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SOME TRANSFORMATIONS OF RENEWAL PROCESSES

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This note deals with homogeneous transformations of stationary renewal point processes on the real axis. Points will be translated successive. The translation of a single point depends on the distance to and the translation of its "foregoer". There are given wide classes of transformations which left the Poisson process invariant. Some characterizations of the Poisson process and of other renewal processes are derived. $M/G/1$-systems with service finish if a new call arrives and with service time dependent on the previous idle time are studied. For such systems necessary and sufficient conditions for the output to be Poissonian were found.

0. Introduction. Let $Y_0, X_1, X_2, \dots$ be independent real random variables, $X_1, X_2, \dots$ being identically distributed according to a common probability distribution P. Let K be a stochastic kernel from R^2 into R. Let $Y_1, Y_2, \dots$ be random variables such that Y_i given $X_i, Y_{i-1}, \dots, X_1, Y_0$ has distribution $K(X_i, Y_{i-1}, .)$. Suppose that there is a unique stationary initial distribution Q for the Markov process $Y_0, Y_1, Y_2, \dots$. Let Y_0 be distributed according to Q. The stationary sequence of random variables $Z_i = X_i + Y_i - Y_{i-1}$, $i = 1, 2, 3, \dots$ may be viewed as a transformation of the sequence $X_1, X_2, \dots$. Call the sequence $X_1, X_2, \dots$ (or the distribution P) K-invariant if $Z_1, Z_2, \dots$ are independent random variables, indentically distributed according to P.

If X_i, Z_i are nonnegative random variables with positive and finite expectations the above procedure describes a class of transformations of stationary renewal processes into stationary point processes. The distances between consecutive points before and after the transformation are X_i and Z_i, respectively, the Y_i are the translations of the points. This is a type of homogeneous translation without overtaking.

In section 2 we give the construction of the transformation and some sufficient conditions for P to be K-invariant.

In sections 3 and 4 we investigate for some interesting classes of kernels the structure of K-invariant distributions.

In particular, Theorems 2 and 4 characterize the Gamma-distribution. Theorem 5 gives a wide class of transformations under which the Poisson process is invariant.

Section 5 deals with input-output-transformations of a single server system with service-finish if a new call arrives and idle time dependent service times.

Let us point out two special results. Let K be the kernel which describes the input-output-transformation of a single server service system with exponential distributed service times and service-finish if a new call arrives. Then P is K-invariant iff P is an Exponential distribution (Theorem 1).

Consider the following transformation. Points move in one direction only, to the left say, the "new" position of a point is equidistributed in the interval which is constituted by the "old" position of this point and the "new" position of its left neighbour. The Poisson processes are the only stationary renewal processes invariant under this transformation (Corollary 3 of Theorem 4).

Some of these transformations are applicable in queueing theory.

1. Notations. Let $[M, \mathfrak{M}]$ be a measurable space. By $[M^n, \mathfrak{M}^n]$, $[M^\infty, \mathfrak{M}^\infty]$ we denote the product spaces $M^n = \times_{i=1}^n M$, $\mathfrak{M}^n = \times_{i=1}^n \mathfrak{M}$, $M^\infty = \times_{i=-\infty}^\infty M$, $\mathfrak{M}^\infty = \times_{i=-\infty}^\infty \mathfrak{M}$. For $P \in \mathbf{P}_{\mathfrak{M}}$ (the set of probability distributions on $[M, \mathfrak{M}]$) P^n, P^∞ denote the product probabilities $\times_{i=1}^n P$, $\times_{i=-\infty}^\infty P$, on $[M^n, \mathfrak{M}^n]$ and $[M^\infty, \mathfrak{M}^\infty]$ respectively.

Let $[R, \mathfrak{L}]$, $[R_+, \mathfrak{L}_+]$, $[R_-, \mathfrak{L}_-]$ $\mathbf{P}, \mathbf{P}_+, \mathbf{P}_-$ be the real axis and the subsets $[0, \infty)$, $(-\infty, 0]$ equipped with the σ-algebras of Borel subsets and the sets of probability distributions on these spaces respectively.

Denote by $l, l_+, l_-, l_A, \Pi_\lambda, \delta_b$ the Lebesgue measure on R, R_+, R_-, the equidistribution on the Borel set A, the exponential distribution with parameter λ and the Dirac measure at the point b, respectively.

Let $\mathbf{F}(\mathfrak{M})$ be the set of $(\mathfrak{M}, \mathfrak{L}_+)$ — measurable bounded functions. χ_A denotes the indicator function of the set $A \in \mathfrak{M}$.

Let $\mathbf{K}(M_1, \mathfrak{M}_2]$ be the set of stochastic kernels from $[M_1, \mathfrak{M}_1]$ into $[M_2, \mathfrak{M}_2]$, i. e. the set of mappings K defined on $M_1 \times \mathfrak{M}_2$, such that $K(\cdot, A) \in \mathbf{F}(\mathfrak{M}_1)$ for all $A \in \mathfrak{M}_2$ and $K(m, \cdot) \in \mathbf{P}_{\mathfrak{M}_2}$ for all $m \in M_1$. If not otherwise specified the domain of integration will be the real axis.

2. The basic model. Definition 1. *Take $K \in \mathbf{K}(R^2, \mathfrak{L})$. Let $P \in \mathbf{P}$, (or P^∞) be called K-transformable if there is one and only one $Q \in \mathbf{P}$ such that for all $f \in \mathbf{F}(\mathfrak{L})$*

$$(1) \qquad \int Q(dy_0) \int P(dx_1) \int K(x_1, y_0, dy_1) f(y_1) = \int Q(dy_1) f(y_1).$$

If P is K-transformable, by the extension theorem of Kolmogorov there is a unique probability distribution $W_{P,K}$ on $[(R^2)^\infty, (\mathfrak{L}^2)^\infty]$ with

$$(2) \qquad \int_{(R^2)^\infty} W_{P,K}(d(\mathbf{x}, \mathbf{y})) h(y_{i-1}, x_i, y_i) = \int Q(dy_{i-1}) \int P(dx_i) \int K(x_i, y_{i-1}, dy_i) h(y_{i-1}, x_i, y_i)$$

for all $h \in \mathbf{F}(\mathfrak{L}^3)$ and all integers i. Now consider the mapping L, $(L((\mathbf{x}, \mathbf{y})))_i = x_i + y_i - y_{i-1}$, $i = \ldots -1, 0, 1, \ldots$, and denote by $Z_{P,K} = W_{P,K} \circ L^{-1}$ the induced probability distribution on $[R^\infty, \mathfrak{L}^\infty]$.

Definition 2. *If P^∞ (or P) is K-transformable we call $Z_{P,K}$ the K-transformation of P^∞ (or P). Further we say that P^∞ (or P) is K-invariant if $Z_{P,K} = P^\infty$.*

Remarks. a. In fact (1) means that Q, respectively $Q \times P$, are stationary initial distributions for the Markov processes $Y_0, Y_1, Y_2, \ldots$ and $[X_1, Y_0]$, $[X_2, Y_1], \ldots$ mentioned in the introduction. Then $W_{P,K}$ is the distribution of the corresponding two-sided stationary Markov process $\ldots [X_{-1}, Y_{-2}], [X_0, Y_{-1}], [X_1, Y_0], [X_2, Y_1], \ldots$

b. If $P \in \mathbf{P}_+$, $\int x P(dx) < \infty$ and $K(x, y, [y-x, +\infty)) = 1$ for all $x \in R_+, y \in R$, it is clear that $Z_{P,K}(R_+^\infty) = 1$. Therefore, P^∞ and $Z_{P,K}$ can be interpreted as the Palm distributions of stationary point processes. The above construction gives then a homogeneous transformation of stationary renewal processes into stationary point processes (see [3; 4; 6]).

In this note however we will use such formulations as "Π_λ is K-invariant" instead of "the stationary Poisson point process with intensity λ is invariant under the homogeneous transformation induced by the kernel K".

Now we give some conditions which guarantee that P is K-invariant.

Definition 3. *Assume that P is K-transformable. We say that P satisfies condition* I_1 *(respectively* I_2*) with respect to K if for all $f, g \in \mathbf{F}(\mathfrak{L})$, we have*

$$(3)\qquad \int Q(dy_0) \int P(dx_1) \int K(x_1, y_0, dy_1) g(y_1) f(x_1+y_1-y_0) = \int Q(dy_0) g(y_0) \cdot \int P(dx_1) f(x_1)$$

respectively

$$(4)\qquad \int Q(dy_0) \int P(dx_1) \int K(x_1, y_0, dy_1) h(y_0, x_1, y_1) = \int Q(dy_0) \int P(dx_1) \int K(x_1, y_0, dy_1) h(y_1, x_1+y_1-y_0, y_0),$$

for all $h \in \mathbf{F}(\mathfrak{L}^3)$ (thereby Q is as in Definition 1).

To be short we will say "P satisfies I_1-K (or I_2-K)".

Remark. Formula (3) means that $Z_1 = X_1 + Y_1 - Y_0$ and Y_1 are independent and distributed according to P and Q respectively. Equation (4) says that $[Y_0, X_1, Y_1]$ and $[Y_1, Z_1, Y_0]$ have the same distributions (cp. [6]).

Proposition 1. a. *If P satisfies I_2-K then P satisfies I_1-K.*

b. *If P satisfies I_1-K then P is K-invariant.*

Proof. Let $f, g \in \mathbf{F}(\mathfrak{L})$ define $h \in \mathbf{F}(\mathfrak{L})$ by $h(y_0, x_1, y_1) = g(y_1) f(x_1+y_1-y_0)$ and use (4). Then (3) follows at once.

To prove b. assume that P satisfies I_1-K. Let $f_1, f_2, \ldots, f_n \in \mathbf{F}(\mathfrak{L})$. By the definition of $Z_{P,K}$ and equation (2) we have

$$\int_{R^\infty} Z_{P,K}(d\mathbf{z}) f_1(z_1) \ldots f_n(z_n) = \int_{(R^2)^\infty} W_{P,K}(d(\mathbf{x}, \mathbf{y})) f_1(x_1+y_1-y_0) \ldots f_n(x_n+y_n-y_{n-1})$$

$$= \int Q(dy_0) \int P(dx_1) \int K(x_1, y_0, dy_1) f_1(x_1+y_1-y_0) \int P(dx_2) \times \ldots \int P(dx_n) \int K(x_n, y_{n-1}, dy_n) f_n(x_n+y_n-y_{n-1}).$$

Now define $g \in \mathbf{F}(\mathfrak{L})$ by

$$g(y_1) = \int P(dx_2) \int K(x_2, y_1, dy_2) f_2(x_2+y_2-y_1) \int P(dx_3) \ldots \int P(dx_n) \int K(x_n, y_{n-1}, \times dy_n) f_n(x_n+y_n-y_{n-1})$$

and use (3). We get

$$\int_{R^\infty} Z_{P,K}(d\mathbf{z}) f_1(z_1) \ldots f_n(z_n) = \int Q(dy_0) g(y_0) \cdot \int P(dx_1) f_1(x_1).$$

Repeating this argument and using that $Z_{P,K}$ is shift-invariant we get $Z_{P,K} = P^\infty$ and b. is proved.

3. C-motions. In this section we consider conditional independent motions of points in one direction without overtaking (C-motions). Roughly speaking the points move independently from each other in one direction to the "left" say, the translation law of a single point depends only on the distance to its left neighbour before the translation. To be more precise let

$$\mathbf{C} = \{ C \in \mathbf{K}(R_+, \mathfrak{L}_-) : C(x, [-x, 0]) = 1 \text{ for all } x \in R_+ \}.$$

For each $C \in \mathbf{C}$ define $K_C \in \mathbf{K}(R^2, \mathfrak{L})$ by $K_C(x, y, \cdot) = C(x, \cdot)$.

Proposition 2. *For all $P \in \mathbf{P}_+$ the following statements hold.*
a. *For each $C \in \mathbf{C}$, $\mathbf{P}$ is K_C-transformable.*
b. *P is K_C-invariant iff for all $f, g \in \mathbf{F}(\mathfrak{L})$*

$$\int P(dx_1) \int C(x_1, dy_1) f(x_1+y_1) g(y_1) \tag{5}$$
$$= \int P(dx_1) \int C(x_1, dy_1) f(x_1+y_1) \cdot \int P(dx_1) \int C(x_1, dy_1) g(y_1).$$

Proof. It is clear that we have for all $f \in \mathbf{F}(\mathfrak{L})$

$$\int P(dx_1) \int K_C(x_1, y_0, dy_1) f(y_1) = \int P(dx_1) \int C(x_1, dy_1) f(y_1).$$

Therefore (1) is fulfilled iff $\int Q(dy) f(y) = \int P(dx_1) \int C(x_1, dy_1) f(y_1)$.

To prove b. we first conclude from (5) that P satisfies $\mathrm{I}_2 - K_C$ (and hence it is K_C-invariant by Proposition 1). Let $f_1, f_2, f_3 \in \mathbf{F}(\mathfrak{L})$. We have using (5)

$$\int Q(dy_0) \int P(dx_1) \int K_C(x_1, y_0, dy_1) f_1(y_0) f_2(x_1+y_1) f_3(y_1)$$
$$= \int Q(dy_0) f_1(y_0) \cdot \int P(dx_1) \int C(x_1, dy_1) f_2(x_1+y_1) f_3(y_1)$$
$$= \int Q(dy_0) f_1(y_0) \cdot \int P(dx_1) \int C(x_1, dy_1) f_2(x_1+y_1) \cdot \int P(dx_1) \int C(x_1, dy_1) f_3(y_1)$$
$$= \int Q(dy_0) f_1(y_0) \int P(dx_1) \int C(x_1, dy_1) f_2(x_1+y_1) \cdot \int Q(dy_1) f_3(y_1).$$

From here we get for all $h \in \mathbf{F}(\mathfrak{L}^3)$

$$\int Q(dy_0) \int P(dx_1) \int K_C(x_1, y_0, dy_1) h(y_0, x_1+y_1, y_1)$$
$$= \int Q(dy_0) \int P(dx_1) \int K_C(x_1, y_0, dy_1) h(y_1, x_1+y_1, y_0).$$

But this is equivalent to (4), because the mapping $(y_0, x_1, y_1) \to (y_0, x_1+y_1, y_1)$ is a one to one and in both directions $(\mathfrak{L}^3, \mathfrak{L}^3)$ — measurable mapping from R^3 onto R^3. Hence P satisfies $\mathrm{I}_2 - K$.

Now suppose that P is K_C-invariant. Then we have for all $s, t \geq 0$

$$\int_{R^\infty} Z_{P,K}(d\mathbf{z}) e^{-sz_1} e^{-tz_2} = \int_{R^\infty} Z_{P,K}(d\mathbf{z}) e^{-sz_1} \int_{R^\infty} Z_{P,K}(d\mathbf{z}) e^{-tz_2}.$$

By definition of $Z_{P,K}$ and formula (2) this is equivalent to

$$\int Q(dy_0) e^{sy_0} \cdot \int P(dx_1) \int C(x_1, dy_1) e^{-s(x_1+y_1)} e^{ty_1} \cdot \int P(dx_1) \int C(x_1, dy_1) e^{-t(x_1+y_1)}$$
$$= \int Q(dy_0) e^{sy_0} \cdot \int P(dx_1) \int C(x_1, dy_1) e^{-s(x_1+y_1)} \cdot \int Q(dy_1) e^{ty_1} \cdot \int P(dx_1) \int C(x_1, dy_1) e^{-t(x_1+y_1)}$$

for all $s, t \geqq 0$.

The first and the last factor of this product are Laplace transforms of the distributions of the nonnegative random variables $-Y_0$ and X_1+Y_1 and therefore nonzero and finite for all $s, t \geq 0$. Now (5) follows easily.

Remark. Because (5) means that $-Y_1, X_1+Y_1$ are independent we can formulate Proposition 2 as follows: $P \in \mathbf{P}_+$ is K_C-invariant for some $C \in \mathbf{C}$ iff there exist independent nonnegative random variables V_1, V_2 such that V_1+V_2 has distribution P and $C(x, \cdot)$ is a version of the conditional distribution of $-V_2$ under condition $V_1+V_2=x$.

Hence there is a one-to-one correspondence between those $C \in \mathbf{C}$ which leave a fixed $P \in \mathbf{P}_+$ invariant and the representations of P as a convolution of two $P_1, P_2 \in \mathbf{P}_+$.

It is more interesting however to seek all K_C-invariant $P \in \mathbf{P}_+$ for fixed $C \in \mathbf{C}$. Theorems 1 and 2 below give all K_C-invariant P for some nice kernels.

Consider the input-output-transformation of a steady state single server system with service-finish if a new call arrives. That is, the points of a stationary renewal process will try to move independently from each other a random distance (to the left say). The interaction consists in stopping the moving by the ("old") position of the left neighbour.

Let $F \in \mathbf{P}_+$ and define $C_F \in \mathbf{C}$ by

$$\int C_F(x, dy) f(y) = \int F(dt) f(-t) \chi_{[0, x]}(t) + f(-x) F([x, +\infty]), \; f \in \mathbf{F}(\mathfrak{L}).$$

Theorem 1. a. *Let* $F = \Pi_\mu$, $\mu > 0$, $P \in \mathbf{P}_+$, $P(\{0\}) < 1$. *P is K_{C_F}-invariant iff $P = \Pi_\lambda$ for some $\lambda > 0$.*

b. *If F has a strong positive density with respect to l_+ and $F \neq \Pi_\mu$ for all $\mu > 0$, then there is no K_{C_F}-invariant $P \in \mathbf{P}_+$.*

Proof. This follows easy from Proposition 2 and the following theorem which can be found in [2].

Theorem. *Let V_1, V_2 independent nonnegative random variables with distributions G and F respectively. Suppose that F has a strong positive density with respect to l_+ and $G(\{0\}) < 1$. Then the random variables* min (V_1, V_2), $V_1 -$ min (V_1, V_2) *are independent iff $G = \Pi_\lambda$ and $F = \Pi_\mu$ for some $\lambda, \mu > 0$.*

Now we consider the following motion. The points move independent of each other. The "new position" of a point is equidistributed in the interval which is bounded by the "old" positions of this point and its left neighbour. Corollary 1 of the following Theorem 2 states that the Erlang distributions order two are the only distributions invariant under this motion.

Theorem 2. *Let S be a measure on $[R_+, \mathfrak{L}_+]$ with a strong positive density s with respect to l_+. Write $S(x)$ instead of $S([0, x])$. Suppose that*

$$\int_0^\infty dx\, \lambda e^{-\lambda x} S(x) = \int S(dx) e^{-\lambda x} = 1 \text{ for some } \lambda > 0. \text{ Define } C_S \in \mathbf{C} \text{ by}$$

$$\int C_S(x, dy) f(y) = [S(x)]^{-1} \cdot \int S(dy) f(-y) \chi_{[0, x]}(y), \; f \in \mathbf{F}(\mathfrak{L}).$$

$P \in \mathbf{P}_+$, $P(\{0\}) < 1$, is K_{C_S}-invariant iff P has a density $p(x) = \lambda e^{-\lambda x} S(x)$ with respect to l_+.

Proof. If P has density $p(x) = \lambda e^{-\lambda x} S(x)$ we obtain for $f, g \in \mathbf{F}(\mathfrak{L})$

$$\int P(dx_1) \int C_S(x_1, dy_1) f(x_1 + y_1) g(y_1) = \int_0^\infty dx\, \lambda e^{-\lambda x} \int S(dy) \chi_{[0, x]}(y) f(x-y) g(-y)$$

$$= \int S(dy) g(-y) \int_y^\infty dx\, \lambda e^{-\lambda x} f(x-y) = \int S(dy) g(-y) e^{-\lambda y} \cdot \int_0^\infty dx\, \lambda e^{-\lambda x} f(x).$$

Hence (5) is fulfilled and P is K_{C_S}-invariant by Proposition 2.

Let on the other hand $P \in \mathbf{P}_+$ with $P(\{0\}) < 1$ be K_{C_S}-invariant. First from (5) it is easy to see that $P(\{0\}) = 0$. Second we note that P has a density p. In fact by Proposition 2 P is a convolution of $P_1 \in \mathbf{P}$ defined by $\int P_1(dy) f(y) = \int P(dx) \int C_S(x, dy) f(-y)$, $f \in \mathbf{F}(\mathfrak{L})$ with a certain $P_2 \in \mathbf{P}_+$ (the distribution of $X_1 + Y_1$). Because the distributions $C_S(x, \cdot)$ have densities for all $x > 0$ P_1 and P have densities also.

Now we obtain using (5) (put there $f(x) = e^{-t_1 x}$, $g(x) = e^{t_2 x}$) for $t_1, t_2 \geq 0$

$$\int_0^\infty \int_0^\infty dy\, dx\, s(y) R(x+y) e^{-t_1 y} e^{-t_2 x}$$

$$= \int_0^\infty \int_0^\infty dy\, dx\, s(y) R(x+y) e^{-t_1 y} \cdot \int_0^\infty \int_0^\infty dy\, dx\, s(y) R(x+y) e^{-t_2 x}$$

(put $R(x)=p(x)/S(x)$). Hence $s(y)R(x+y)=\int_0^\infty dx\, s(y)R(x+y)\cdot\int_0^\infty dy\, s(y)R(x+y)$, $l_+\times l_+$-a. e. Because s is strong positive it cancels out. Multiplying by $e^{-t_1x}e^{-t_2y}$ and integrating over $R_+\times R_+$ yields for $t_1, t_2\geqq 0$

$$t_1t_2\cdot\int_0^\infty dx R(x)\,x\cdot\int_0^\infty dx R(x)\,(e^{-t_1x}-e^{-t_2x})$$

$$=(t_2-t_1)\int_0^\infty dx\, R(x)(1-e^{-x_1}\cdot\int_0^\infty dx R(x)\,(1-e^{-t_2x}).$$

Dividing by t_2-t_1, letting t_2 tend to t_1 and defining L by $L(t)=\int_0^\infty dx R(x)\,(1-e^{-tx})$ we get

$$L^2(t)=-ct^2\frac{d}{dt}L(t),\ L(o)=0,\ c=\int_0^\infty xR(x)dx.$$

Solving this equation we find $L(t)=ct/(1+cdt)$, $d>0$, and $p(x)=S(x)\,(cd^2)^{-1}e^{-x/cd}$. Now $\int S(dx)\lambda e^{-\lambda x}=1$ implies $d=1$ and $\lambda=1/c$. The Theorem is proved.

Corollary 1. *Let $C\in\mathbf{C}$ defined by*

$$\int C(x, dy)f(y)=\frac{1}{x}\int_0^x dy f(-y),\ f\in\mathbf{F}(\mathfrak{L}).$$

$P\in\mathbf{P}_+$, $P(\{0\})<1$, is K_C-invariant iff $P=\Pi_\lambda*\Pi_\lambda$ for some $\lambda>0$.

Proof. Choose in Theorem 2 $S=\lambda^{-1}l_+$.

4. D-motions. A C-motion of the previous section we can imagine as a simultaneous motion of all points (all the information needed is the position of points before the motion). Now we consider a class of motions with successive translation of the points. A point will "wait" until its left neighbour is translated and then "move" (without overtaking) according to a distribution $D(d,\cdot)$ which depends on the distance d between this point and the "new" position of its left neighbour.

Let $\mathbf{D}=\{D\in\mathbf{K}(R,\mathfrak{L})$. $D(v,[-v,+\infty))=1$ for all $v\in R\}$ and define for $D\in\mathbf{D}$ $K_D\in\mathbf{K}(R^2,\mathfrak{L})$ by $K_D(x,y,\cdot)=D(x-y,\cdot)$. It is not to expect that each $P\in\mathbf{P}_+$ is K_D-transformable for all $D\in\mathbf{D}$. We remember that K-transformability means that the Markov process of translations has a unique stationary initial distribution. In some cases we can find a special stationary initial distribution and want to prove uniqueness. This we can establish by the following

Proposition 3. *Suppose that $P\in\mathbf{P}_+$ has a strong positive density with respect to l_+. Let $D\in\mathbf{D}$. There exists at most one $Q_D\in\mathbf{P}$ such that $\int Q_D(dy_0)\int P(dx_1)\int D(x_1-y_0, dy_1)f(y_1)=\int Q_D(dy_1)f(y_1)$ for all $f\in\mathbf{F}(\mathfrak{L})$.*

To prove this it is sufficient to show that all stationary Markov processes with transition function $H(y,\cdot)=\int P(dx)D(x-y,\cdot)$ are ergodic. In fact one can show that $H(y,A)=\chi_A(y)$ Q—a. e. implies $Q(A)=0$ or $Q(A)=1$ for every stationary initial distribution Q, i. e. all invariant sets are trivial. We omit the details of the proof because it is only "technical".

Before turning to special D-motions we collect some useful properties of such P which satisfy condition I_1-K_D.

Lemma 1. *If $P\in\mathbf{P}$ satisfies I_1-K_D then the following equations are valid*

a. *for all $f\in\mathbf{F}(\mathfrak{L}^2)$*

$$\int Q(dy_0)\int P(dx_1)\int D(x_1-y_0, dy_1)f(y_1, x_1-y_0)=\int Q(dy_0)\int P(dx_1)f(y_0, x_1-y_0);$$

b. *for all $g\in\mathbf{F}(\mathfrak{L}^3)$*

$$\int Q(dy_0)\int P(dx_1)\int D(x_1-y_0,\, dy_1)g(y_0,\, x_1-y_0,\, y_1)$$
$$=\int Q(dy_0)\int P(dx_1)\int D(x_1-y_0, dy_1)g(y_1,\, x_1-y_0,\, y_0)\,;$$

c. *If P has a density p with respect to l then for all $h\in \mathbf{F}(\mathfrak{L})$*

$$\int Q(dy_0)\, p(v+y_0)\int D(v,\, dy_1)h(y_1)=\int Q(dy_0)\, p(v+y_0)h(y_0),\ \ l-a.\,e.$$

Proof. Equation a. follows from (3) replacing there $f(x)\cdot g(y)$ by $f(y,\, x-y)$ Now let $g\in\mathbf{F}(\mathfrak{L}^3)$. Using a. twice we can write

$$\int Q(dy_0)\int P(dx_1)\int D(x_1-y_0,\, dy_1)g(y_0,\, x_1-y_0,\, y_1)$$
$$\int Q(dy_0)\int P(dx_1)\int D(x_1-y_0,\, dt)\int D(x_1-y_0,\, dy_1)g(t,\, x_1-y_0,\, y_1)$$
$$=\int Q(dy_0)\int P(dx_1)\int D(x_1-y_0,\, dy_1)\int D(x_1-y_0,\, dt)g(t,\, x_1-y_0,\, y_1)$$
$$=\int Q(dy_0)\int P(dx_1)\int D(x_1-y_0,\, dt)g(t,\, x_1-y_0,\, y_0).$$

It remains to show c. Let $h, k\in\mathbf{F}(\mathfrak{L})$, define $g\in\mathbf{F}(\mathfrak{L}^3)$ by $g(y_0, x_1, y_1)=h(y_0)\,.\,k(x_1)$ and use b. Then c. follows immediately.

Because b. is equivalent to (4) we have (see Proposition 1)

Proposition 4. *Let $P\in\mathbf{P}$, $D\in\mathbf{D}$. Then P satisfies I_1-K_D iff P satisfies I_2-K_D.*

The following theorem gives invariant distributions for some interesting motions.

Theorem 3. *Suppose that S is as in Theorem 2. Define $D_S\in\mathbf{D}$ by*

$$\int D_S(v,\, dy)f(y)=(S(v))^{-1}\,.\int_0^v S(dy)f(y-v),\ \ f\in\mathbf{F}(\mathfrak{L}).$$

The distribution $P\in\mathbf{P}_+$ with density $p(x)=s(x)e^{-\lambda x}$ is K_{D_S}-invariant.

Proof. For $f,\, g\in\mathbf{F}(\mathfrak{L})$ we obtain

$$\int dve^{-\lambda v}g(v)\int_0^v S(dy_0)\int D(v;\, dy_1)f(y_1)=\int dve^{-\lambda v}g(v)\int_0^v S(dy_0)f(y_0-v),$$

$$\int S(dy_0)e^{-\lambda y_0}\int_{y_0}^\infty dve^{-\lambda(v-y_0)}g(v)\int D(v,\, dy_1)f(y_1)$$

$$=\int S(dy_0)e^{-\lambda y_0}\int_{y_0}^\infty dve^{-\lambda(v-y_0)}g(v)f(y_0-v),$$

$$\int P(dy_0)\int \Pi_\lambda(dz)g(z+y_0)\int D(z+y_0,\, dy_1)f(y_1)=\int P(dy_0)\int\Pi_\lambda(dz)g(z+y_0)f(-z).$$

Hence (3) is fulfilled with $Q(A)=\Pi_\lambda(-A)$, $K=K_{DS}$, P. Thus, P is K_{DS}-invariant by Propositions 1 and 3.

Corollary 2. *Suppose that $D_\alpha(v,\cdot)$ has density $v^{-1}\alpha(1+x/v)^{\alpha-1}\times\chi_{[-v,0]}(x)$, $\alpha>0$, with resp. to l_+. Then $P=\Gamma(\alpha,\lambda)$ (the Gamma-distribution) is K_{D_α}-invariant.*

Proof. Choose $s(x)=\lambda^\alpha\Gamma(\alpha)^{-1}x^{\alpha-1}$, $\lambda>0$, in Theorem 3.

The following theorem gives a characterization of the Gamma-distribution as the only K_{D_α}-invariant distribution.

Theorem 4. *Let be $D_\alpha\in\mathbf{D}$, $\alpha>0$ such that $D_\alpha(v,\cdot)$ has density $v^{-1}\alpha\times(1+x/v)^{\alpha-1}\cdot\chi_{[-v,0]}(x)$ with respect to l. Then $P\in\mathbf{P}_+$, $P(\{0\})<1$ is K_{D_α}-invariant iff P has density $p(x)=\lambda^\alpha\Gamma(\alpha)^{-1}\chi^{\alpha-1}e^{-\lambda x}$ for some $\lambda>0$.*

Proof. The K_{D_α}-invariance of the Gamma-distribution follows from Corollary 2.

Now assume that P is K_{D_α}-invariant, i. e. $Z_{P,K_{D_\alpha}}=P^\infty$. Using (2) the equation

$$\int_{R^\infty} Z_{P,K_{D_\alpha}}(dz)e^{-sz_1}e^{-tz_2}=\int P(dx_1)e^{-sx_1}\cdot\int P(dx_1)e^{-tx_1}$$

has the following form

$$\int Q(dy_0)\int P(dx_1)\int D_\alpha(x_1-y_0,dy_1)e^{-s(x_1+y_1-y_0)}\int P(dx_2)\int D_\alpha(x_2-y_1,dy_2)e^{-t(x_2+y_2-y_1)}$$
$$=\int P(dx_1)e^{-sx_1}\int P(dx_1)e^{-tx_1}.$$

Because we have $\int D_\alpha(v,dy)e^{-t(v+y)}=\alpha\cdot t^{-\alpha}\int_0^t dze^{-vz}z^{\alpha-1}$ we get after some elementary calculations

$$\alpha^2t^{-\alpha}\int_0^t dz\bar{P}(z)z^{\alpha-1}(s-z)^{-\alpha}\int_z^s dw(w-z)^{\alpha-1}\bar{P}(w)\bar{Q}(-w)=\bar{P}(s)\bar{P}(t)$$

for all $s>t>0$, where $\bar{P}$, $\bar{Q}$ are defined by $\bar{P}(t)=\int P(dx)e^{-tx}$, $\bar{Q}(t)=\int Q(dx)e^{-tx}$. Differentiation by t and s leeds to

$$\alpha^2\bar{P}(t)\bar{P}(s)\bar{Q}(-s)=((s-t)\bar{P}'(s)+\alpha\bar{P}(s))(t\bar{P}'(t)+\alpha\bar{P}(t))$$

for all $s>t>0$. If t tends to zero this gives $\alpha\bar{P}(s)\bar{Q}(-s)=s\bar{P}'(s)+\alpha\bar{P}(s)$, $s>0$. Finally we get $\alpha(s-t)^{-1}(\bar{P}(t)\bar{P}'(s)-\bar{P}'(t)\bar{P}(s))=\bar{P}'(s)\bar{P}'(t)$, $s>t>0$. Letting t tend to zero we obtain the differential equation

$$\bar{P}'(s)(\alpha-\bar{P}'(0)\cdot s)=\alpha\bar{P}'(0)\bar{P}(s),\quad \bar{P}(0)=1,$$

which has the solution $\bar{P}(s)=\alpha^\alpha(\alpha-\bar{P}'(0)s)^{-\alpha}$. Because $P(\{0\})<1$ we have $\bar{P}'(0)<0$ and $\bar{P}(s)=\lambda^\alpha(\lambda+s)^{-\alpha}$ for some $\lambda>0$. Hence P is a Gamma-distribution with parameters α and λ.

For $\alpha=1$ we get the following

Corollary 3. *Let $D\in\mathbf{D}$ such that $D(v,\cdot)=l_{[-v,0]}$. $P\in\mathbf{P}_+$, $P(\{0\})<1$ is K_D-invariant iff $P=\Pi_\lambda$ for some $\lambda>0$.*

On this way we get an interesting transformation which left the Poisson process invariant: Points will be translated successive in one direction without overtaking, according to the equidistribution on the "free" interval.

The next theorem gives a broad class of transformations which left the Poisson process invariant.

Theorem 5. *$P=\Pi_\lambda$ satisfies I_1-K_D iff there is a measure S on $[R,\mathfrak{L}]$ such that $\int S(dy)e^{\lambda y}=1$ and*

$$S([-v,+\infty))\int D(v,dy)f(y)=\int S(dy)f(y)\chi_{[-v,+\infty]}(y)$$

for all $f\in\mathbf{F}(\mathfrak{L})$.

Proof. Suppose first that S,D are as above. For $f,g\in\mathbf{F}(\mathfrak{L})$ we get

$$\int dve^{-\lambda v}g(v)\int S(dy_0)\chi_{[-v,+\infty)}(y_0)\int D(v,dy_1)f(y_1)$$
$$=\int dve^{-\lambda v}g(v)\int S(dy_0)f(y_0)\chi_{[-v,+\infty)}(y_0)$$

$$=\int S(dy_0)e^{\lambda y_0}\int_{-y_0}^{\infty} dv e^{-\lambda(v+y_0)}g(v)\int D(v, dy_1)f(y_1)$$

$$=\int S(dy_0)e^{\lambda y_0}\int_{-y_0}^{\infty} dv e^{-\lambda(v+y_0)}f(y_0)g(v).$$

Defining Q_0 by $\int Q_0(dy)h(y)=\int S(dy)e^{\lambda y_0}h(y)$, $h\in \mathbf{F}(\mathfrak{L})$, we have

$$\int Q_0(dy_0)\int \Pi_\lambda(dx_1)\int D(x_1-y_0,dy_1)g(x_1-y_0)f(y_1)=\int Q_0(dy_0)\int \Pi_\lambda(dx_1)g(x_1-y_0)f(y_0).$$

Hence (3) is valid with $P=\Pi_\lambda$, $Q=Q_0$, $K=K_D$. By Proposition 3 Q is unique. Therefore Π_λ satisfies I_1-K_D.

Now suppose that Π satisfies I_1-K_D. By Lemma 1, c. we have for all $h\in\mathbf{F}(\mathfrak{L})$

$$\int Q(dy_0)e^{-\lambda y_0}\chi_{[-v,\infty)}(y_0)\int D(v, dy_1)h(y_1)=\int Q(dy_0)e^{-\lambda y_0}h(y_0)\chi_{[-v,+\infty)}(y_0),\ l\text{—a. e.}$$

Defining S by $\int S(dy)f(y)=\int Q(dy)f(y)e^{-\lambda y}$, $f\in\mathbf{F}(\mathfrak{L})$, we get $\int S(dy)e^{\lambda y}=1$ and

$$S([-v, +\infty))\int D(v, dy_1)h(y_1)=\int S(dy_0)h(y_0)\chi_{[-v,+\infty)}(y_0),\ l\text{—a. e.}$$

For each measure S on $[R, \mathfrak{L}]$ with $\int S(dy)e^{\lambda y}=1$ we get a transformation which left the Poisson process invariant.

If $S=\Pi_\mu$, $\mu>\lambda$, then

$$D(v,\cdot)=\begin{cases}\Pi_\mu, & v<0,\\ \delta_v*\Pi_\mu, & v\geqq 0,\end{cases}$$

and Theorem 5 gives a long known result: The output of a steady state $M|M|1|\infty$ service system is Poissonian too with the same parameter as the input (see, e. g. [1]).

If $S=\lambda l_-$ we get $D(v,\cdot)=l_{[-v,0]}$ and we have the case treated in Corollary 3.

Remark concerning the lattice case. Let $\mathbf{P}_L$ denote the set of distributions on $\{0, 1, 2, \ldots\}$, $\mathbf{D}_L$ the set of stochastic matrices $D(v, y)$ with $\sum_{y=-v}^{\infty} D(v, y)=1$ for all $v=\cdots -1, 0, 1\ldots$

For $P\in\mathbf{P}_L$ and $D\in\mathbf{D}_L$ one can show results similar to Proposition 3, Lemma 1, c.,Theorems 3 and 5, where the role of the Lebesgue measure and the exponential distribution play the counting measure on the integers and the geometric distribution respectively.

5. A single server system. This section is deduced to a class of transformations which seem to be applicable in queueing theory. Consider a single server system with Poisson input. There are no waiting places. If an arriving customer finds the counter occupied the running service is finished and the new customer has service time distribution $E[(0,\cdot)\in\mathbf{P}_+$. If an arriving customer finds the counter free the service time distribution will be $E(t,\cdot)\in\mathbf{P}_+$, t being the time the counter was free. By the output we mean both completed and not completed services. Theorem 6 below gives all service systems of such type which have Poissonian output.

To be precise let $E\in\mathbf{K}(R_+, \mathfrak{L}_+)$ and define $K_E\in\mathbf{K}(R_+\times R_-, \mathfrak{L})$ by

$$\int K_E(x_1, y_0, dy_1)f(y_1)$$
$$=f(0)E(-y_0, [x_1, +\infty))+\int E(-y_0, dy_1)f(y_1-x_1)\chi_{[0,x_1]}(y_1),\ f\in\mathbf{F}(\mathfrak{L}).$$

We introduce the following functions

$$\bar{E}(s)=\int E(0,dz)e^{-sz},\ \ \bar{\bar{E}}(s)=\int_0^\infty d(y)e^{-sy}\int E(y,dz)e^{-sz},\ \bar{F}(s)=\int F(dz)e^{-sz},\ \ F\in\mathbf{P}_+.$$

Now we can formulate

Theorem 6. *$P=\Pi_\lambda$ is K_E-invariant iff $s\bar{E}(s)(1-\lambda\bar{\bar{E}}(\lambda))=\lambda\bar{E}(\lambda)(1-s\bar{\bar{E}}(s))$ for all $s\geq 0$.*

The proof of Theorem 6 we give at the end of this section. Because it is not easy to see which kernels fulfill the condition of Theorem 6 we give some corollaries.

Corollary 4. *Let $E(0,.)=F_0\in\mathbf{P}_+$ and $E(y,)\cdot\equiv F_1\in\mathbf{P}_+$ for all $y>0$. Then $P=\Pi_\lambda$ is K_E-invariant iff $s\bar{F}_0(s)(1-\bar{F}_1(\lambda))=\lambda\bar{F}_0(\lambda)(1-\bar{F}_1(s))$ for all $s\geq 0$.*

Proof. Obvious from Theorem 6.

We give some distributions which satisfy the condition of Corollary 4. If $F_0=F_1=\Pi_\mu$, $\mu>0$, we have the case treated in Theorem 1.

We can also choose $F_0=\Pi_\mu$, $F_1=p\delta_0+(1-p)\Pi_\mu$, $\mu>0$, $0\leq p<1$. $F_0=l_{[0,b]}$, $F_1=p\delta_0+(1-p)\delta_b$, $b>0$, $0\leqq p<1$, induce an interesting motion. We may describe the case $p=0$ in the following manner: An arriving call which finds the server empty tries to speak a time of a units. A call which finds the server occupied finishes the service of its "foregoer" and will be shorter: its service time distribution is $l_{[0,b]}$.

Corollary 5. *Let $p\in\mathbf{F}(\mathfrak{L}_+)$, $F_0,F_1\in P_+$ and let E be defined by $E(y,\cdot)=p(y)\delta_0+(1-p(y))F_1$, $y>0$, $E(0,\cdot)F_0$. $P=\Pi_\lambda$ is K_E-invariant iff for some $c>0$ and all $s\geqq 0$*

$$s\bar{F}_0(s)=c(1-\bar{F}_1(s))(1-s\int_0^\infty dz\,p(z)e^{-sz}).$$

Proof. Obvious from Theorem 6.

For example we can take $p(y)=e^{-\mu y}$, $F_1=\delta_b$, $F_0=\Pi_\mu * l_{[0,b]}$, $\mu>0$, $b>0$, or $p(y)=(1-p)e^{-\sigma py}$, $F_1=\Pi_\sigma$, $F_0=\Pi_\mu$, $0<p\leq 1$, $\sigma>\mu>0$.

Proof of Theorem 6. We first show that Π_λ is K_E-transformable for all E. Indeed we have for all $s\geqq 0$

$$\int\Pi_\lambda(dx_1)\int K_E(x_1,y_0,dy_1)e^{sy_1}=1-s(\lambda+s)^{-1}\int E(-y_0,dy_1)e^{-\lambda y_1},$$

$Q\in\mathbf{P}$ fulfills condition (1) iff $Q\in\mathbf{P}_-$ and

$$\int Q(dy)e^{sy}=1-s(\lambda+s)^{-1}\int Q(dy_0)\int E(-y_0,dy_1)e^{-\lambda y_1}.$$

Hence for each E exists a unique Q_E which fulfills (1), namely $Q_E(A)=(1-c_E)\delta_0(-A)+c_E\Pi_\lambda(-A)$, $A\in\mathfrak{L}_-$, with $c_E=(1-\bar{E}(\lambda)-\bar{\bar{E}}(\lambda))^{-1}\bar{E}(\lambda)$.

Now let Π_λ be K_E-invariant. Then we have in particular for all $s\geqq 0$

$$\lambda(\lambda+s)^{-1}=\int Q_E(dy_0)\int\Pi_\lambda(dx_1)\int K_E(x_1,y_0,dy_1)e^{-s(x_1+y_1-y_0)}$$
$$=\int Q_E(dy_0)(\lambda+s)^{-1}(\lambda+se^{-sy_0}\int E(-y_0,dz)e^{-(\lambda+s)z}$$
$$=\lambda(\lambda+s)^{-1}(\lambda+(1-c_E)s)(\lambda+s)^{-1}+s(\lambda+s)^{-1}((1-c_E)\bar{E}(\lambda+s)+c_E\lambda\bar{\bar{E}}(\lambda+s)).$$

It follows immediately that for all $s\geqq 0$ we have $s\bar{E}(s)(1-\lambda\bar{\bar{E}}(\lambda))=\lambda\bar{E}(\lambda)(1-s\bar{\bar{E}}(s))$. On the other hand we can compute

$$r(s, t) = \int Q_E(dy_0) \int \Pi_\lambda(dx_1) \int K_E(x_1, y_0, dy_1) e^{-s(x_1+y_1-y_0)} e^{ty_1}$$

$$= \lambda(\lambda+s)^{-1}((\lambda+s)^{-1}(\lambda+(\lambda-c_E)s)+(\lambda+t)^{-1}(s-t)\,((\lambda-c_E)\bar{E}(\lambda+s) + c_E\bar{\bar{E}}(\lambda+s))).$$

If we assume that $s\bar{E}(s)(1-\lambda\bar{\bar{E}}(\lambda))=\lambda\bar{E}(\lambda)(1-s\bar{\bar{E}}(s))$ for all $s \geqq 0$ we get $r(s, t)=(\lambda+s)^{-1}(\lambda+(1-c_E)s)\lambda(\lambda+t)^{-1}$. Hence Π_λ satisfies I_1-K_E and therefore Π_λ is K_E-invariant by Proposition 1.

R e m a r k concerning a special lattice case. Consider kernels $K_L \in \mathbf{K}(\{0, 1, 2, ...\} \times \{0, 1\}, \mathfrak{P}(\{0, 1\}))$ and let $P \in \mathbf{P}_L$ (see the remark at the end of section 4). The condition of nonovertaking is expressed by $K_L(0, 0, 1)=0$. To be short we will denote $P(\{n\})=p_n$, $K_L(n, 1, 0)=s_n$, $K_L(n, 0, 1)=l_n$, $n=0, 1, 2, \ldots$

By simple computations (sp [5]) we get the following

T h e o r e m 7. *For every K_L of the above type the following statements hold:*

a. *every $P \in \mathbf{P}_L$ is K_L-transformable;*

b. *if $l_n > 0$ for all $n=1, 2, \ldots$, then $P \in \mathbf{P}_L$ is K_L-invariant iff there is a constant $c>0$ such that $\sum_{n=1}^{\infty} c^n(l_1 \cdot \ldots \cdot l_n)^{-1} \; s_0 \cdot \ldots \cdot s_{n-1} < \infty$ and $p_n = c^n(l_1 \cdot \ldots \cdot l_n)^{-1} s_0 \cdot \ldots \cdot s_{n-1} \cdot p_0$, $n=1, 2, \ldots$*

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