bar{G}'$, $s\in[-M,M]$. *Let* $f\in C^2(\bar{G}')$ *its second derivatives being Lipschitz continuous and*

$$f(x,t)=0 \tag{26}$$

for $(x,t)\notin G$. *Let* $\Gamma=\partial\Omega\times[0,T]$ *be* S_2 *for the operator* (25). *Then there exists a real* τ *with* $0<\tau\leq T$, *such that if* $G_\tau=\Omega\times(0,\tau)$, *the boundary value problem* $M(u)=f$ *in* G_τ *and*

$$u|_{\Omega\cup(\partial\Omega\times[0,\tau])}=0$$

has unique classical solution.

Proof. Let the coefficients of the operator

$$M_0(v;u)\equiv b^{ij}(x,t,v)u_{ij}+b^{00}(x,t)u_{tt}+b^i(x,t,v)u_i+b^0(x,t)u_t-b(x,t,v)u \tag{27}$$

are defined in R^{n+1}, coincide in $\bar{G}$ with the coefficients of the operator (25) and $b^{ij}(x,t,s)\xi_i\xi_j+b^{00}\xi_0^2\geq 0$ $(\xi_0,\xi_1,\ldots,\xi_n)\in R^{n+1}$, $b^0(x\,t)=-1$ for $t\leq T$. Furthermore let there exist real numbers R' and R'' such that $R''>R'$, $Q_{R'}\supset\bar{G}$, the boundary $\partial Q_{R'}$ is S_2 for the operator (27) and outside the cube $Q_{R''}$ the operator (27) has the form $M_0(v;u)=\mu\Delta u$ $(1/4\leq\mu<1/3)$. Such extension exists, since the coefficients of (25) are defined in $G' \supset \bar{G}$. Now let the coefficients of the operators

$$(28)\qquad M_\varepsilon(v;\,u) \equiv b_\varepsilon^{ij}(x,\,t,\,v)u_{ij} + b_\varepsilon^{00}(x,\,t)u_{tt} + \varepsilon\Delta u$$
$$+ b_\varepsilon^{i}(x,\,t,\,v)u_i + b_\varepsilon^{0}(x,\,t)u_t - b_\varepsilon(x,\,t,\,v)u$$

be appropriate smooth approximations for the coefficients of (27) in R^{n+1}. Such approximations may be constructed for instance by using modifiers [6], and they can be choosed so that outside the cube $Q_{R''}$ the operators $M_\varepsilon(v;u)$ will have the form $M_\varepsilon(v;\,u) = (\mu+\varepsilon)\Delta u$. If now

$$(29)\qquad M_k = \max\{|D^{\alpha+p}b_\varepsilon^{ij}|,\ |D^{\alpha+p}b_\varepsilon^{00}|,\ |D^{\alpha+p}b_\varepsilon^{i}|,\ |D^{\alpha+p}b_\varepsilon^{0}|,\ |D^{\alpha+p}b_\varepsilon|\},$$

$|\alpha|+v=k$, $k=1,\,2,\,3$, the constants M_k exist and are determined by the maximums of the derivatives up to second order of the coefficients of (25) and by the Lipschitz constants of these derivatives. Let $f_\varepsilon(x,t)$ be sufficiently smooth approximations for $f(x,\,t)$. Because of (26) we can assume that

$$(30)\qquad \operatorname{supp} f_\varepsilon \subset Q_{R'},\quad f_\varepsilon(x,\,t)=0\ \ (t<0).$$

Let $\eta_k = \max_{|\alpha|=k}|D^\alpha f_\varepsilon(x,\,t)|$, $k=1,\,2,\,3$. These constants are determined from the correspondent derivatives of f and the Lipschitz constants of the second derivatives. If $F_{\varepsilon,\lambda}(x,\,t)=e^{-\lambda t}f_\varepsilon(x,\,t)$ $(\lambda>0)$ (30) and Taylor's formula imply

$$f_\varepsilon(x,\,t) = t\,\frac{\partial f_\varepsilon}{\partial t}(x,\,\tau') = \frac{t^2}{2!}\,\frac{\partial^2 f_\varepsilon}{\partial t^2}(x,\,\tau'') = \frac{t^3}{3!}\,\frac{\partial^3 f_\varepsilon}{\partial t^3}(x,\,\tau'''),$$

whence $|F_{\varepsilon,\lambda}(x,\,t)|\leq e^{-\lambda t}t\eta_1 = e^{-\lambda t}(\lambda t)\eta_1/\lambda \leq m\eta_1/\lambda$, where $m=\max_{\xi\geq 0}e^{-\xi}\xi$. Therefore

$$(31)\qquad \|F_{\varepsilon,\lambda}\|_{C^0(R^{n+1})}\leq m'\eta_1/\lambda,$$

the inequalities

$$(32)\qquad \|F_{\varepsilon,\lambda}\|_{C^0(R^{n+1})}\leq m'\eta_2/\lambda^2,$$

$$(33)\qquad \|F_{\varepsilon,\lambda}\|_{C^1(R^{n+1})}\leq m'\eta_1,\quad \|F_{\varepsilon,\lambda}\|_{C^1(R^{n+1})}\leq m'\eta_2/\lambda,$$

$$(34)\qquad \|F_{\varepsilon,\lambda}\|_{C^2(R^{n+1})}\leq m'\eta_2,$$

$$(35)\qquad \|F_{\varepsilon,\lambda}\|_{C^3(R^{n+1})}\leq m'\eta_3,$$

where m' denotes a constant depending on the dimension $n+1$ and on the maximums of the functions $e^{-\xi}$, $\xi e^{-\xi}$, $\xi^2 e^{-\xi}$ for $\xi\geq 0$ being obtained similarly.

If u is a smooth function in the halfspace $\{t\leq\tau\}$, it can be extended to a smooth function $\tilde{u}$ in R^{n+1} (as in [5, § 4.8]) in such a manner that

$$(36)\qquad \|\tilde{u}\|_{C^3(R^{n+1})}\leq\theta\,\|u\|_{C^3(\{t\leq\tau\})},$$

where θ is independent of τ and u, provided the right-hand side is defined.

If for a function u, with $|D^0u|_\tau$, $|D^1u|_\tau$, $|D^2u|_\tau$, are denoted the maximums of the function and its derivatives for $t\leq\tau$ (cf. (8)), then for the function

$$(37)\qquad w=e^{\lambda t}u\quad(\lambda>0)$$

hold the inequalities

$$(38)\qquad \begin{cases} |D^0 w|_\tau \leq e^{\lambda\tau} |D^0 u|, \\ |D^1 w|_\tau \leq e^{\lambda\tau}(|D^1 u| + \lambda |D^0 u|), \\ |D^2 w|_\tau \leq e^{\lambda\tau}(|D^2 u| + 2\lambda |D^1 u| + \lambda^2 |D^0 u|). \end{cases}$$

Let now

$$(39)\qquad L_{\varepsilon,\lambda}(v, u) \equiv M_\varepsilon(v; u) - \lambda u,$$

where for $a(x, t, s) \geqq \lambda_0$ (it is possible $\lambda_0 < 0$ also) the constant $\lambda > 0$ satisfies the inequality

$$(40)\qquad \lambda + \lambda_0 > 2 + n_1 M_1 + n_2 M_2 + (n_1 M_1 + n_2 M_2)\theta m' \eta_1 (1 + (\lambda + \lambda_0)^{-1/2}) + n_2 M_2 \theta^2 m'^2 \eta_1^2 (1 + (\lambda + \lambda_0)^{-1/2})^2 + n_2 M_1 \theta m' \eta_2 (3 + (\lambda + \lambda_0)^{-1/2}),$$

where M_1 and M_2 are defined with (29) and n_1 and n_2 are connected with the dimension $n+1$ as in propositions 2—4.

Let $R > R''$ be such that in Q_R hold the hypotheses of propositions 1—4 for λ defined as above, let $\{\varepsilon_\nu\}_{\nu=0}^\infty$ be a sequence of real numbers with $im_{\nu\to\infty}\varepsilon_\nu = 0$ and $\varepsilon_\nu \geqq \varepsilon_{\nu+1} > 0$, and let

$$(41)\qquad \tau = \min(T, \lambda^{-1} \ln(\lambda + \lambda_0)^{1/2}).$$

Let us consider the sequences of functions $\{u_\nu\}_{\nu=0}^\infty$, $\{\widetilde{u}_\nu\}_{\nu=0}^\infty$ and $\{v_\nu\}_{\nu=0}^\infty$ defined as follows: $u_0 = \widetilde{u}_0 = v_0 = 0$ and if u_ν and $\widetilde{u}_\nu$ are already defined, then let $v_{\nu+1}$ be the unique solution of the boundary value problem $L_{\varepsilon_{\nu+1},\lambda}(\widetilde{u}_\nu; v_{\nu+1}) = F_{\varepsilon_{\nu+1},\lambda}$ in Q_R and $v_{\nu+1}|_{\partial Q_R} = 0$; now by definition

$$(42)\qquad u_{\nu+1} = e^{\lambda t} v_{\nu+1}$$

and $\widetilde{u}_{\nu+1}$ is a smooth extension of $u_{\nu+1}$ across the hyperplane $\{t = \tau\}$, i. e.

$$(43)\qquad \widetilde{u}_{\nu+1}(x, t) = u_{\nu+1}(x, t)$$

for $t \leq \tau$. From (36)—(38) and (42) follow the inequalities

$$(44)\qquad \begin{cases} |D^0 \widetilde{u}_\nu| \leq \theta |D^0 u_\nu|_\tau \leq \theta e^{\lambda\tau} |D^0 v_\nu|, \\ |D^1 \widetilde{u}_\nu| \leq \theta |D^1 u_\nu|_\tau \leq \theta e^{\lambda\tau}(|D^1 v_\nu| + \lambda |D^0 v_\nu|), \\ |D^2 \widetilde{u}_\nu| \leq \theta |D^2 u_\nu|_\tau \leq \theta e^{\lambda\tau}(|D^2 v_\nu| + 2\lambda |D^1 v_\nu| + \lambda^2 |D^0 v_\nu|) \end{cases}$$

for every ν. We shall prove for every ν the inequalities

$$(45)\qquad \lambda + \lambda_0 > 2 + n_1 M_1 + n_2 M_2 + (n_1 M_1 + n_2 M_2) |D^1 \widetilde{u}_\nu| + n_2 M_2 |D^1 \widetilde{u}_\nu|^2 + n_2 M_1 |D^2 \widetilde{u}_\nu|,$$

$$(46)\qquad |v_\nu| \leq (\lambda + \lambda_0)^{-1} \| F_{\varepsilon_\nu,\lambda} \|_{C^0(\overline{Q}_R)},$$

$$(47)\qquad |D^1 v_\nu|^2 \leq \max_{\overline{Q}_R} \Sigma_{|\alpha|=1} (D^\alpha v_\nu)^2 \leq (\lambda + \lambda_0)^{-1} \| F_{\varepsilon_\nu,\lambda} \|^2_{C^1(\overline{Q}_R)},$$

(48) $$|D^2 v_\nu|^2 \leq \max_{\overline{Q}_R} \Sigma_{|\alpha|=2}(D^\alpha v_\nu)^2 \leq (\lambda+\lambda_0)^{-1} \| F_{\varepsilon_\nu,\lambda} \|^2_{C^2(\overline{Q}_R)}.$$

Indeed, (45)—(48) hold for $\nu=0$ since $u_0=\widetilde{u}_0=v_0=0$ and (40). Let them hold for some ν. Since (45) and (40) imply that all the hypotheses of propositions 1—3 are satisfied, the inequalities (46)—(48) hold for $\nu+1$. From the right-hand side of (45), (41), (46)—(48) for $\nu+1$, (31)—(34) and (40) follows that (45) holds for $\nu+1$, i. e. (45)—(48) hold for every ν. Now (45), (47) and proposition 4 imply

$$R_{v_{\nu+1}} \leq n_2^2 M_2^2 m'^2 \eta_2^2 (\lambda+\lambda_0)^{-3} \max R_{\widetilde{u}_\nu} + M'.$$

But from $\max R_{\widetilde{u}_\nu} \leq \theta^2 \max_{(\{t\leq\tau\})} R_{u_\nu} \leq \theta^2 e^{2\lambda\tau} R_{v_\nu} + M''$, where according to the already proved M'' does not depend on ν follows $\max R_{v_{\nu+1}} \leq \alpha \max R_{v_\nu} + M'''$, where according to (40) and (41), $0<\alpha<1$. Now the uniform boundedness of the sequence $\{R_{v_\nu}\}_{\nu=0}^\infty$ follows directly, whence follows the same for $\{R_{u_\nu}\}_{\nu=0}^\infty$ and $\{R_{\widetilde{u}_\nu}\}_{\nu=0}^\infty$. By using (40), (46)—(48) and considering the differences

$$L_{\varepsilon_{\nu+1},\lambda}(\widetilde{u}_\nu\,;\, v_{\nu+1}) - L_{\varepsilon_\nu,\lambda}(\widetilde{u}_{\nu-1}\,;\, v_\nu) = F_{\varepsilon_{\nu+1},\lambda} - F_{\varepsilon_\nu,\lambda}$$

similarly to [1, lemma 5] is proved that $\{\widetilde{u}_\nu\}_{\nu=1}^\infty$, and cosequently $\{u_\nu\}_{\nu=1}^\infty$ and $\{v_\nu\}_{\nu=1}^\infty$ are uniformly convergent in $\widetilde{Q}_R$.

The Arzelá-Ascoli theorem and (46)—(48) imply that we can chose an uniformly convergent subsequence $\{v_{\nu_\mu}\}_{\mu=1}^\infty$ such that

$$v = \lim_{\mu\to\infty} v_{\nu_\mu}, \quad \lim_{\mu\to\infty} D^\alpha v_{\nu_\mu} = D^\alpha v, \quad |\alpha| \leq 2.$$

Since the sequence $\{\widetilde{u}_{\nu_\mu}-1\}_{\mu=1}^\infty$ is also uniformly convergent, we can pass to limits in the equalities $L_{\varepsilon_{\nu_\mu},\lambda}(\widetilde{u}_{\nu_\mu-1}\,;\, v_{\nu_\mu}) = F_{\varepsilon_{\nu_\mu},\lambda}$ and for $\widetilde{u} = \lim_{\mu\to\infty} \widetilde{u}_{\nu_\mu-1}$ we obtain $M_0(\widetilde{u}\,;\, v) - \lambda v = F_\lambda$ in Q_R, where $F_\lambda(x,t) = e^{-\lambda t} f(x,t)$. The equality $v|_{\partial Q_{R'}}=0$ is obtained by integrating two times by parts the identity

(49) $$M_0(\widetilde{u}\,;\, v)v - \lambda v^2 = 0$$

over the set $Q_R \setminus Q_{R'}$ as in [1, 2] (and as usually done in the linear case [7]), since λ is sufficiently large, $f(x,t)\equiv 0$, $\partial Q_{R'}$ is S_2 for the operator (27) and the derivatives of the functions $\widetilde{u}$ and v are bounded by constants as in (47) and (48). Now applying the same procedure again over the set $\{(x,t) : -R < t < \tau,\ |x_k| < R',\ k=1,\dots n\} \setminus \overline{G}$ we get $v|_{\Omega\cup\Gamma}=0$, since $\Omega\cup\Gamma$ is S_2 for the operator (27) and (26) holds for the function f. From (42) follows that

$$u = \lim_{\mu\to\infty} u_{\nu_\mu} = e^{\lambda t} v$$

and now (27) implies that the function $u = e^{\lambda t} v$ satisfies the equation

$$a^{ij}(x,t,\widetilde{u})u^{ij} + a^i(x,t,\widetilde{u})u^i - u_t - a(x,t,\widetilde{u})u = f(x,t)$$

in G and $u|_{\Omega\cup\Gamma}=0$. Since (43) implies that $u(x,t) = \widetilde{u}(x,t)$ for $0\leq t\leq\tau$, this proves the existence.

The uniqueness is proved similarity as in [1, theorem 2] or [8]. More precisely, if u is the solution constructed above and v is some solution, by considering the function $w=e^{\lambda t}(u-v)$ in the region $\Omega\times(0, \tau)$ and applying the maximum principle for parabolic equations with appropriate $\lambda>0$ one obtains that $w\equiv 0$.

R e m a r k. The solution u obtained above has derivatives up to second order, its second derivatives being Lipschitz continuous. At that, as seen from the proof, the maximums of the derivatives and the Lipschitz constants are determined from the coefficients and the right-hand side of the equation only and do not depend on the region. The same is true for the constant λ and consequently for the interval $[0, \tau]$ in which the existence is proved. These facts are used in the proof of the next theorem.

T h e o r e m 2. *For* $0<\beta_k\in R$, $k=1,\dots, m$, $0<\delta$ *and*

$$P=\{y\in R^m: |y_k|<\beta_k,\quad k=1,\dots, m\},\ P_\delta=\{y\in R^m: |y_k|<\beta_k+\delta,\ k=1,\dots, m\},$$
$$H=R^n_x\times P,\quad H_\delta=R^n_x\times P_\delta$$

let the coefficients of the operator

$$L(u)\equiv a^{ij}(x, y, u)u_{ij}+a^i(x, y, u)u_i+b^k(x, y, u)\partial u/\partial y_k-c(x, y, u)u \tag{50}$$

be defined, two times continuously differentiable and bounded together with their derivatives in $\bar{H}_\delta\times[-M, M]$, *the second derivatives being Lipschitz continuous. Furthermore, let*

$$a^{ij}(x, y, s)\xi_i\xi_j\geq 0\quad (\xi\in R^n,\ (x, y)\in\bar{H}_\delta,\quad s\in[-M, M]), \tag{51}$$
$$b^m(x, y, s)\leq\mu_0<0 \tag{52}$$

and the coefficients $b^k(x, y, s)$ *have constant signes on the hyperplanes* $y_k=\pm\beta_k$ $(k=1,\dots, m-1)$. *If* $f\in C^2(\bar{H}_\delta)$ *has bounded derivatives up to second order, its second order derivatives being Lipschitz continuous and satisfies compatibility conditions of the form*

$$f(x, y)=\frac{\partial f}{\partial y_k}(x, y)=\frac{\partial^2 f}{\partial y_k^2}(x, y)=0$$

for $y_k=\pm\beta_k$ $(k=1,\dots, m-1)$ *if this part of the boundary is* S_2 (*for the definition of* S_1 *and* S_2 *see* (4) *and* (5)) *and* $f(x, y)=0$ *for* $y_m<0$, *then there exists a constant* η *with* $0<\eta\leq\beta_m$, *such that the equation* $M(u)=f(x, y)$ *has unique bounded classical solution in the region* $H\cap\{(x, y): 0<y_m<\eta\}$ *satisfying the boundary values* $u|_{S_2\cup\{y_m=0\}}=0$.

P r o o f. Application of theorem 1 for regions of the form

$$Q_k=\{(x, y): |x_i|<R,\ i=1,\dots, n;\ |y_k|<\beta_k,\ k=1,\dots, m-1;\ 0<y_m<\beta_m\}$$

for appropriate modifications of the operator (50), so that the part of the boundary $x_i=\pm R$ is S_2 (cf. 2, theorem 3).

Let $\{R_\nu\}_{\nu=0}^{\infty}$ be a sequence of real numbers with $R_\nu\to\infty$. Theorem 1 mplies the existence of a sequence of functions $\{u_\nu\}_{\nu=0}^{\infty}$, which are uniformly bounded together with their derivatives up to second order, the second order derivatives having the same Lipschitz constants for $0<y_m<\eta$. From this sequence can be choosed a subsequence which is uniformly convergent on every compact subset of H. The limit of this subsequence is the desired solution.

The uniqueness is proved as in [8] or as in [2, theorem 5]. More precisely if u is the solution obtained and v is some bounded solution, by considering the function $w=e^{\lambda y_m}(u-v)$ it is established that w satisfies equation of the form

$$a^{ij}(x, y, v)\frac{\partial^2 w}{\partial x_i \partial x_j}+a^i(x, y, v)\frac{\partial w}{\partial x_i}+b^k(x, y, v)\frac{\partial w}{\partial y_k}$$
$$+(\lambda b^m(x, y, v)-c(x, y, v))w=\Phi(x, y),$$

where for an appropriate $\lambda>0$ the quantity $\lambda b^m(x, y, v)-c(x, y, v)$ can be made sufficiently large and negative according to (52). Further on the proof goes by applying a variant of the maximum principle for ultraparabolic equations.

For $m=1$, theorem 2 is close to the results obtained in [8]; for $n=1$, $m=2$ it supplies a proof for the results, anounced in [9].

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