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A CLASS OF FACTORIAL DOMAINS

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In this complement to recent work of H. Kleppe and D. Laksov (1979) and of V. Marinov (1979), we record the factoriality of the coordinate ring of the affine variety, defined by the vanishing of all the subpfaffians of order $2t+2$ of an $s\times s$ matrix of indeterminates $(2\leqq 2t+2\leqq s)$.

The proof uses a pleasantly unsophisticated variation of an argument appearing in a particular case in [2, § 14], and developed in full generality by Bruns in his computation of the divisor class groups of the generic determinantal varieties [1]. Our reference for the theory of class groups is [2], and we refer to [4] for the results on pfaffian ideals.

Theorem. *Let R be a commutative Noetherian ring and $X=\{X_{ij}\}_{1\leqq i<j\leqq s}$ be a set of indeterminates. Let S denote the factor ring of the polynomial ring $R[X]$ modulo the ideal I, generated by all the $(2t+2)\times(2t+2)$ subpfaffians of the alternating $s\times s$ matrix X, having X_{ij} as above-diagonal elements.*

Then S is a normal domain iff R is a normal domain. In this case, $\mathrm{Cl}(S)\simeq\mathrm{Cl}(R)$. In particular, S is factorial iff R is factorial.

Corollary. *S is Gorenstein iff R is Gorenstein.*

For a completely different proof of the corollary see [4, § 6].

The homomorphism $R\to S$ being faithfully flat [4, proposition 14], it transports up and down both the Gorenstein [6, theorem 1′] and the normality [3, (6. 5. 4)] properties, provided the fibers satisfy the corresponding condition. Hence, noting that the property of being a domain passes from R to S [4, theorem 12], the corollary (resp. the first assertion of the theorem) will follow if we show that in case R is a field, S is Gorenstein (resp. normal). Gorensteinness is a consequence of Murthy's theorem [2, (12. 3)], since S is a factorial (by the theorem above) Cohen-Macauley [4, theorem 15] homomorphic image of the regular ring $R[x]$. As for normality, we have by [4, theorem 17] that S is Cohen-Macauley and regular in co-dimension $2(s-2t)\geqq 4$, hence Serre's criterion applies. All that remains to be proved is the isomorphism of the class groups.

We write $X(i_1,\dots,i_k)$ for the alternating submatrix of X, formed by the elements lying on the intersection of rows and columns with indices $i_1<i_2<\dots<i_k$; its pfaffian is denoted $p(i_1,\dots,i_k)$, with the abbreviation $p=p(1, 2,\dots, 2t)$. For $s_1<s_2<\dots<s_t<s_{t+1}=s$ such that $s_j\geqq 2j-1$, $Pf(X; s_1,\dots, s_t)$ is the ideal, generated by all $p(i_1,\dots,i_k)$ with $i_{2j}\leqq s_j$; in particular, when $s_j=2j-1$ $(1\leqq j\leqq t)$ one gets $I=Pf(X; s_1,\dots, s_t)$. We set $T=R[\{X_{ij}\}_{1\leqq i<j\leqq s;\ i\leqq 2t}]\subset R[X]$. In order to establish our claim we shall prove the existence of the following isomorphisms:

$$\mathrm{Cl}(S) \xrightarrow{\alpha} \mathrm{Cl}(S[p^{-1}]) \xrightarrow{\beta} \mathrm{Cl}(T[p^{-1}]) \xrightarrow{\gamma} \mathrm{Cl}(T) \xleftarrow{\delta} \mathrm{Cl}(R).$$

α) Setting $s_j = 2j-1$ for $1 \leqq j \leqq t$, $s_t = 2t$, we obtain $I+(p) = Pf(X;\, s_1, \dots, s_t)$. It follows from [4, theorem 12] that p generates in S a height one prime ideal, hence it is irreducible. Nagata's theorem [2, (7.3)] implies that α is a natural isomorphism.

β) Let T' denote the image of T in S under the natural map f. If $i > j > 2t$, by expanding $p(1, 2, \dots, 2t, j, i)$ along the last row, one obtains the relation

$$X_{ij}\, p = \Sigma_{k=1}^{2t} (-1)^k X_{ik}\, p\,(1, \dots, \widehat{k}, \dots, 2t),$$

which implies $T'[p^{-1}] = S\,[p^{-1}]$. On the other hand, f is injective, because of the equalities $\dim T = \dim R + \binom{s}{2} - \binom{s-2t}{2} = \dim S$ (the second one holding by 4, theorem 12]). We conclude that $T[p^{-1}] \simeq S[p^{-1}]$. Note that this isomorphism can also be obtained by specializing [4, proposition 10].

γ) The indecomposability of p in the polynomial ring is classical, but can also be deduced as in α) by remarking that $(p) = Pf(X(1, 2, \dots, 2t);\, a_1, \dots, a_{t-1})$ with $a_j = 2j-1$ $(1 \leqq j \leqq t-1)$. Once more apply Nagata's theorem.

δ) This is of course Gauss' lemma [2, (8. 1)].

In conclusion we note that dropping the Noetherian assumption, only minor changes in the proof are needed in order to establish a similar result for Krull domains.

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