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SIMPLE GROUPS OF ORDER $2^a.3.5.q^b$

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The aim of the paper is to give a characterization of certain finite simple groups by means of their order. The main result is as follows.

Let G be a simple group of order $2^a.3.5.q^b$, q prime. Then G is isomorphic to one of the groups $L_2(5)$, $L_2(9)$, $L_2(11)$, $L_2(16)$, $L_2(31)$, and $U_4(2)$.

1. Introduction. In the last ten years considerable progress has been made in classifying finite simple groups whose orders are divisible by exactly three distinct primes. Because of the joint work of a number of persons it seems that these groups are close to classification. On the other hand, from the group order point of view, only special cases of simple groups of order divisible by more than three primes have been settled. The purpose of this paper is to prove a result in this direction. More precisely, the following theorem is obtained.

Main theorem. *Let G be a simple group of order $2^a.3.5.q^b$, q prime. Then G is isomorphic to one of the groups $L_2(5)$, $L_2(9)$, $L_2(11)$, $L_2(16)$, $L_2(31)$, and $U_4(2)$.*

The simple groups of the order considered here arose naturally in the course of investigation on finite groups admitting a maximal subgroup isomorphic to A_5, the alternating group of degree 5. An application of the present result to the determination of a particular class of such groups is given in [24].

The bulk of our proof is the component case. The theory of both simple groups of component type and simple groups of low 2-local 3-rank is the object of intensive study. Nevertheless, it seems, at least to author's knowledge, that the present result cannot be deduced from the collection of results so far published.

The organization of the proof of the Main theorem is as follows. In Section 2 we list a number of known preliminary results, mostly modular character-theoretic, which are repeatedly used in the proof, as well as prove two elementary arithmetical facts which are necessary to determine the block structure of G. Section 3 imposes initial restrictions on G and, in particular, shows that if G is not one of the groups listed in Main theorem, it must be of component type. Section 4 is preparatory and establishes further odd-local properties of a minimal counterexample to Main theorem. Section 5 completes the proof by showing such a counterexample is necessarily of non-component type and thus contradicting Section 3.

2. Notation and preliminary results. Notation used in this paper is fairly standard and follows [12]. In addition, $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G and $B(p)$ the principal p-block of G. Character means an

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ordinary irreducible complex character. Notation like $B(3)=(1, 64, 65)$ means that $B(3)$ consists of three characters of degrees 1,64 and 65. If N is an integer, $\nu_p(N)$ denotes the exponent of the exact power of the prime p which divides N. A group element is called p-regular if its order is prime to p and p-singular in the contrary case. $N(H)$ and $C(H)$ always stand for the normalizer, respectively centralizer, of a subgroup H in the whole group. A group is said to be of non-component type if its 2-local subgroups are all 2-constrained, and of component type otherwise. Finally, we shall freely use the so-called bar convention for homomorphic images.

We now list a number of preliminary results which are needed for the proof of the Main theorem. In all of them G denotes a finite simple group and p a prime divisor of $|G|$.

Lemma 2.1. *Let $P\in \mathrm{Syl}_p(G)$ and $|P|=p$. Then $B(p)$ consists of $r=|N(P): C(P)|$ nonexceptional characters χ_i, $1\leq i\leq r$, and of an exceptional family of $s=(p-1)/r$ algebraically conjugate characters $\chi^{(i)}$, $1\leq i\leq s$, all of the same degree z_0. There are signs $\delta_i=\pm 1$, $\delta_0=\pm 1$, such that $z_i=\chi_i(1)\equiv\delta_i$, $sz_0\equiv\delta_0 \pmod p$; $i=1,\dots,r$. Furthermore, for every p-singular element x in G $\chi_i(x)=\delta_i$, $1\leq i\leq r$, and for every p-regular element x $\chi^{(i)}(x)=-\delta_0\sum_{k=1}^{r}\delta_k\chi_k(x)$, $1\leq i\leq s$. Finally, the so-called degree equation for $B(p)$ holds* ([4])

$$\sum_{k=1}^{r}\delta_k z_k+\delta_0 z_0=0.$$

Lemma 2.2 ([5]). *A character χ belongs to $B(p)$ iff*

$$|G: C(x)|\,\chi(x)/\chi(1)\equiv|G: C(x)| \pmod \omega)$$

for every $x\in G$; here ω is an appropriate prime ideal divisor of p.

Lemma 2.3 ([23; 6]). *If the character χ lies in $B(p)$ then so does every algebraically conjugate of χ.*

Lemma 2.4 ([10]). *If χ is a character of degree $p^n, n>0$, then χ is not in $B(p)$.*

Lemma 2.5 ([2]). *If the character χ belongs to $B(p)$ and χ is rational, then $\chi(1)\geq p^n(p-1)$, where $n=\nu_p(\chi(1))$.*

Lemma 2.6 ([10; 23]). *If x is any p-singular element in G then $\sum\chi(1)\chi(x)=0$, where the sum is taken over all characters in $B(p)$ (block-section orthogonality).*

Lemma 2.7 ([4; 23]). *If $\nu_p(|G|)=1$, $q\neq p$ is a prime divisor of $|G|$, and there is no element of order pq in G, then, in the notation of Lemma 2.1, $\sum\delta_i z_i\equiv 0 \pmod{q^b}$, where the sum is over all characters in $B(p)\cap B(q)$ and z_0 occurs at most once; here $b=\nu_q(|G|)$ (block separation condition).*

Lemma 2.8 ([9, p. 717]). *Let $P\in Syl_p(G)$ and $|P|=p$. If the character χ is in $B(p)$ then $\chi(1)$ divides $|G: C(P)|$.*

Lemma 2.9 ([6]). *Let x be a p-element in G and y is an element in $O_{p'}(C(x))$. Then for every character χ in $B(p)$, $\chi(xy)=\chi(x)$.*

Lemma 2.10 ([5]). *If $P\in \mathrm{Syl}_p(G)$ and $C(P)\subseteq P$, then $B(p)$ is the only p-block of full defect.*

Lemma 2.11 ([19]). *If G has a character of degree $p^n, n>0$, then $C(P)\subseteq P$ for $P\in \mathrm{Syl}_p(G)$.*

Lemma 2.12 ([12, Lemma 4.3.1]). *If $x\in G$ and χ is a character of G such that $(|G: C(x)|, \chi(1))=1$, then $\chi(x)=0$.*

Lemma 2.13. *If x is an element in G and χ is a character of G, then $\chi(x)\equiv\chi(x_1)\,(\mathrm{mod}\,\omega)$, where x_1 is the p-regular part of x and ω is as in Lemma 2.2. Moreover, if x is an involution, then $\chi(x)\equiv\chi(1)$ (mod 4).*

Proof. This is perhaps well-known. However, we note about the second part of the lemma that as G is simple, any involution of G has an even number of eigenvalues equal to 1, which yields the congruence.

Lemma 2.14. *If G has a character of degree p then $\nu_p(|G|)=1$.*

Proof (cf. [12, p. 171]). Let $P\in \mathrm{Syl}_p(G)$ and χ be the character of degree p. Consider the restriction of χ on P. Since G is simple and $\chi(1)$ is prime, it is easily seen that P is Abelian. Then Lemma 2.12 implies that χ vanishes on $P^{\#}$. Now $(\chi|_P, 1_P)$ shows that $|P|$ must divide $\chi(1)$, that is, $|P|=p$.

Lemma 2.15 ([12, Theorem 4.7.13]). *If χ is a character of G of p-defect zero then χ vanishes on all p-singular elements of G.*

Lemma 2.16 ([20]). *If G has a character of degree n which is rational, then*

$$\nu_p(|G|)\leq \sum_{i\geq 0}\left[\frac{n}{p^i(p-1)}\right].$$

Finally, we need two simple arithmetical lemmas.

Lemma 2.17. *Let x and y be positive integers greater than 1 and of the form $2^\alpha.5^\varepsilon.p^\beta$, p prime >3, $0\leq\varepsilon\leq 1$. If $1+x=y$ then either $x=4, y=5$ or $x=64, y=65$ or one of x, y is equal to p.*

Proof. Clearly x and y are mutually prime and we are led to three possibilities.

If x or y is 5^ε, then $x=4$, $y=5$.

If x or y is 2^α, we have $2^\alpha=5^\varepsilon.p^\beta\pm 1$. $\varepsilon=0$ yields, as well-known, $\beta=1$ as $p\neq 3$. So x or y is p. Let $\varepsilon=1$. Then $2^\alpha\equiv\pm 1$ (mod 5) which implies α is even and then $2^2\equiv 1$ (mod 3) yields $2^\alpha=5.p^\beta-1$. Further, $2^4\equiv 1$ (mod 5) shows α is not divisible by 4 whence $\alpha=2(2\gamma+1)$. Now $5.p^\beta=2^{2(2\gamma+1)}+1=(2^{2\gamma+1}-2^{\gamma+1}+1)(2^{2\gamma+1}+2^{\gamma+1}+1)$. The two factors are mutually prime which forces $2^{2\gamma+1}-2^{\gamma+1}+1=5$. Thus, $\gamma=1$, $\alpha=3$, and hence $x=64, y=65$.

If x or y is p^β, we have $p^\beta=5^\varepsilon.2^\alpha\pm 1$. Assume $\beta>1$ whence $\varepsilon=1$ and $\alpha>1$. If β is even, $p^\beta\equiv 1$ (mod 3) shows we have $p^\beta=5.2^\alpha-1$ which in turn yields $p^\beta\equiv -1$ (mod 4), an impossibility. So β is odd and we have $5.2^\alpha=(p\mp 1)$. $(p^{\beta-1}\pm\cdots\pm p+1)$. The second factor is odd and hence must be 1. This gives $\beta=1$, a contradiction.

The lemma is proved.

Lemma 2.18. *The solutions to the Diophantine equation*

$$(*)\qquad \delta_1.3^{\varepsilon_1}13^i+\delta_2.3^{\varepsilon_2}2^j+\delta_3.3^{\varepsilon_3}2^k13^l=63,$$

where

$$(**)\qquad 1<3^{\varepsilon_1}13^i\equiv\delta_1,\ 1<3^{\varepsilon_2}2^j\equiv\delta_2,\ 1<3^{\varepsilon_3}2^k13^l\equiv\delta_3\ (\mathrm{mod}\ 5),$$

and $\delta_1,\delta_2,\delta_3=\pm 1$; $0\leq\varepsilon_1,\varepsilon_2,\varepsilon_3\leq 1$; $i,l\leq 5$, are $-3.13+3.2+3.2^5=63$, $-13^2-3.2^3+2^8=63$, *and* $3.13^3-3.2^7-3.2^{11}=63$.

Proof. Assume first that $l>0$. Then reduction of $(*)$ modulo 13 implies $\delta_2.3^{\varepsilon_2}2^j\equiv -2$ (mod 13). If $\varepsilon_2=0$ this becomes $2^{j-1}\equiv-\delta_2$ (mod 13) and hence j is odd, while $(**)$ yields $2^j\equiv\delta_2$ (mod 5) and we reach a contradiction. So $\varepsilon_2=1$. Then $2^{j+1}\equiv-\delta_2$ (mod 5) by $(**)$, and also $3.2^{j-1}\equiv-\delta_2$ (mod 13) by the above, that is, $2^{j-3}\equiv\delta_2$ (mod 13). But the latter congruence is inconsistent with the former for both $\delta_2=\pm1$.

Thus, $l=0$ and hence $k>0$ by (**). Suppose one of j and k is 1. Let $k=1$. Then $\varepsilon_3=1$, $\delta_3=1$ by (**) and (*) becomes $\delta_1 . 3^{\varepsilon_1}13^i+\delta_2 . 3^{\varepsilon_2}2^j=57$. It is easily checked that $j=1$ is not possible. So $j>1$ and we obtain $\delta_1 . 3^{\varepsilon_1}\equiv 1$ (mod 4). Now $\varepsilon_1=0$ yields $\varepsilon_2=0$ and also $\delta_1=1$ whence $13^i\equiv 1$ (mod 5) so that $i=4$ and $\delta_2 . 2^j=-28504$, an absurd. Thus $\varepsilon_1=1$ and hence $\varepsilon_2=1$, $\delta_1=-1$. Then $13^{i+1}\equiv -1$ (mod 5) yields $i=1$ or 5 and $\delta_2\cdot 2^j=19+13^i$. This is possible only for $i=1$, giving the first solution of (*).

Let now $j, k\geq 2$. If $\varepsilon_1=0$, reduction of (*) modulo 4 implies $\delta_1=-1$ and hence (**) produces $i=2$. This leads to $\delta_2 . 3^{\varepsilon_2}2^j+\delta_3 . 3^{\varepsilon_3}2^k=2^3 . 29$. Now $j=k=2$ is not possible and then j (or k) must be 3 which yields the second solution of (*).

Finally, if $\varepsilon_1=1$, we have as above $\delta_1=1$, $i=3$, and (*) reads $\delta_2 . 3^{\varepsilon_2}2^j+\delta_3 . 3^{\varepsilon_3}2^k=-2^7 . 51$. Here j, k cannot both be ≤ 6, since the left side of the equation would be greater than the right one. Thus, we may take $j=7$ which leads to the third solution of (*). The lemma is proved.

3. **Proof of Main theorem. Initial reduction.** Let G be a simple group of order $2^a . 3 . 5 . q^b$, q prime. If $q\leq 5$, results of Brauer [8] and Herzog [18] yield $G\simeq A_5$, A_6 or $U_4(2)$. So we may suppose throughout $q\geq 7$ and $b>0$.

Lemma 3.1. *Either* $G\simeq L_2(11)$, $L_2(16)$ *or* $L_2(31)$, *or* $q=13$ *and* $B(3)=(1, 64, 65)$.

Proof. Let $P\in \mathrm{Syl}_3(G)$. Since G is simple, $N(P)\neq C(P)$ by Burnside's well-known theorem [12, Theorem 7.4.3] so that $|N(P): C(P)|=2$. Lemma 2.1 implies that $B(3)$ consists of three characters, of degrees $1, x, y$, and the degree equation is $1+x-y=0$. As x and y are prime to 3 and greater than 1, Lemma 2.17 yields the possibilities for $B(3)$.

The characters in $B(3)$ are the only ones of their degrees in $B(3)$ and so they are rational by Lemma 2.3. If $B(3)=(1, 4, 5)$ then Lemma 2.16 yields $b=0$, a contradiction.

Assume now that $B(3)$ contains a character of degree q. Then $b=1$ by Lemma 2.14. If $q\geq 17$ a result of Brauer and Tuan shows that $G\simeq L_2(q)$, $q=2^n\pm1$ or $G\simeq L_2(q-1)$, $q-1=2^n$ ([10]). Comparing orders we have $G\simeq L_2(31)$ or $G\simeq L_2(16)$. If $q<17$ then the degree equation for $B(3)$ shows $q=7$ or 11. In the first case Lemma 2.16 yields $a\leq 11$. In the second case G has a rational character of degree 11 and a result of Feit [11, Section 8.3] implies that G has a subgroup of index 11 or 12. Then G is contained in the symmetric group S_{12} so that $a\leq 10$. In both cases $|G|<10^6$ and M. Hall's list [16] gives $G\simeq L_2(11)$.

We note that in the above paragraph one could as well quote the result announced in [1].

Thus, to prove Main theorem, we must establish the following result.

Theorem 1. *There exists no simple group of order* $2^a . 3 . 5 . 13^b$ *with* $b>0$.

Lemma 3.2. *If* G *is a simple group of order* $2^a . 3 . 5 . 13^b$, $b>0$, *then* G *is of component type.*

Proof. Assume false so that all 2-local subgroups of G are 2-constrained. Lemma 3.1 shows that G has a character of degree 64.

$2\notin\pi_1$ (in Thompson's terminology) by a well-known old result [12, Theorem 7.6.1]. If $2\in\pi_2$, then [22] is applicable and implies $G\simeq L_2(t)$, $L_3(3)$, $U_3(3)$,

$U_3(4)$, A_7, M_{11}, J_2 or J_3. This contradicts the order of G. Thus $2 \notin \pi_3 \cup \pi_4$ and hence for every involution i of G $O(C(i))=1$ by a theorem of Gorenstein-Goldschmidt [14, p. 74]. Another result of Gorenstein [13] then yields $2 \notin \pi_4$.

Thus, all 2-local subgroups of G are 2-constrained and $2 \notin \pi_4$. Now a theorem of Wales [26] tells us that every nonprincipal 2-block of G has defect zero. Since the character of degree 64 cannot lie in $B(2)$ by Lemma 2.4, this implies $a=6$.

Now Sylow's theorems yield a unique possibility for the number of Sylow 13-subgroups of G, namely, 40. But this is impossible by a theorem of M. Hall [15]. This contradiction proves the lemma.

4. Odd-local structure of a minimal counterexample to theorem 1. In this section we shall derive further properties of a minimal possible simple group of order $2^a.3.5.13^b$, $b>0$. They will be needed in the next section for the proof of Theorem 1.

Throughout this section G is a counterexample to Theorem 1 of least possible order. It follows from the preceding section that G is of component type and $B(3)=(1, 64, 65)$.

Lemma 4.1. *Let π be an element of order 3, i an involution in G, and χ a nonprincipal character in $B(3)$. Then*

$$|G|\,|C(\pi)|\,(\chi(1)-\chi(i))^2=2^6.5.13.|C(i)|^2N(i)^2,$$

where $N(i)$ is a positive integer. Furthermore, $29 \leqq \chi(1)-\chi(i) \leqq 96$.

Proof. This is a group order formula due to Brauer [7, Section IX] and applicable here in view of the simplicity of G, Lemma 2.1, and $B(3)=(1, 64, 65)$.

Lemma 4.2. *Every proper simple section of G is isomorphic to A_5.*

Proof. This follows from the minimality of G, nonexistence of simple groups of orders $2^i.5.13^j$ [25] and $2^i.3.13^j$ [18], and the fact that A_5 is the only simple group of order $2^i.3.5$ [8].

Lemma 4.3. *G has no element of order* 15.

Proof. This follows from Lemma 2.8 as there is a character of degree 65 in $B(3)$.

Lemma 4.4. *A subgroup of G is nonsolvable iff its order is divisible by* 15.

Proof. This follows from Lemmas 4.2 and 4.3 and the existence of Hall subgroups of every possible order in a solvable group, as any group of order 15 is cyclic.

Lemma 4.5. $B(5)=(1, 2^6, 2^7.3, 2^{11}.3, 3.13^3)$.

Proof. The character of degree 64 is rational and Lemma 2.16 yields $b \leqq 5$. Since G has no element of order 15 by Lemma 4.3, we may apply 3—5 block separation (Lemma 2.7). Thus, the characters of degrees 1 and 64 are in $B(5)$, while that of degree 65 is not by Lemma 2.1. If $R \in \mathrm{Syl}_5(G)$ and $|N(R): C(R)|=2$, then the degree equation for $B(5)$ is $1-64+z=0$ which is impossible, as $z=63$ does not divide $|G|$. Therefore $|N(R): C(R)|=4$ and the degree equation is of the form $1-64+\delta_1 z_1+\delta_2 z_2+\delta_3 z_3=0$ or $\delta_1 z_1+\delta_2 z_2+\delta_3 z_3=63$. Here $\delta_i=\pm 1$, $z_i>1$, $z_i \equiv \delta_i \pmod 5$; $i=1, 2, 3$. We can obviously suppose z_1 odd and then, as z_1 cannot be 3, z_2 not divisible by 13. Now Lemma 2.18 yields the possibilities for $B(5)$.

The first one is immediately rejected, since the character of degree 6 is rational and then Lemma 2.16 gives $b=0$, a contradiction.

If the second case of Lemma 2.18 holds, G has a rational character of degree 24 and Lemma 2.16 shows that $b=2$. If G has no element of order 39, we may apply 3—13 block separation. The character of degree 65 is not in $B(13)$ by Lemma 2.5 and we have $1+64\equiv 0 \pmod{13^b}$, that is, $b=1$, a contradiction. Thus, let x be an element of order 39. Apply now block-section orthogonality (Lemma 2.6). Since the principal character belongs to $B(13)$, there must be a nonprincipal character, χ_1, in $B(13)$ such that $\chi_1(1)$ is odd and $\chi_1(x)\neq 0$. The latter condition implies that $\chi_1(1)$ is prime to 3. Moreover, as $b=2$, a theorem of Brauer [3, Theorem 3] shows $\chi_1(1)$ is not divisible by 13. Eventually, $\chi_1(1)=5$. Now Lemma 2.11 implies that a Sylow 5-subgroup of G is self-centralizing and then Lemma 4.4 yields the centralizer of every involution in G is solvable and hence 2-constrained. Equivalently, every 2-local subgroup of G is 2-constrained [13]. This contradicts Lemma 3.2 and so proves the present lemma.

Lemma 4.6. *Let T be a 13-subgroup of G of order at most 13^3. If an element ϱ of order 5 normalizes T then ϱ centralizes T.*

Proof. ϱ acts on $T/\Phi(T)$ which is elementary Abelian of order at most 13^3. Hence its automorphism group is $GL(n, 13)$. Since $|GL(n, 13)|$ is prime to 5 for $n\leq 3$, it follows that ϱ induces the trivial automorphism on $T/\Phi(T)$ and hence on T by a theorem of Burnside [12, Theorem 5. 1. 4].

Lemma 4.7. *Every proper subgroup of G is 13-constrained.*

Proof. Assume false and let H be a counterexample. By considering $H/O_{13'}(H)$ we may suppose $O_{13'}(H)=1$. Put $T=O_{13}(H)$, $C=TC_H(T)$, and $\bar{C}=C/T$; then $\bar{C}\neq 1$. It is well-known from the theory of non-constrained groups (or easily verified) that $O_{13}(\bar{C})=O_{13'}(\bar{C})=1$. Let now K be a minimal nontrivial normal subgroup of $\bar{C}$. Then K must be a nonabelian simple group and hence $K\simeq A_5$ by Lemma 4.2. But this contradicts the condition $O_{13'}(\bar{C})=1$, proving the lemma.

Lemma 4.8. $4\leq b\leq 5$.

Proof. We have already seen that $3\leq b\leq 5$ so that we have only to prove $b\neq 3$. Assume false and let π be an element of order 3 in G. Now the formula in Lemma 4.1 yields $\nu_{13}(|C(\pi)|)$ is even. If $\nu_{13}(|C(\pi)|)=O$, G has no element of order 39 and again 3—13 block separation (see Lemma 4.5) implies $b=1$, an impossibility. Thus, $\nu_{13}(|C(\pi)|)=2$. Let $T\in \mathrm{Syl}_{13}(C(\pi))$, $Q\in \mathrm{Syl}_{13}(G)$, and $T\subset Q$.

We shall first show that Q is Abelian. For, if not, Q is not contained in $C(T)$, that is, $T\in \mathrm{Syl}_{13}(C(T))$, and 13-constraint of $C(T)$ (Lemma 4.7) means $C(T)=T\times O_{13'}(C(T))$. As $O_{13'}(C(T))=O_{13'}(N(T))$ and $Q\subset N(T)$, Q normalizes $O_{13'}(C(T))$. But $\pi\in O_{13'}(C(T))$ and by the Frattini argument Q normalizes and hence centralizes a subgroup of order 3 in $O_{13'}(C(T))$. This contradicts $\nu_{13}(|C(\pi)|)=2$ and this shows that Q is Abelian.

Further, $C(Q)=Q$. Indeed, otherwise there is an involution i in $C(Q)$, as Q cannot centralize an element of order 3 or 5 by Lemma 2.8 (which actually shows G has no element of order 65). Using block-section orthogonality we can construct, as in Lemma 4.5, a character, χ_1, of degree 65 in $B(13)$. Here, $\chi_1(1)$ must be divisible by exactly the first power of 13, and also by 5, as the arguments in Lemma 4.5 show. Since $\chi_1(i)$ is an integer and $|G: C(i)|$ is prime to 13 by assumption, Lemma 2.2 reduces to $\chi_1(i)/\chi_1(1)\equiv 1 \pmod{13}$.

So $\chi_1(i)$ is a non-zero multiple of 13 and the congruence forces $\chi_1(i)=65=\chi_1(1)$, which is impossible for a simple group. This proves $C(Q)=Q$.

Let now χ be the character of degree 65 in $B(3)$. χ is not in $B(13)$, as already mentioned, and $B(13)$ is the only 13-block of full defect, as $C(Q)=Q$ (Lemma 2.10). So χ belongs to a 13-block whose defect group, D, is of order 13^2. Further, $O_{13}(C(D))=D$ [5] so that Q is not normal in $C(D)$. If now $x\in D^{\#}$ and $O_{13'}(C(x))=1$, we must have $Q=O_{13}(C(x))$ by the 13-constraint and hence Q is normal in $C(x)\supset C(D)$, a contradiction. Thus, $O_{13'}(C(x))=O_2(C(x))$ (by the above) $\neq 1$. Let E be a minimal nontrivial normal 2-subgroup of $C(x)$. E is elementary Abelian and $Q/C_Q(E)$ is a nontrivial automorphism group of E. This yields at once $|E|=2^{12e}, e>0$. This is, however, impossible.

For, let ζ be the character of degree 64 in $B(3)$. Then ζ is in $B(13)$, as ζ is of full 13-defect. Now for every i in $E^{\#}$, $i\in O_{13'}(C(x))$ and we have, because of Lemmas 2.13 and 2.9, $64=\zeta(1)\equiv\zeta(x)=\zeta(ix)\equiv\zeta(i)\pmod{13}$, as ζ is rational. Lemma 2.13 also implies $\zeta(i)\equiv\zeta(1)\equiv 0\pmod 4$. Thus, $\zeta(i)\equiv 12\pmod{52}$. Since $-32\leqq\zeta(i)\leqq 35$ by Lemma 4.1, we have eventually $\zeta(i)=12$ for every i in $E^{\#}$. Now $(\zeta|_E, 1_E)$ shows $|E|\leqq 4$. This contradicts the above paragraph and completes the proof of the lemma.

Lemma 4.9. *G has elements of orders 39 and 65. $B(13)$ contains characters of degrees 3.13^k and 5.13^l; k, l positive integers.*

Proof. We have already shown how to prove the existence of elements of order 39 and characters of degree 5.13^l in $B(13)$.

The character of degree 3.13^3 in $B(5)$ (Lemma 4.5) is rational (Lemma 2.3) and hence does not belong to $B(13)$ (Lemma 2.5). If G has no element of order 65,5—13 block separation yields the unique possibility $-3.13^3-1-2^6-3.2^7-3.2^{11}\equiv 0\pmod{13^b}$ or $b=3$ which is not possible by Lemma 4.8.

Thus, G has elements of order 65. Now, using block-section orthogonality and Lemma 2.4, one can construct characters of degree 3.13^k in $B(13)$ in the same way as for those of degree 5.13^l (cf. Lemma 4.5).

Lemma 4.10. *If $Q\in \mathrm{Syl}_{13}(G)$ then $C(Q)\subseteq Q$. If $x\in Z(Q)^{\#}$ then $|C(x)|$ is prime to 15.*

Proof. Assume $C(x)$ contains an element, π, of order 3. Let χ_1 be a character of degree 3.13^k in $B(13)$; such a character does exist by the preceding lemma. Since $(\chi_1(1), |G: C(x)|)=1$, Lemma 2.12 tells us $\chi_1(x)=0$. Then Lemma 2.2 implies $0=\chi_1(x)/\chi_1(1)\equiv 1\pmod{\omega}$. This contradiction proves that $C(x)$ possesses no elements of order 3.

Similarly, using a character of degree 5.13^l in $B(13)$, one proves that $C(x)$ possesses no elements of order 5.

Now, if $C(Q)\subseteq Q$, then there must be an involution, i, in $C(Q)$. Put i in the formula of Lemma 4.1. Since $\nu_{13}(\chi(1)-\chi(i))\leqq 1$, this yields $\nu_{13}(|C(\pi)|)\geqq b-1$. So the centralizer of some elment of order 3 contains a maximal subgroup of Q and hence contains a central element of Q. This is impossible by the above and completes the proof of the lemma.

Lemma 4.11. *If ϱ is an element of order 5 in G, then $\nu_{13}(|C(\varrho)|)=1$.*

Proof. Let $T\in \mathrm{Syl}_{13}(C(\varrho))$. Lemma 4.9 shows that $|T|\geqq 13$. Let $x\in T^{\#}$ and χ be the character of degree 65 in $B(3)$. Lemma 2.13 implies $\chi(x)\equiv 0\pmod{13}$ and also $0=\chi(\varrho x)\equiv\chi(x)\pmod 5$ as χ is of 5-defect 0 (Lemma 2.15). Thus, $\chi(x)\equiv 0\pmod{65}$, whence $\chi(x)=0$ for every x in $T^{\#}$. Now $(\chi|_T, 1_T)$ shows $|T|=13$.

L e m m a 4.12. *The centralizer of every nontrivial* 13-*subgroup in* G *is solvable.*

P r o o f. It is sufficient to prove the statement for a subgroup T of order 13. Assume it is not true. Set $C=C(T)$; thus $|C|$ is a multiple of 15 by Lemma 4.4. Let $D=O_{13'13}(C)$ so that $D=O_{13'}(C)U$, $U\in \mathrm{Syl}_{13}(D)$. Assume that $|D|$ is prime to 5. $C=DN_C(U)$ by the Frattini argument and then there is an element ϱ of order 5 in $N_C(U)$. Thus, ϱ normalizes U and centralizes $T\cdot|U|\leq 13^4$ as otherwise T is in the centre of a Sylow 13-subgroup of G and Lemma 4.10 shows C is solvable. Hence $|U/T|\leq 13^3$ and then ϱ centralizes U/T (see Lemma 4.6). So ϱ stabilizes the normal series $U\supset T\supset 1$ and [12, Lemma 5.3.2] implies that ϱ centralizes U. However, this contradicts the 13-constraint of C. Therefore, 5 divides $|D|$ and consequently 3 also divides $|D|$, as in the contrary case D and C/D would be solvalbe, yielding solvability of C.

Thus, we have shown there is an element, π, of order 3 in D, that is, in $O_{13'}(C)$. Furthermore, Lemmas 4.10 and 2.10 together imply that the character, ζ, of degree 3.2^7 from $B(5)$ is in $B(13)$. As ζ is rational and vanishes on all 3-singular elements, Lemma 2.9 yields $3.2^7=\zeta(1)\equiv\zeta(x)=\zeta(\pi x)=0 \pmod{13}$ for $x\in T^{\#}$, an absurd. The lemma is proved.

5. **G is of non-component type.** In view of Lemma 3.2, the proof of Theorem 1 will be completed once we have shown the following result.

T h e o r e m 2. *If* G *is a minimal counterexample to Theorem* 1 *then* G *is of non-component type.*

This last section is devoted to the proof of Theorem 2. In what follows we continue to assume that G is a minimal counterexample to Theorem 1.

L e m m a 5.1. *Let* H *be a proper non* 2-*constrained* 2-*local subgroup of* G *of largest possible order.*

(i) H *is a maximal* 2-*local subgroup of* G;

(ii) $H=N(V)$, $V=O_2(H)$, *and* $O(H)=O_{13}(H)$;

(iii) *There is a characteristic subgroup,* L, *of* H *such that* $\bar H=H/O(H)$ *has the following properties:* $[\bar V,\bar L]=1$, $\bar L'=\bar L$, $|Z(\bar L)|\leq 2$, $\bar L/Z(\bar L)\cong A_5$, *and* $\bar L\cong SL_2(5)$ *or* A_5.

P r o o f. Assume $H\subset H_1$, where H_1 is a proper 2-local subgroup of G. The maximal choice of H implies that H_1 is 2-constrained. An immediate consequence of a theorem of Gorenstein [13, Theorem 4] tells us that every 2-local subgroup in H_1 is also 2-constrained. This is not the case for H and thus proves (i). Now the first statement of (ii) is obvious, and the second follows from the fact that $O(H)$ is solvable (here easily verified) while $H/O(H)$ must not be.

(*iii*) collects well-known properties of non 2-constrained groups, see [14], together with Lemma 4.2 and results of Schur on perfect central extensions of $L_2(5)$ [21].

We now fix the notation of this lemma. In addition, π and ϱ will denote two arbitrary elements of L of orders 3 and 5, respectively. As shows the next lemma, we can remove the bars in Lemma 5.1. Then, the various parts of Lemma 5.1 will be freely used throughout this section without any further comment.

L e m m a 5.2. $O(H)=1$.

P r o o f. Assume false. Then $O(H)$ is a nontrivial 13-group.

If $|O(H)|\geq 13^4$, take an involution, i, in $V.[V,L]\subseteq V\cap O(H)=1$ yields $L\subseteq C(i)$. Consider once again the formula in Lemma 4.1, with $\chi(1)=65$. Now

$65-\chi(i)$ is prime to 13. For, otherwise we have $65-\chi(i)$ divisible by 5.13 as 5 divides $|C(i)|$ but not $|C(\pi)|$. So $\chi(i)=0$, incompatible with Lemma 2.13. Thus, $65-\chi(i)$ is prime to 13 and, since $O(H)$ centralizes i, the formula yields $|C(\pi)|$ is dividible by 13^4. We reach a contradiction to Lemma 4.10.

Thus, $|O(H)|\leq 13^3$. Now ϱ centralizes $O(H)$ by Lemma 4.6. It follows from Lemma 4.11 that $|O(H)|=13$. $C(O(H))$ is solvable by Lemma 4.12 and $N(O(H))/C(O(H))$ is cyclic. Consequently $N(O(H))$ is solvable and hence so does H, since $H\subseteq N(O(H))$. This is a contradiction proving the lemma.

Lemma 5.3. *$C_H(L)=V$ and $H/V\subseteq S_5$.*

Proof. $|C_H(L)|$ is prime to 15 as $Z(L)$ is a 2-group; it is also prime to 13, since the centralizer of every nonidentity 13-element is solvable by Lemma 4.12. Thus, $C_H(L)$ is a normal 2-subgroup in H and hence $C_H(L)=V$.

Assume now that an element $x\in H\setminus V$ acts trivially on $L/Z(L)$. Then $[x, L]\subseteq Z(L)$ so that $[L, x, L]=[x, L, L]=1$. Now the Three-subgroup lemma yields $[L, L, x]=1$, or $[L, x]=1$ as $L'=L$. This is impossible by the above paragraph and hence H/V acts nontrivially on $L/Z(L)\simeq A_5$. Then $H/V\subseteq \mathrm{Aut}(A_5)\simeq S_5$.

Lemma 5.4. *V is of rank at most two.*

Proof. Let ϑ be an involution in V and χ the character of degree, say, 65 in $B(3)$. Lemma 2.13 implies $\chi(\vartheta)\equiv\chi(\vartheta\pi)\equiv -1 \pmod 3$. Similarly, $\chi(\vartheta)\equiv\chi(\vartheta\varrho)=0 \pmod 5$. Furthermore, $\chi(\vartheta)\equiv 1 \pmod 4$, again by Lemma 2.13. Thus, $\chi(\vartheta)\equiv 5 \pmod{60}$. Since $-31\leq\chi(\vartheta)\leq 36$ by Lemma 4.1, we have $\chi(\vartheta)=5$ for, every involution ϑ in V. Let now W be an elementary Abelian subgroup in V. Then $(\chi|_W, 1_W)$ shows $60\equiv 0 \pmod{|W|}$ or $|W|\leq 4$.

Lemma 5.5. *$V\in \mathrm{Syl}_2(C(\varrho))$.*

Proof. Suppose $V\notin \mathrm{Syl}_2(C(\varrho))$. Then there is a subgroup W in $C(\varrho)$ with $|W: V|=2$. Now V is normal in W and hence $W\subseteq N(V)=H$. Then Lemma 5.3 implies that W/V is a subgroup of order 2 in S_5. But W centralizes ϱ, while S_5 has no element of order 2 centralizing an element of order 5. This contradiction proves the lemma.

Lemma 5.6. *If $\vartheta\in\Omega_1(Z(V))$ then $C(\vartheta)\subseteq H$.*

Proof. $\Omega_1(Z(V))$ is an elementary Abelian normal subgroup in H of order at most 4 so that ϑ has at most 3 conjugates in H. Since π centralizes ϑ, it remains $|H: C_H(\vartheta)|\leq 2$. In particuler, $C_H(\vartheta)$ is normal in H.

Further, $C(\vartheta)$ is not 2-constrained. For, in the contrary case $N_{C(\vartheta)}(V)$ is also 2-constrained by the result of Gorenstein already quoted in Lemma 5.1. But then $C_H(\vartheta)=H\cap C(\vartheta)=N_{C(\vartheta)}(V)$ is 2-constrained. As $C_{H(\vartheta)}$ is normal in H, it is clear that $O_2(C_H(\vartheta))=V$ and $O(C_H(\vartheta))\subseteq O(H)=1$. It follows $V\supseteq C_{C_H(\vartheta)}(V)$ by the 2-constraint. This is obviously impossible as $L\subseteq C_H(\vartheta)$.

Thus, $C(\vartheta)$ is not 2-constrained and then the maximality of H yields $|H|\geq|C(\vartheta)|$. Hence $|C(\vartheta): C_H(\vartheta)|\leq 2$ so that $C_H(\vartheta)$ is normal in $C(\vartheta)$. Therefore, $C(\vartheta)\subseteq N(V)=H$.

Lemma 5.7. *V is cyclic of order at most 4.*

Proof. Let $C=C(\varrho)$. Lemma 4.11 shows that $|C|$ is divisible by exactly the first power of 13. If $O_2(C)\neq 1$, any 13-element in C normalizes and hence centralizes $\Omega_1(Z(O_2(C)))$, which is of order at most 4 (Lemma 5.4). But $\Omega_1(Z(O_2(C)))$ is a normal subgroup in $V\in \mathrm{Syl}_2(C)$ (Lemma 5.5) and so has a nontrivial intersection with $Z(V)$. This contradicts Lemma 5.6 as $|H|$ is prime to 13 (Lemma 5.3). Thus, $O_2(C)=1$. As $C=\langle\varrho\rangle\times D$, where D is solvable, it follows $O_{13}(C)=O_{13}(D)\neq 1$. Now $C_V(O_{13}(C))$ is normal in V and, as

$C_V(O_{13}(C)) \cap Z(V)$ must be trivial, we have $C_V(O_{13}(C)) = 1$. Hence $V \subseteq \text{Aut}(O_{13}(C))$, the latter group being cyclic of order 12. This proves the lemma.

Lemma 5.8. $|V| = 2$.

Proof. Let $S_1 \in \text{Syl}_2(C(\pi))$ with $S_1 \supseteq V$ and set $W = N_{S_1}(V)$.

Let now i in Lemma 4.1 be a central involution of G. Then the formula yields $\nu_2(|C(\pi)|) \geq a-6$, as $\nu_2(\chi(1)-\chi(i)) \leq 6$. The existence of a character of degree 3.2^{11} (in $B(5)$) shows $a \geq 11$ and hence $|S_1| \geq 2^5$.

We now proceed as in Lemma 5.5. So $W \subset H$ and W/V is a 2-subgroup of S_5 centralizing an element of order 3. This is possible only if $|W/V| \leq 2$ and then $|W| \leq 8$ (Lemma 5.7). This shows, together with the above paragraph, that W is properly contained in S_1 and $|W: V| = 2$.

Thus, there is an element, x, in $N_{S_1}(W) \setminus W$ such that $x^2 \in W$. Then $(V^x \cap V)^x = V^{x^2} \cap V^x = V \cap V^x$. If v is the involution in V (Lemma 5.7), $C(v) = N(V) = H$ and, as $x \notin H$, we must have $V \cap V^x = 1$. Now $VV^x \subseteq W$, $|VV^x| = |V|^2 \leq |W| \leq 8$ which is possible only if $|V| = 2$.

Lemma 5.9. *G is not of component type.*

Proof. We have so far seen $H = C(v)$, $\langle v \rangle = V = O_2(H)$, and $H/V \cong A_5$ or S_5. We must actually have $H/V \cong S_5$, since otherwise $W = V$ in Lemma 5.8, as A_5 has no element of order 6. Then $V \in \text{Syl}_2(C(\pi))$ which is impossible (see Lemma 5.8). Consider $H_1 = \langle v \rangle L$. $H_1 \cong SL_2(5)$ or $Z_2 \times A_5$ and $|H: H_1| = 2$.

Suppose first that $H_1 \cong SL_2(5)$. Then v is the only involution in $S_1 \in \text{Syl}_2(H_1)$, as Sylow 2-subgroups of $SL_2(5)$ are quaternion. If now z is a central involution of G lying in $S \in \text{Syl}_2(H)$, $|H|$ shows that $z \neq v$ and so $z \in S \setminus S_1$. Thus $S = S_1 \times \langle z \rangle$. But then $S/\langle v \rangle$ is elementary Abelian which is not the case for S_5.

Therefore, $H_1 \cong Z_2 \times A_5$. Now $S_1 \in \text{Syl}_2(H_1)$ is elementary Abelian of order 8. Since the four-subgroups in A_5 are self-centralizing and S cannot be Abelian, we have $C(S_1) = C_H(S_1) = S_1$. Now a result of Harada [17, Theorem 2] implies that G has sectional 2-rank at most 4 and then a theorem of Gorenstein and Harada [27] yields the possibilities for G. All these are incompatible with the order of G, giving the final contradiction.

This proves the lemma and thus completes the proof of Theorem 2, Theorem 1, and Main theorem.

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