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BETWEENNESS AND TERNARY OPERATIONS ON PARTIAL LATTICES

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A partial lattice structure with the ternary operation of betweenness is constructed. The structure found in the paper generalizes the structure of simple graphic algebras.

1. Introduction and basic concepts. The purpose of this paper is to generalize Draškovičová's considerations on betweenness and ternary relations on lattices [3] for partial lattices. This work is also a generalization of the paper [5] on simple graphic algebras. All the results here are based on the paper [3]. As a general reference we have used Birkhoff's monograph [2], to which the reader is referred. The ternary operations on modular lattices are also considered recently by Kolibiar and Marcisova in [4].

Let S be a join-semilattice, briefly a semilattice. The set $[a)=\{x \mid x\geqq a, x, a\in S\}$ is called the principal filter of S generated by a. If for each $a\in S$, $[a)$ is a sublattice of S, then S is called a partial lattice. The observations of the following paragraph are from Draškovičová's paper [3].

Let (M, d) be a metric space, $a, b, c, x\in M$, and let us denote: $[x; a, b, c] = d(x, a)+d(x, b)+d(x, c)$. $V(a, b, c)$ is the set of all elements $x\in M$ for which the function $f(x)=[x; a, b, c]$ reaches its minimum. In a metric space (M, d) the relations abc and $d(a, b)+d(b, c)=d(a, c)$ are equivalent and moreover, in a metric lattice the relation abc is equivalent to the relation

$$(a\wedge b)\vee(b\wedge c)=b=(a\vee b)\wedge(b\vee c). \tag{1}$$

As in [3], we write abc also then when (1) holds in an arbitrary lattice L, where $B(a, b)$ denotes the set $\{x \mid axb\}$ and $B(a, b, c)=B(a, b)\cap B(a, c)\cap B(b, c)$. In a metric space (M, d), $B^m(a, b, c)$ denotes the set of all elements $x\in M$ for which axb, bxc and cxa hold. Accordingly, $B(a, b, c)=B^m(a, b, c)$ in a metric lattice. We collect the three first theorems of Draškovičová into a lemma:

Lemma 1. (i) *In a metric space* (M, d), *if* $B^m(a, b, c)\neq\emptyset$, *then* $B^m(a,b,c) = V(a, b, c)$.

(ii) *A lattice* L *is modular iff* $B(a, b, c)\neq\emptyset$ *for any three elements* $a, b, c\in L$.

(iii) *In a metric lattice* L, $B(a, b, c)=V(a, b, c)$ *for any three elements* $a, b, c\in L$.

As it is well known, a metric lattice is also modular.

Let S be a partial lattice. If S can be embedded into a metric lattice $L(S)$ such that for any three elements $a, b, c\in S$, $B^m_{L(S)}(a, b, c)\cap S\neq\emptyset$, where the set $B^m_{L(S)}(a, b, c)$ is determined in the lattice $L(S)$, then the betweenness relation $B_{L(S)}(a, b, c)$ of $L(S)$ determines a betweenness relation $B_S(a, b, c)$

$=B_{L(S)}(a, b, c)\cap S$ in the partial lattice S. In the following we shall consider the structure of a partial lattice S for which the betweenness $B_S(a, b, c)=B_{L(S)}(a, b, c)\cap S$ satisfies: $B_S(a, b, c)\neq\emptyset$ for any three elements $a, b, c\in S$.

2. The betweenness relation and partial lattices. We describe the properties of S in a serie of lemmas.

Lemma 2. *Let S be a partial lattice such that $B_S(a, b, c)=B_{L(S)}(a, b, c)\cap S\neq\emptyset$ for any three $a, b, c\in S$. Then $[p)$ is a sublattice of $L(S)$ for each $p\in S$.*

Proof. Let $a, b\in[p)$. According to [2, Chapt. V:7, Ex. 1], $d(a, x)+d(b, x)=d(a, b)$ in $L(S)$ iff $x\in[a\mathbf{\wedge} b, a\mathbf{\vee} b]$; we denote in this proof the meet and join of $L(S)$ by $\mathbf{\wedge}$ and $\mathbf{\vee}$, respectively.

Consider the triplet $a, b, a\wedge b$ of S. If $x\in B_{L(S)}(a, b, a\wedge b)$, then $x\in[a\mathbf{\wedge} b, a\mathbf{\vee} b]\cap[a\mathbf{\wedge} b, a]\cap[a\mathbf{\wedge} b, b]$. If $x\neq a\mathbf{\wedge} b$, there would be a common lower bound x of a and b in $L(S)$, $x>a\mathbf{\wedge} b$, which is a contradiction. Hence $x=a\mathbf{\wedge} b$. But because $\{x\}=B_{L(S)}(a, b, a\wedge b)$, then $a\mathbf{\wedge} b\in S$, as $B_S(a, b, a\mathbf{\wedge} b)\neq\emptyset$. Clearly $a\mathbf{\wedge} b=a\mathbf{\wedge} b$. The proof is similar for $a\vee b$. Thus $[p)$ is a sublattice of $L(S)$.

Lemma 3. *Let S be a partial lattice where $B_S(a, b, c)=B_{L(S)}(a, b, c)\cap S\neq\emptyset$ for any three elements $a, b, c\in S$. Then there is for any three $a, b, c\in S$ an element $p\in S$ such that $(a\vee b)\wedge(a\vee c)\wedge(b\vee c)\in[p)\subset S$.*

Proof. As before, if $x\in B_{L(S)}(a, b, c)$, then $x\in[a\wedge b, a\vee b]\cap[a\wedge c, a\vee c]\cap[b\wedge c, b\vee c]$. Accordingly, $x\geqq(a\wedge b)\vee(a\wedge c)\vee(c\wedge b)$ and, on the other hand, $x\leqq(a\vee b)\wedge(a\vee c)\wedge(b\vee c)$. So for each $x\in B_S(a, b, c)=B_{L(S)}(a, b, c)\cap S$ it holds $(a\vee b)\wedge(a\vee c)\wedge(c\vee b)\in[x)$ because $x\in S$, $[x)$ is a sublattice of S and because $x\leqq a\vee b, a\vee c, b\vee c$. Hence the lemma.

As $L(S)$ is a metric lattice, it is modular. The following lemma gives a class of partial lattices which can be embedded into a modular lattice.

Lemma 4. *Let S be a partial lattice. The lattice $\mathfrak{F}(S)$ of all filters of S is modular iff $[p)\subset S$ is a modular sublattice of S for each $p\in S$.*

Proof. We previously defined the principal filter $[a)$ of S olny. By a filter F of S is meant a non-empty subset of S such that (i) if $a\in F$ and $x\geqq a$, then $x\in F$, and (ii) if $a, b\in F$ and $a\wedge b\in S$, then $a\wedge b\in F$. Let $F, K\in\mathfrak{F}(S)$. The meet $F\wedge K$ is defined to be the set theoretical intersection of F and K, and the join $F\vee K$ is the smallest filter of S containing the filters F and K. $F\vee K$ can be characterized also as follows (see Abbot [1, Sect. 5]): $F\vee K=\{x\mid x\geqq f\wedge k,\ f\in F,\ k\in K \text{ and } f\wedge k\in S\}$. If F and K are filters of S, then $f\vee k\in F\wedge K$, and hence $F\wedge K$ exists for any finite meet of filters of S. Clearly $\mathfrak{F}(S)$ is a lattice. So we must show the modularity of $\mathfrak{F}(S)$: if $F, K, J\in\mathfrak{F}(S)$ and $F\subset J$ then $(F\vee K)\wedge J\subset F\vee(K\wedge J)$.

Let $j\in(F\vee K)\wedge J \Longleftrightarrow j\in J$ and $j\in F\vee K$. So $j\geqq f\wedge k$ for some j and $k, f\in F\subset J$ and $k\in K$. As $f, j\geqq f\wedge k$, $f\wedge j$ exists and it belongs to J. Then $k\vee(f\wedge j)$ is in K and J according to the definition of a filter, whence it is also in $K\wedge J$. According to the modularity of $[f\wedge k)$ in S, $f\wedge(k\vee(f\wedge j))=(f\wedge k)\vee(f\wedge j)\leqq j$, where $f\wedge(k\vee(f\wedge j))\in F\vee(J\vee K)$. Hence $j\in F\wedge(J\vee K)$, and the modularity of $\mathfrak{F}(S)$ follows.

Conversely, let $\mathfrak{F}(S)$ be a modular lattice. Clearly $[p)\subset S$ is a sublattice of $\mathfrak{F}(S)$for each $p\in S$. If $[p)$ is not a modular sublattice of S, then it, and so also $\mathfrak{F}(S)$, contains a non-modular sublattice, which contradicts the modularity of $\mathfrak{F}(S)$. Hence $[p)$ is a modular sublattice of S.

If there is a greatest element 1 in S, then $[1)$ is the least element of $\mathfrak{F}(S)$

In every finite modular lattice L the height function $h[x]$ (the length h of the maximum chain $0<x_1<x_2<\cdots<x_h=x$; see [2]) determines a valuation on L such that $h[x]$ defines a distance function with respect to which L is a metric lattice. The observations above suggest the following theorem:

Theorem 1. *Let S be a finite partial lattice. If $[p)$ is a modular sublattice of S for each $p\in S$ and there is an element $q\in S$ such that $a\vee b$, $a\vee c$, $b\vee c\in[q)$ for each three elements $a, b, c\in S$, then the betweenness relation $B^m([a), [b), [c))$ of the lattice $\mathcal{F}(S)$ induces a betweenness relation $B_S(a, b, c)$ in S such that $B_S(a, b, c)\neq\emptyset$ for each three elements $a, b, c\in S$.*

Proof. $\mathcal{F}(S)$ is a metric lattice with respect to the distance function generated by the height function $h[x]$ in $\mathcal{F}(S)$. Then $([a)\vee[b))\wedge([a)\vee[c))\wedge([b)\vee[c))\in B_{\mathcal{F}(S)}([a), [b), [c))$ for any three $[a), [b), [c)\in\mathcal{F}(S)$. But then the assumptions of the theorem imply that $B_S(a, b, c)\neq\emptyset$ for any three $a, b, c\in S$.

3. **A ternary operation characterization.** Let N be a set with a ternary operation (abc). If (abc) satisfies in N the formulas

$$(baa)=a,\quad ((ade)b(cde))=(a(bed)(ced)),$$

N is called a ternary modular semilattice, briefly a *TMS* (see [3, sect. 5]). As proved by Draškovičová [3, Thm. 6], if N is a *TMS* with a least element 0 and where there is an element $u\in N$ for any two elements $a, b\in N$ such that $(0au)=a$ and $(0bu)=b$, then N can be translated into a modular lattice $L=(N, \wedge, \vee)$ with 0 as its least element. Conversely, if L is a modular lattice with a least element 0, then it determines a *TMS* with respect to the operation

$$(abc)=((b\vee c)\wedge a)\vee(b\wedge c)=(b\vee c)\wedge(a\vee(b\wedge c)).$$

In the following we shall show what kind of a ternary subsystem S generates in the *TMS* of $\mathcal{F}(S)$. This will be done in two theorems.

Theorem 2. *Let S be a partial lattice such that $1\in S$, $[p)$ is a modular sublattice of S for each $p\in S$, and for any three elements $a, b, c\in S$ there is an element $q\in S$ such that $a\vee b$, $a\vee c$, $b\vee c\in[q)\subset S$. Then the elements of S constitute a subalgebra of the TMS generated by the modular lattice $\mathcal{F}$(S) with the least element [1).*

Proof. We must only show that the elements in S are closed under the operation (abc) of the *TMS* of $\mathcal{F}(S)$, i. e. if $a, b, c\in S$, then $([a)\ [b)\ [c))=([b)\wedge[c))\vee([a)\wedge([b)\vee[c)))=[n)$ for some element n of S.

According to the definition of a filter in S, $[b)\wedge[c)=[b\vee c)$. We obtain two cases for $[b)\vee[c)$: 1° $b\wedge c\in S$ and 2° $b\wedge c\notin S$.

1° If $b\wedge c\in S$, then $[b)\vee[c)=[b\wedge c)$ and $[a)\wedge([b)\vee[c))=[a\vee(b\wedge c))$. Further, $([b)\wedge[c))\vee([a)\wedge([b)\vee[c)))=[b\vee c)\vee[a\vee(b\wedge c))=[(b\vee c)\wedge(a\vee(b\wedge c)))$; the element $(b\vee c)\wedge(a\vee(b\wedge c))$ exists in S according to the properties of S.

$$2°[b)\vee[c)=\{x\mid x\geq k\wedge g,\ k\geq b,\ g\geq c \text{ and } k\wedge g\in S\}.$$

If now $a\geq k\wedge g$ for some $k\wedge g\in[b)\vee[c)$, then $[a)\wedge([b)\vee[c))=[a)$ and then $b\vee c\in[a)$ or $a\in[b\vee c)$. So $([b)\wedge[c))\vee([a)\wedge([b)\vee[c)))=[b\vee c)\vee[a)=[(b\vee c)\wedge a)=[b\vee c)$ or $[a)$, and the lemma is valid in this case.

Obviously $[a)\wedge([b)\vee[c))=\{y\mid y=a\vee x,\ x\in[b)\vee[c)\}=T$. $a\vee b$, $a\vee c\in[a)$, and as $[a)$ is a sublattice of S, $(a\vee b)\wedge(a\vee c)\in[a)\subset S$. On the other hand, $a\vee b\in[b)$ and $a\vee c\in[c)$, whence $(a\vee b)\wedge(a\vee c)\in[b)\vee[c)$. Consequently, $[(a\vee b)\wedge(a\vee c))\subset T$. Let $z\in T$. Then $z=a\vee x\geq a\vee(k\wedge g)$, where $x\geq k\wedge g$, $k\geq b$ and

$g \gtrsim c$. Assume now that $a \leqq a \vee x < (a \vee b) \wedge (a \vee c)$. Then $k \wedge g \urcorner > a$, as in the other case $(a \vee k) \wedge (a \vee g) < (a \vee b) \wedge (a \vee c)$, which is a contradiction. The case $k \wedge g \leqq a$ leads also to the desired result. So it remains only the case $k \wedge g$ and a are non-comparable.

If $a \vee b = a \vee c = b \vee c$, then $a \vee x = x$ and $[a) \wedge ([b) \vee [c)) = [x)$ or we find a non-modular sublattice of S with five elements. Thus $b \vee c \in [x)$ and so $([b) \wedge [c)) \vee [x) = [b \vee c)$, wh re $b \vee c \in S$. The case $a \vee b = a \vee c > b \vee c$ is obvious. If $a \vee b \neq a \vee c$, then $a \vee b > k > k \wedge g$ and $a \vee b > (b \vee a) \wedge (a \vee c) > x \vee a > k \vee g$ constitute a non-modular sublattice of S, whence $x \vee a \geq (b \vee a) \wedge (a \vee c)$. Thus in this case $T \subset [(a \vee b) \wedge (a \vee c))$, and so $T = [(a \vee b) \wedge (a \vee c))$. But then $([b) \wedge [c)) \vee ([a) \wedge ([b) \vee [c))) = [b \vee c) \vee [(a \vee b) \wedge (a \vee c)) = [(b \vee c) \wedge (a \vee b) \wedge (a \vee c)) = [n)$, where $n \in S$ according to the properties of S. This completes the proof.

Theorem 3. *Let a set N be a TMS with respect to theternary operation (abc) such that it can be translated into a modular lattice $L = (N, \wedge, \vee)$ with the greatest element* 1. *Let $S \subset N$ $(1 \in S)$ be a closed subset under (abc). Then the elements of S can be translated into a partial lattice, where for each $p \in S$, the set $[p) \subset S$ is a modular sublattice of S and where there is an element $q \in S$ such that $a \vee b, a \vee c, b \vee c \in [q) \subset S$ for each three elements $a, b, c \in S$.*

Proof. We shall show the existence of the element $q \in S$ only; the other assertions hold obviously.

Let $a, b, c \in S$. As S is closed under (abc), then (abc) and (bca) belong to S. Moreover, $((abc)\ 1\ (bca)) \in S$ and

$$((abc)\ 1\ (bca)) = (abc) \vee (bca) = ((b \vee c) \wedge a) \vee (b \wedge c) \vee ((c \vee a) \wedge b) \vee (a \wedge b) = (b \wedge c)$$
$$\vee [(c \vee a) \wedge (b \vee c) \wedge (a \vee b)] \vee (a \wedge b) = (c \vee a) \wedge (b \vee c) \wedge (a \vee b).$$

By putting this element equal to q, the theorem follows.

The two theorems above characterize S in terms of ternary operations.

REFERENCES

1. J. C. Abbot. Semi-Boolean algebra. *Математич. Вестник*, 4, 1967, 177—198.
2. G. Birkhoff. Lattice theory. Providence, 1961.
3. H. Draškovičová. Über die Relation „zwischen" in Verbänden. *Mat.-Fyz. Casopis SAV*, 16, 1966, 13—20.
4. M. Kolibiar, T. Marcisova. On a question of J. Hashimoto. *Mat. Casopis SAV*, 24, 1974, 179—185.
5. J. Nieminen. Join-semilattices and simple graphic algebras. *Math. Nachr.*, 77, 1977, 87—91.

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