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APPROXIMATION OF SOLUTIONS TO NONLINEAR VARIATIONAL TIME DEPENDENT PROBLEMS

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We examine a family of nonlinear elliptic variational problems, parametrized by a time parameter $t\in[0, T]$. Such problems are crucial in the derivation of error bounds for Galerkin procedures pertaining to nonlinear parabolic and hyperbolic equations. Consequently, we develop a general theory for existence, uniqueness and differentiability of approximate solutions. Moreover, we derive error estimates which are optimal in the energy norm.

1. Introduction. This paper is concerned with existence-uniqueness theory and error estimation for nonlinear variational problems, which arise from the elliptic part of a time dependent differential operator. Variational problems of this type appear naturally in the error analysis of Galerkin procedures for nonlinear time dependent problems. Typical instances are equations of parabolic ($j=1$) or hyperbolic ($j=2$) type

$$D_t^j u(x, t)+L(u(x, t))=f(x, t),\ x\in\Omega\subset R^n,\quad t\in[0, T], \tag{1.1}$$

where L is a nonlinear elliptic differential operator of order $2m$. It is the application to problems such as (1.1), which motivates the theory developed here (for details on these applications we refer to Papatheodorou [6]). In addition, this theory is general: in the existing literature one may find only special cases (e. g. $m=1$), restricted by unrealistic assumptions such as that L is uniformly Lipschitz continuous with respect to its functional argument.

As a motivation and introduction to the general formulation we use the example

$$L(\varphi(x, t))\equiv\sum_{k=0}^{m}(-1)^k D_x^k f_k(x, t, D_x^k\varphi(x, t)). \tag{1.2}$$

In order to obtain a variational problem, one multiplies (1.1) by some admissible function v and integrates over the domain Ω of the space variable x. Then, if X is an appropriate space of functions, the elliptic part L gives rise to a mapping

$$E^0:[0, T]\times X\times X\longrightarrow R$$
$$(t, \varphi^0, v)\longrightarrow E^0[t](\varphi^0; v),$$

defined by

$$E^0[t](\varphi; v)=\sum_{k=0}^{m}\int_\Omega f_k(x, t, D_x^k\varphi^0)D_x^k v\,dx. \tag{1.3}$$

We are concerned with projections $Q_0u[t]$, of the function $u[t]\equiv u(.,t)$, on a finite dimensional subspace S of X, defined by the requirement

$$E^0[t](Q_0u[t];\ v)=E^0[t](u[t];\ v),\quad \text{all}\quad v\in S. \tag{1.4}$$

Such "elliptic projections" have been previously used (see e. g. Houstic [4]) but only for linear problems. In most linear cases, if $Du[t]$ exists ($D\equiv\partial u/\partial t$), and belongs to X, then $D(Q_0u[t])$ automatically exists, belongs to S and is the elliptic projection of $Du[t]$ on S. In other words, if $u[t]$ is in the domain $\mathfrak{D}\subset X$ of $D=\partial/\partial t:\mathfrak{D}\to X$, then $Q_0u[t]$ is also in $\mathfrak{D}$ and

$$DQ_0u[t]=Q_0Du[t]. \tag{1.5}$$

Since (1.5) is instrumental in the analysis of the Galerkin methods, we ask whether it holds true for the nonlinear situation. When E^0 is nonlinear in $Q_0u[t]$, as in (1.3), some of the fundamental difficulties that arise are the following:

i) The existence of $Q_0u[t]$ is in doubt, and the existence of the time derivatives of $Q_0u[t]$ does not follow automatically from the existence of the corresponding derivatives of $u[t]$. Still, for the error analysis, one needs to differentiate (1.4) in t and obtain these derivatives as solutons to "derivative problems" that are similar to (1.4).

ii) Differentiation of (1.4) introduces new mapping

$$E^j\equiv\frac{d^j}{dt^j}E^0,\quad \text{or}\quad E^{j+1}=\frac{d}{dt}E^j. \tag{1.6}$$

However, the definition of E^1, for instance, via (1.6) is inappropriate. For the chain rule introduces $DQ_0u[t]$ into the right side of (1.6) which, by the preceding comment, is not known to exist. Hence, the "derivative forms" are introduced differently.

To obtain the form E^1, one formally writes $\frac{d}{dt}\{E^0[t](\varphi^0;\ v)\}$ using the chain rule. In the resulting expression one replaces $D\varphi^0[t]$ by an arbitrary function φ^1, which is not required to be related to $\varphi^0[t]$. In this manner, one obtains $E^1[t](\varphi^0,\ \varphi^1;\ v)$. Similarly, one defines $E^2[t](\varphi^0,\ \varphi^1,\ \varphi^2;\ v)$ through $E^1[t](\varphi^0,\ \varphi^1;\ v)$. Symbolically, with $\vec{\Phi}^j\equiv(\varphi^0,\ldots,\varphi^j)\in X^{j+1}$, $\vec{\Phi}^{j+1}\equiv(\varphi^0,\ldots,\varphi^j,\varphi^{j+1})=(\vec{\Phi}^j,\ \varphi^{j+1})\in X^{j+2}$, E^{j+1} is defined via E^j by

$$E^{j+1}[t](\vec{\Phi}^{j+1};\ v)\equiv\frac{d}{dt}\{E^j[t](\vec{\Phi}^j;\ v)\}\,|_{D\varphi^j=\varphi^{j+1}}. \tag{1.7}$$

In this work we take X to be the Sobolev space $H_0^m(\Omega)$, (cf. Aubin [1]), where Ω is a bounded domain in R^n. The norm of this space is denoted by $\|\cdot\|_m$. The index j in (1.7) is taken from the set $\{0,\ldots,J\}$, where J is a fixed nonnegative integer. (For the applications to elliptic, parabolic and hyperbolic problems, the theory, to be developed, applies with $J=0$, 1 and 2 respectively, [6]).

The forms E^j are considered as mappings

$$E^j:[0,\ T]\times X^{j+1}\times X\longrightarrow R \tag{1.8}$$

$$(t,\ \vec{\Phi}^j,\ v)\longrightarrow E^j[t](\vec{\Phi}^j;\ v),$$

which are, in general, nonlinear in $\vec{\Phi}^j = (\varphi^0, \ldots, \varphi^j) \in X^{j+1}$ and are assumed to be linear in $v \in X$.

Now, if $Q_0 u[t]$ is already defined by (1.4), one defines, e. g. $Q_1 Du[t] \in S$ by

$$E^1[t](Q_0 u[t],\ Q_1 u[t];\ v) = E^1[t](u[t],\ Du[t];\ v) \quad \text{all} \quad v \in S.$$

However:

iii) Even if the original mapping E^0 is elliptic, or monotone, in $\vec{\Phi}^0 = \varphi^0$, the derivative forms E^j are no longer "monotone" in $\vec{\Phi}^j = (\varphi^0, \ldots, \varphi^j)$. Nevertheless, we show that $Q_j(D^j u[t])$, $j > 0$, is still well defined and lies in the domain of D. In addition, we establish the commutativity of the diagram

$$\begin{array}{ccc} D^j u[t] & \xrightarrow{\ D \equiv \partial/\partial t\ } & D^{j+1} u[t] \\ \Big\downarrow Q_j & & \Big\downarrow Q_{j+1} \\ Q_j D^j u[t] & \xrightarrow[D]{} & DQ_j D^j u[t] = Q_{j+1} D^{j+1} u[t] \end{array}$$

i. e. in analogy with (1.5), we prove that on the range of D^j

$$DQ_j = Q_{j+1} D. \tag{1.9}$$

iv) Finally, uniform Lipschitz continuity (of the coefficient functions f_k for instance) although widely used in the literature, does not usually hold for φ^j on the entire real line. Hence, before taking the Lipschitz constant as independent of a functional argument, one must show that this argument lies in a ball of radius r and take the constant to depend only on r.

The precise hypotheses and the formulation of the variational problems of definition of $Q_j D^j u[t]$ are presented in Section 2. The existence-uniqueness and differentiability theory of the projections is developed in Section 3. In the same section we establish the commutativity relationship (1.9). Optimal error estimates for norms of the error $(I - Q_j) D^j u[t]$ are derived in Section 4. Applications to nonlinear parabolic and hyperbolic problems are presented in [6].

2. Hypotheses and problem formulation. The hypotheses that we use are observed in fairly general differential operators, which, for each fixed t, are elliptic. The form (1.3) may serve as an example. For an easier illustration of the comments (i)—(iii) of the introduction, and as a motivation for the general formulation to be presented below, we suggest the special case of (1.1), when $m = 1$, $f = 0$ and $f_1(x, t, z) = z^3 + z$, i. e.

$$\begin{aligned} E^0[t](\vec{\Phi}^0;\ v) &= E^0[t](\varphi^0;\ v) = \int_0^1 [(D_x\varphi^0)^3 + D_x\varphi^0] D_x v\,dx, \\ E^1[t](\vec{\Phi}^1;\ v) &= E^1[t](\varphi^0,\ \varphi^1;\ v) = \int_0^1 [3(D_x\varphi^0)^2 + 1] D_x\varphi^1 D_x v\,dx. \end{aligned} \tag{2.1}$$

If for each $t \in [0, T]$ a function $\varphi[t] \equiv \varphi(\cdot, t)$ is in X and if $\sup\{\|\varphi[t]\|_m : t \in [0, t]\} \leqq r$, then we say that φ belongs to the ball $B(r)$.

First, we make an assumption on the smoothness of the "given" functions $u[t] \in X$:

Hypothesis (HO). *For all* $t \in [0, T]$, $l = 0, 1, 2$, $D_t^l u[t]$ *exists as an element of* X *and for some* $r > 0$ $D_t^l u[t] \in B(r)$.

Our second assumption represents the "loss of ellipticity" in passing to derivative forms. At the same time, it also represents the observation that some kind of "almost monotonicity" is preserved. For $\vec{\Phi}^k = (\varphi^0, \ldots, \varphi^k) \in X^{k+1}$, let

$$\|\vec{\Phi}^k\|_m \equiv \sum_{i=0}^{k} \|\varphi^i\|_m. \tag{2.2}$$

Hypothesis (Hl). *Each* E^j, $j = 0, \ldots, J$, *is "hypomonotone" in* $\vec{\Phi}^j \in X^{j+1}$, *i. e. there exists a* $b > 0$, *and given* Φ^j, $\Psi^j \in X^{j+1}$, *there exists a* $c_j > 0$, *which may depend on* $\vec{\Phi}^j$, $\vec{\Psi}^j$ *such that*

$$b^2 \|\varphi^j - \psi^j\|_m^2 - c_j^2 \|\varphi^j - \psi^j\|_m \|\vec{\Phi}^{j-1} - \vec{\Psi}^{j-1}\|_m \tag{2.3}$$
$$\leqq E^j[t](\vec{\Phi}^j;\ \varphi^j - \psi^j) - E^j[t](\vec{\Psi}^j;\ \varphi^j - \psi^j)$$

for all $t \in [0, T]$. *If* $\vec{\Phi}^j$, $\vec{\Psi}^j \in B(r)^{j+1}$, *for some* $r > 0$, *then* c_j *may be taken as dependent only on* r.

Notice that when $j = 0$, the empty sum in $\|\vec{\Phi}^{j-1} - \vec{\Psi}^{j-1}\|_m$, (see (2.2)), is to be taken equal to zero. Then (Hl) reduces to a standard monotonicity, or $H^m(\Omega)$-ellipticity, assumption on $E^0[t]$.

Next, an inequality of Cauchy-Schwartz-Bunyakovskii type is assumed.

Hypothesis (H2). *Given* $\vec{\Phi}^j$, $\vec{\Psi}^j \in X^{j+1}$, *there exists a* $d_j > 0$, *which may depend on* $\vec{\Phi}^j$, $\vec{\Psi}^j$ *such that*

$$|E^j[t](\vec{\Phi}^j;\ v) - E^j[t](\vec{\Psi}^j;\ v)| \leqq d_j^2 \|\vec{\Phi}^j - \vec{\Psi}^j\|_m \|v\|_m \tag{2.4}$$

or all $t \in [0, T]$ *and all* $v \in X$. If $\vec{\Phi}^j$, $\vec{\Psi}^j \in B(r)^{j+1}$, *for some* $r > 0$, *then* d_j *may be aken as dependent only on* r. *If* $\varphi^i = \psi^i$ *for* $i = 0, \ldots, j-1$ *the* d_j *may be taken as independent of the particular elements* $\vec{\Phi}^j$, $\vec{\Psi}^j$.

The fourth hypothesis is concerned with the explicit dependence of each E^j on $t \in [0, T]$.

Hypothesis (H3). *For each fixed* $\vec{\Phi}^j$ *and* v, E^j *is uniformly Lipschitz continuous in* $t \in [0, T]$, *in the sence that there exists a constant* $\gamma_j > 0$, *which may depend on* $\vec{\Phi}^j$, *such that*

$$|E^j[t](\vec{\Phi}^j;\ v) - E^j[t'](\vec{\Phi}^j;\ v)| \leqq \gamma_j^2 |t' - t| \ \|v\|_m \tag{2.5}$$

for all t, $t' \in [0, T]$, $v \in X$. *If* $\Phi^j \in B(r)^{j+1}$, *for some* $r>0$, *then* γ_j *may be taken as dependent only on* r.

Next, we use an assumption, which represents the symbolic expression (1.7) or, the fact that $E^{j+1}[t](\vec{\Phi}^{j+1}; v)$ is in some sense the total time derivative of $E^j[t](\vec{\Phi}^j; v)$.

For this, let the symbol $\tau(t)$ stand for a sequence $\{t_k\}_1^\infty \subset [0, T]-\{t\}$, whose limit is t, and consider the abbreviations

$$\varphi_k \equiv \varphi[t_k], \quad \Delta\varphi_k[t] \equiv \varphi_k - \varphi[t], \quad \partial_k\varphi[t] \equiv \Delta\varphi_k[t]/(t_k - t), \quad t_k \in \tau(t). \tag{2.6}$$

We first illustrate the connection between E^0 and E^1 in the case of the Example (2.1). By application of the mean value theorem, we find that for each x there exists a number $\psi_k[t](x)$, which lies between $D_x\varphi^0[t](x)$ and $D_x\varphi_k^0(x)$ such that

$$A_k \equiv E^0[t_k](\varphi_k^0; v) - E^0[t](\varphi^0[t]; v) - (t_k - t)E^1[t](\varphi^0[t], \partial_k\varphi^0[t]; v)$$
$$= 3(t_k - t)\int_0^1 \{(\psi_k[t](x))^2 - (D_x\varphi^0[t](x))^2\}\partial_k(D_x\varphi^0[t])D_x v\,dx.$$

Next, suppose that for the given t, for some sequence $\tau(t)$, and for some $\varphi^0[s] \in X$, $s \in [0, T]$, there exists an element $f_\tau[t] \in X$ such that $\lim_{k\to\infty}\partial_k D_x\varphi^0[t] = D_x f_\tau[t]$, uniformly on $\bar{\Omega}$.

Note that this is not an assumption on existence of $D(D_x\varphi^0[t])$, as $f_\tau[t]$ is allowed to depend on the sequence $\tau(t)$. Applying the uniform convergence, we now get that given $\varepsilon>0$ there exists a $k_0(\varepsilon)$ such that for $k \geq k_0(\varepsilon)$ we have $|A_k| \leq c'\varepsilon|t_k - t|\,\|v\|_m$, where $c' \equiv 3(1+\|D_x f_\tau[t]\|_m)$ depends on $\|f_\tau[t]\|_m$, but if $f_\tau[t] \in B(r)$ then we may take $c' = 3(1+r)$.

This observation goes through for more general forms. We formulate it as our fifth hypothesis:

Hypothesis (H4). *For every* $t \in [0, T]$, *any sequence* $\tau(t)$, *any* $\Phi^j[s] \in X^{j+1}$, $s \in [0, T]$, *for which there exists an* $f_\tau[t] \in X$ *such that*

$$\lim_{k\to\infty} \partial_k D_x^\beta \varphi^j[t] = D_x^\beta f_\tau[t], \quad |\beta| \leq m, \quad \textit{uniformly on } \bar{\Omega}, \tag{2.7}$$

and any $\varepsilon>0$, *there exists a* $k_0(\varepsilon)$ *and a constant* c', *which may depend on* $\|f_\tau[t]\|_m$, *such that*

$$|E^j[t_k](\vec{\Phi}_k^j; v) - E^j[t](\vec{\Phi}^j[t]; v) - (t_k - t)E^{j+1}[t](\vec{\Phi}^j[t], f_\tau[t]; v)| \leq c'\varepsilon|t_k - t|\,\|v\|_m$$

for all $k \geq k_0(\varepsilon)$ *and all* $v \in X$. *If* $f_\tau[t] \in B(r)$, *for some* $r>0$, *then* c' *may be taken as dependent only on* r.

Our last hypothesis may be translated to boundedness of the coefficients (and some of their derivatives) of the differential operator on $\bar{\Omega}\times[0, T]$:

Hypothesis (H5). *There exists a constant* $\beta \neq 0$ *such that*

$$|E^j[t](\vec{O}^j; v)| \leq \beta^2\|v\|_m, \quad \textit{all } v \in X, \quad \textit{all } t \in [0, T]. \tag{2.8}$$

where $\vec{O}^j \equiv (0, \ldots, 0) \in X^{j+1}$.

Finally, we formulate the family of the nonlinear variational problems, which give rise to the projections Q_j. As always, S is a finite dimensional subspace of X. In this formulation, we use the abbreviations $\vec{U}^i[t]\equiv(u^0[t], \dots, u^i[t])$, $Q^i\vec{U}^i[t]\equiv(Q_0u^0[t], \dots, Q_iu^i[t])$.

For each fixed j and each fixed $t\in[0, T]$ consider:

Problem P(j ; t). Given $\vec{U}^j[t]\in X^{j+1}$, and, in case $j\geqq 1$, given $Q^{j-1}\vec{U}^{j-1}[t]\in S^j$, determine $Q_ju^j[t]\in S$ by the requirement that $Q^j\vec{U}^j[t]\equiv(Q^{j-1}\vec{U}^{j-1}[t], Q_ju^j[t])$ satisfies

$$E^j[t](Q^j\vec{U}^j[t];\ v)=E^j[t](\vec{U}^j[t];\ v),\quad \text{all}\quad v\in S. \tag{2.9}$$

In the sequel, the expression "$Q_jD^ju[t]$ solves $P(j;\ t)$" applies to the case $u^i[t]=D^iu[t]$, $i=0, \dots, j$.

3. Existence, uniqueness and differentiability theory. The first result establishes the existence and uniqueness of solution to Problem $P(j;\ t)$. Its proof is based on a result of Minty [5] and Browder [2] and a modification of an argument used by Ciarlet, Schultz and Varga [3].

Let S be a reflexive Banach space with norm $\|\cdot\|_S$. If S^* denotes the conjugate of S, and $s^*\in S^*$, consider the canonical map $(s^*, s)_S=s^*(s)$, $s\in S$.

An operator $T:S\to S^*$ is said to be

(a) *strongly monotone, iff there exists a $b>0$ such that $b^2\|\varphi-\psi\|_S\leqq(T(\varphi)-T(\psi), \varphi-\psi)_S$, for all $\varphi, \psi\in S$.*

(b) *finitely continuous, iff it is continuous on finite dimensional subspaces of S with the weak topology of S^*.*

Equivalently: *if V is any finite dimensional subspace of S and if $\{v_n\}_1^\infty$, is any sequence in V that converges to $v\in V$, then for any $w\in S$ the sequence $\{(T(v_n), w)_S\}_{n=1}^\infty$ converges to $(T(v), w)_S$.*

We have (Browder [2], Minty [5]);

Lemma 3.1. *If T is strongly montone and finitely continuous then for any $s_*\in S^*$ the problem $(T(w), v)_S=(s^*, v)_S$, $v\in V$, has a unique solution $w\in S$.*

Now fix t and j. With the notation of (2.9) define the operator $T:S\to S^*$ by

$$(T(\varphi), v)_S\equiv E^j[t](Q^{j-1}\vec{U}^{j-1}[t], \varphi;\ v) \tag{3.1}$$

for $\varphi, v\in S$, and the functional $s^*\in S$ by $(s^*, v)_S\equiv E^j[t](\vec{U}^j[t];\ v)$. Then (2.9) is equivalent to $(T(\varphi), v)_S=(s^*, v)_S$.

Taking S to be the finite dimensional space of Problem $P(j;\ t)$, with $\|.\|_S\|\cdot\|_m$, we have:

Theorem 3.1. *If* (H1) *and* (H2) *hold then $Q_jD^ju[t]$ is well defined by* (2.9).

Proof. We show that $P(j;\ t)$ has unique solution. By the preceding discussion and by Lemma 3.1, it suffices to show that the operator T of (3.1) is (i) strongly monotone, and (ii) finitely continuous. Applying (H1) with $\vec{\Phi}^j=(Q^{j-1}\vec{U}^{j-1}[t], \varphi)$, $\vec{\Psi}^j=(Q^{j-1}\vec{U}^{j-1}[t], \psi)$ we get the strong monotonicity condition. To show (ii), apply (H2) with $\vec{\Phi}^j=(Q^{j-1}\vec{U}^{j-1}[t], v_n)$, $\vec{\Psi}^j=(Q^{j-1}\vec{U}^{j-1}[t], v)$ to get

$$|(T(v_n), w)_S - (T(v), w)_S| \leqq d_j^2 \| v_n - v \|_m \| w \|_m,$$

where, according to (H2), d_j is independent of n. This concludes the proof of the Theorem.

Taking $u^i[t] = D^i u[t]$ in $P(j;\, t)$ and "solving" successively for $j=0, \ldots, J$, we have well defined projections $Q_j D^j u[t]$ of $D^j u[t]$, $j=0, \ldots, J$. The next task is to show the differentiability of the projections. Toward this direction we need:

Lemma 3.2. *If* (H1) *and* (H2) *hold then the unique solution* $Q_j D^j u[t]$ *of* $P(j;\, t)$ *satisfies*

$$\| Q_j D^j u[t] \|_m \leqq c_j \left\{ \| D^j u[t] \|_m + \sum_{i=1}^{j-1} \| (I-Q_i) D^i u[t] \|_m \right\},$$

where the constant c_j *depends on* $\vec{U}^j[t] \equiv (u[t], \ldots, D^j u[t])$ *and* $\vec{W}^{j-1}[t] \equiv (Q_0 u[t], \ldots, Q_{j-1} D^{j-1} u[t])$. *If for some* $\varrho > 0$, $\vec{U}^j[t] \in B(\varrho)^{j+1}$ *and* $\vec{W}^{j-1}[t] \in B(\nu)$, *then* c_j *may be taken as dependent only on* ϱ.

Proof. Apply (H1) with $\vec{\Phi}^j = \vec{W}^j[t] \equiv (\vec{W}^{j-1}[t], Q_j D^j u[t])$ and $\vec{\Psi}^j = (\vec{W}^{j-1}[t], 0)$ This gives

$$b^2 \| Q_j D^j u[t] \|_m^2 \leqq E^j[t](\vec{W}^j[t];\; Q_j D^j u[t]) - E^j[t](\vec{W}^{j-1}[t], 0;\; Q_j D^j u[t]).$$

Since $Q_j D^j u[t]$ solves $P(j, t)$, the first term of the right side is equal to $E^j[t](\vec{U}^j[t];\; Q_j D^j u[t])$. The proof is completed by (H2).

Lemma 3.3. *If* (H0), (H1) *and* (H2) *hold, then there exists a* $\varrho > 0$, *depending only on* r *of* (H0), *such that* $Q_j D^j u[t] \in B(\varrho)$, *all* $j=0, \ldots, J$, *all* $t \in [0, T]$.

Proof. Trivially, by finite induction on the inequality of Lemma 3.2. Now pick any $t \in [0, T]$ and keep it fixed. Also, pick any sequence $\tau(t) = \{t_k\}_1^\infty \subset [0, T] - \{t\}$, whose limit is t. With the notation of (2.6) we have:

Lemma 3.4 *If* (H0) *through* (H5) *hold, then there exists a* $\tilde{\varrho} > 0$, *depending only on* r *of* (H0), *such that* $\partial_k(Q_j D^j u[t]) \in B(\tilde{\varrho})$ *for sufficiently large* k *and for all* $j=0, \ldots, J$.

Proof. Let $w^i[t] \equiv Q_i D^i u[t]$, $i=0, \ldots, j$,

$$\alpha_k \equiv E^j[t](\vec{W}_k^j;\; \Delta_k w^j) - E^j[t_k](\vec{W}_k^j;\; \Delta_k w^j),\quad \beta_k \equiv E^j[t](\vec{U}_k^j;\; \Delta_k w^j) - E^j[t](\vec{U}^j[t];\; \Delta_k w^j)$$

On the one hand, (H1) applied with $\vec{\Phi}^j = \vec{W}_k^j$, $\vec{\Psi}^j = \vec{W}^j[t]$ and the fact that w_k^j and $w^j[t]$ solve $P(j;\, t_k)$ and $P(j;\, t)$, respectively, give

$$b^2 \| \Delta_k w^j[t] \|_m^2 - c_j^2 \| \Delta_k w^j[t] \| \| \Delta_k \vec{W}^{j-1}[t] \|_m \leqq \alpha_k + \beta_k, \tag{3.2}$$

where c_j depends on $\vec{W}_k^j$, $\vec{W}^j[t]$ and, hence, by Lemma 3.3, only on ϱ. On the other hand, we get from (H3) that

(3.3) $$|\alpha_k| \leqq \gamma^2 \, |t_k - t| \; \| \Delta_k w^j[t] \|_m,$$

where, similarly, γ depends only on ϱ. Also, by (H4), applied with $\vec{\Phi}_k^j = \vec{U}_k^j$, $\vec{\Phi}^j = U^j[t]$ and by (H0) we get

(3.4) $$|\beta_k| \leqq c' \, |t_k - t| \; \| \Delta_k w^j[t] \|_m + |t_k - t| \; |E^{j+1}[t](\vec{U}^{j+1}[t]; \Delta_k w^j[t])| ,$$

for $k \geqq k_0(1)$, where, again similarly, c' depends only on ϱ. Applying (H2), the triangle inequality, and (H5) we also have

$$|E^{j+1}[t](\vec{U}^{j+1}[t]; \Delta_k w^j[t])|$$

$$\leqq |E^{j+1}[t](\vec{Q}^j; \Delta_k w^j[t])| + d^2_{j+1} \| \vec{U}^{j+1}[t] \|_m \| \Delta_k w^j[t] \|_m \leqq \tilde{c} \| \Delta_k w^j[t] \|_m,$$

where $\tilde{c}$ depends only on ϱ. Hence, by (3.4)

(3.5) $$|\beta_k| \leqq \tilde{c} \, |t_k - t| \; \| \Delta_k w^j[t] \|_m,$$

where $\tilde{c}$ depends only on ϱ.

Now use (3.3) and (3.5) into (3.2), divide through by $|t_k - t| \; \| \Delta_k w^j[t] \|_m$ and observe that, in Lemma 3.3, ϱ depends only on r, to conclude the proof.

Theorem 3.2. *If* (H0) *through* (H5) *hold and if the subspace S is such that $D_x^\beta v$ is continuous on $\bar{\Omega}$, $|\beta| \leqq m$, for all $v \in S$, then the time derivative $D(Q_j D^j u[t])$ at t exists and equals $Q_{j+1} D^{j+1} u[t]$.*

Proof. We show that $\lim_{k\to\infty} \partial^k (Q_j D^j u[t])$ exists independently of the choise of $\tau(t)$ and is equal to $Q_{j+1} D^{j+1} u[t]$. Suppose for a moment that we have proved the following:

(*) For each β, $|\beta| \leqq m$, the sequence $A^\beta = \{D_x^\beta(\partial_k Q_j D^j u[t])\}_{k=1}^\infty$ is uniformly bounded in the norm $\|\varphi\|_\infty \equiv \sup\{|\varphi(x)| : x \in \bar{\Omega}\}$ and equicontinuous on $\bar{\Omega}$.

Then, every subsequence A_1^β of A^β is uniformly bounded in the $\|\cdot\|_\infty$ — norm and equicontinuous. Hence, A_1^β has in turn a subsequence, which converges uniformly on $\bar{\Omega}$ to a function $f^\beta \in S$. Since this is true for all β, $|\beta| \leqq m$, we have $f^\beta = D_x^\beta f$, where $f = f^0$. Hence, by (H4),

(3.6) $$\lim_{k\to\infty} \{(E^j[t_k](\vec{W}_k^j; v) - E^j[t](\vec{W}^j[t]; v)/(t_k - t)\} = E^{j+1}[t](\vec{W}^j[t], f; v)$$

for all $v \in S$, where $w^i[t] = Q_i D^i u[t]$. On the other hand, in the left side of (3.6), $\vec{W}_k^j$, $\vec{W}^j[t]$ may be replaced by $\vec{U}_k^j$, $\vec{U}^j[t]$, respectively, because $\vec{W}_k^j$, $\vec{W}^j[t]$ solve the Problems $P(j; t_k)$ and $P(j; t)$. Hence, by (H0) and (H4) this limit is also equal to $E^{j+1}[t](\vec{U}^{j+1}[t]; v)$ for all $v \in S$. Consequently, f solves $P(j+1; t)$ and, by the uniqueness Theorem 3.1, $f = Q_{j+1} D^{j+1} u[t]$. Hence, f is independent of the choice of the subsequences A_1^β, A_2^β. This shows that the entire sequence A^β converges to f. Since f is also independent of $\tau(t)$, the result of this Theorem follows.

It remains to show statement (*): dropping the first few indices k, we get from Lemma 3.4 that the sequence A^β is uniformly bounded in the L_2-norm. By the equivalence of norms on the finite dimensional S, it is also uniformly bounded in the $\|\cdot\|_\infty$-norm. Next, pick a basis $\{B_i\}_{i=1}^N$ of S, which is orthonormal in the L_2-norm. Writing $Q_jD^ju[t](x)=\Sigma_{i=1}^N\gamma_i^j(t)B_i(x)$, we first have by Lemma 3.4 that $\Sigma_{i=1}^N|\partial_k\gamma_i^j(t)|^2<\tilde{\varrho}$, so that $\max_i\sup_t|\partial_k\gamma_i^j(t)|<\sqrt{\tilde{\varrho}}$ for all k. From this, for any $x, y\in\bar{\Omega}$ we get

$$|D_x^\beta\partial_kQ_jD^ju[t](x)-D_x^\beta\partial_kQ_jD^ju[t](y)|\leq\sqrt{\varrho}\sum_{i=1}^N|D_x^\beta(B_i(x)-B_i(y))|.$$

By the assumed smoothness of the elements of S, $\Sigma_{i=1}^N D_x^\beta B_i$ is continuous on $\bar{\Omega}$, hence, A^β is equicontinuous. This concludes the proof of the Theorem.

4. A priori Error Estimates. The next two theorems provide bounds for some norms of the error $(I-Q_j)D^ju[t]$ of Problem $P(j;\,t)$. The rates of convergence are shown to be the same as those of the linear cases.

Although one could carry these estimates on to higher time derivatives, we stop at the point of interest to applications, i. e. at $j=1$.

Let $z^j[t]$ be the best approximation to $D^ju[t]$ by element of S, i. e. $\|z^j[t]-D^ju[t]\|_m=\inf\{\|v-D^ju[t]\|_m:\ v\in S\}$.

For the corresponding $(j+1)$-tuples $\vec{Z}^j[t]=(z^0[t],\ \ldots,\ z^j[t])$ and $\vec{U}^j[t]=(u^0[t],\ldots,\ D^ju^j[t])$ we have, according to the general notation used so far,

$$\|\vec{z}^j-\vec{U}^j\|_m=\sum_{i=0}^j\|z^i[t]-u^i[t]\|_m=\sum_{i=0}^j\inf_{v\in S}\|v-D^iu[t]\|_m.$$

Hence, let

$$\varrho_m(\vec{U}^j[t];\ S)=\|\vec{Z}^j-\vec{U}^j\|_m. \tag{4.1}$$

Assume that all the Hypotheses (H0) through (H5) hold.

Theorem 4.1. *There exists a constant B, depending only on r of* (H0), *such that* $\|(I-Q_j)D^ju[t]\|_m\leq B\varrho_m(\vec{U}^j[t];\ S)$ *for all* $t\in[0,\ T]$ *and all* $j=0,\ldots,\ J$.

Proof. By (4.1) each $z^i[t]$ lies in some ball $B(\varrho)\subset X$, where ϱ depends only on r. The same is true for $D^ju[t]$ and, by Lemma 3.3, for $Q_jD^ju[t]$. Hence, the constants involved below depend only on r. Applying (H1) with $\vec{\Phi}^j=(Qu[t],\ldots,\ Q_jD^ju[t])$, $\vec{\Psi}^j=(z^0[t],\ \ldots,\ z^j[t])$, taking into account that $Q_jD^ju[t]$ solves $P(j;\,t)$ and applying (H2) with the same $\vec{\Psi}^j$ and with $\vec{\Phi}^j=(u[t],\ \ldots,\ D^ju[t])$ we find $\|Q_jD^ju[t]-z^j[t]\|_m\leq\hat{B}\varrho_m(\vec{U}^j[t];\ S)$. The proof becomes complete by the triangle inequality.

Now let $\varrho_m(D^iu[t];\ S)\equiv\inf_{v\in S}\|D^iu[t]-v\|_m=\|D^iu[t]-z^i[t]\|_m$.

Theorem 4.2. *If B is as in Theorem* 4.1, *then*

$$\sup_{t\in[0,\ T]}\|(I-Q_0)u[t]\|_m+\sup_{t\in[0,T]}\|(I-Q_1)Du[t]\|_m$$

$$\leq\|(I-Q_0)u[0]\|_m+\|(I-Q_1)Du[0]\|_m+2B\sum_{i=0}^2\left[\int_0^T\varrho_m(D^iu[t];\ S)^2dt\right]^{1/2}.$$

Proof. If $\varphi[t]$ is any sufficiently smooth function, start with

$$D\varphi^2[t](x)=2\varphi[t](x)D\varphi[t](x)\leq\varphi^2[t](x)+(D\varphi[t])^2(x),$$

integrate the left side from 0 to $t\leq T$ and the right from 0 to T, then integrate on Ω, change the order of integration and apply the elementary inequality $\sqrt{|a|+|b|}\leq\sqrt{|a|}+\sqrt{|b|}$. Doing this with all the derivatives $D^\beta\varphi$, $|\beta|=m$, and taking the sup over $t\in[0, T]$ we find

$$\sup_{t\in[0,\,T]}\|\varphi[t]\|_m\leq\|\varphi[0]\|_m+\sum_{k=0}^{1}\left[\int_0^T\|D^k\varphi[t]\|_m^2\right]^{1/2}.$$

Apply this inequality with $\varphi[t]=(I-Q_i)D^i u[t]$, $i=0$, 1 and use Theorem 4.1 to complete the proof.

Conclusions. Under the given set of five hypotheses, the analogies with the linear case, in existence, smoothness and commutativity of the projections with $D=\partial/\partial t$, hold. In addition, the rates of convergence are the same as those of the linear case.

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