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UMBILIC POINTS ON COMPACT SURFACES OF REVOLUTION

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The paper describes a class of surfaces whose only umbilic points are the poles.

It is well known that a surface of revolution of the usual space R^3 is obtained by rotating a planar curve (the profile curve) about an axis of revolution lying in the plane of the curve and rigidly connected with the curve, through a complete revolution. If the profile curve is a simple closed regular curve and the axis of rotation is an axis of the profile curve and intersects it at two points only, at a right angle [2, p. 228], then the surface of revolution is a regular compact surface homeomorphic to S^2.

We take in the space R^3 a rectangular Cartesian coordinate system $Oxyz$ and we choose the z-axis to be the axis of rotation of the surface of revolution. Therefore, the surface is parameterized by

$$F(s,t)=(h(s)\cos t,\ \ h(s)\sin t, g(s)). \tag{1}$$

The curves of the surface, $s=$const, $t=$const are the *parallels* and *meridians* of the surface, correspondingly. The coefficients E, F, G of the first and L, M, N of the second fundamental forms of the surface are

$$E=(h')^2+(g')^2,\ F=0,\ G=h^2, \tag{2}$$

$$L=(h'g''-g'h'')/\sqrt{(h')^2+(g')^2},\ M=0,\ N=hg'/\sqrt{(h')^2+(g')^2}.$$

It is well known that on a compact surface of revolution (homeomorphic to S^2) the two points of intersection with the axis of rotation (that is the poles of the surface) are umbilic points of the surface. The question now is whether there are any other umbilic points except the poles on a compact surface of revolution [3, p. 290]. In this paper we find a class of such surfaces whose only umbilic points are the poles. This is a conclusion of the following

Proposition. *If the profile curve of a compact surface of revolution (homeomorphic to S^2) is a strictly convex simple closed curve with exactly four vertices, then there are no umbilics other than the poles on this surface of revolution.*

First, we shall prove the following

Lemma. *A point p of a surface of revolution, that doesn't lie on the axis of rotation, is an umbilic point iff the center of curvature at this point of the meridian which passes through p is a finite or infinite point of the axis of rotation.*

A proof of the above lemma exists in [1, p. 328]. Here we give another geometric proof.

Proof. Let M be the surface of revolution and $f(s,t)=(h(s)\cos t,\ h(s)\sin t,\ g(s))$ be the parametric representation, as we have seen above. The meridian

c_1 of the surface M, which goes through the point $p=f(s_1,t_1)$ has parametric representation

(3) $$c_1(s)=(h(s)\cos t_1,\ h(s)\sin t_1,\ g(s)).$$

We suppose that $k(p)\neq 0$ is the curvature of the meridian $c_1(s)$ (3) at the point p. We denote by c_2 the parallel of the surface, which goes through the point p and by $\bar{c}_2$ thecurve which is obtained from the intersection of the surface M with the plane (II): $[r(x,y,z)-f(s_1,t_1),f_2(s_1t_1),\nu(p)]=0$, where r is the position vector of the generic point of the plane, $\nu(p)$ is the normal vector to the surface M at the point $p=f(s_1,t_1)$ and $f_2(s_1,t_1)=\frac{\partial f}{\partial t}(s_1,t_1)$ is the tangent vector at p of the parallel c_2, which goes through the point p. The point p will be an umbilic point of the surface M iff the meridian c_1 and the curve $\bar{c}_2$ have the same curvature at the point p. The curve $\bar{c}_2$ and the parallel c_2 have the same tangent $f_2(s_1,t_1)=\frac{\partial f}{\partial t}(s_1,t_1)$ at p. Thus, from Meusnier's theorem [3, p. 76] we get that

(4) $$R=R_\nu\cos\varphi,$$

where φ is the angle of the principal normals of the curves $\bar{c}_2$ and c_2. R, R_ν are the radiuses of the curvature of the same curves c_2 and $\bar{c}_2$, correspondingly. The principal normals, at the point p, of the curves $\bar{c}_2$, c_1 and c_2 lie on the meridian plane (II): $[r(x,y,z)-f(s_1,t_1),f_1(s,t),\nu(p)]=0$, which goes through the point p. Also from (4) we have that the center of curvature of the curve $\bar{c}_2$ lies on the axis of revolution. Thus, the point p is umbilic iff the center of curvature of the meridian c_1 lies on the axis of revolution.

Let now $k(p)=0$. In this case the point p will be an umbilic point iff it is a flat point of the surface M. In this case the coefficients of the second fundamental form must be zero at the point p. Namely, $L(s_1\,t_1)=0$, $M(s_1,t_1)=0$, $N(s_1,t_1)=0$. Hence we take from (2) that $g'(s_1)=0$ and $g''(s_1)=0$. So the center of curvature of the meridian c_1 at the point p is the infinite point of the axis of rotation.

Notation. Since M is a surface of revolution and the tangent vectors $f_1(s,t)=\frac{\partial f}{\partial s}(s,t)$, $f_2(s,t)=\frac{\partial f}{\partial t}(s,t)$ of the meridians and parallels are the directions of the principal curvatures it is implied that every point of the parallel which passes through p is umbilic iff p is umbilic.

It is well known that the *evolute curve* of a given planar curve γ is the locus of the centers of curvature of the given curve γ.

Corollary. *A compact surface of revolution has and other umbilic points than the poles, iff the evolute curve of the profile curve of the surface has and other common points with the axis of rotation of the surface, except the centers of curvature which correspond at the common points of the profile curve and the axis of rotation.*

Proof of proposition. Let

(5) $$\gamma:[0,L]\to R^2: s\to\gamma(s)=(x=h(s),\ z=g(s))$$

be the profile curve, which is parameterized by the arclength $|\gamma'|^2=(h'(s))^2+(g'(s))^2=1$, and $\gamma(0)=P_1$, $\gamma(L_1)=P_2$, $\gamma(L/2)=P_3$, $\gamma(L-L_1)=P_4$ are the vertices of the curve γ. We notice that the poles P_1 and P_3 of the surface M are vertices of the curve γ because γ is symmetric to the axis of revolution.

The evolute curve of the curve γ is

$$r:[0, L] \to R^2 : s \to r(s) = \gamma(s) + n(s)/k(s), \tag{6}$$

where $k(s)$ is the curvature of the curve γ and $n(s)$ is the unit normal, at s, of the curve γ.

From (6) we take

$$r(s) = (x = h(s) - g'(s)/k(s),\ z = g(s) + h'(s)/k(s)) \tag{7}$$

and $r(0) = (x = 0,\ z = g(0) + h'(0)/k(0))$, $r(L/2) = (x = 0,\ z = g(L/2) + h'(L/2)/k(L/2))$.

The proposition will be true if we show that

$$h(s) - g'(s)/k(s) \neq 0 \text{ for every } s \in (0, L/2), \tag{8}$$

which means that the evolute curve of γ (5) doesn't intersect the axis of revolution oz.

From (7) we have

$$x' = g'(s)k'(s)/(k(s))^2. \tag{9}$$

Also since the curve γ (5) is convex we have $g'(s) < 0$ for every $s \in (0, L/2)$ and either

$$k'(s) < 0\ \forall s \in (0, L_1) \text{ and } k'(s) > 0\ \forall s \in (L_1, L/2) \tag{10}$$

or

$$k'(s) > 0\ \forall s \in (0, L_1) \text{ and } k'(s) < 0\ \forall s \in (L_1, L/2). \tag{11}$$

If the relation (10) is valid then we conclude that for the relation (9) we have $x'(s) > 0\ \forall s \in (0, L_1)$ and $x'(s) < 0\ \forall s \in (L_1, L/2)$. It means that the function $h(s) - g'(s)/k(s)$ is monotone and of course it is strictly increasing on the domain $(0, L_1)$ and strictly decreasing on the domain $(L_1, L/2)$: Since for $s = 0$ or $s = L/2$ the function $h(s) - g'(s)/k(s)$ is equal to zero we conclude that the relation (8) is valid. The same consideration for the relation (11) shows the truth of the relation (8).

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