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THE BEHAVIOUR OF CARDINAL FUNCTIONS ON INVERSE SPECTRA CONSISTING OF TOPOLOGICAL SPACES

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Let $S=\{X_\alpha, \omega^\beta_\alpha, \alpha\in A\}$ be a well ordered spectrum consisting of topological spaces and Φ be any cardinal function, defined on the class of topological spaces. Let further τ be an infinite cardinal and $\Phi(X_\alpha)<\tau$ for every $\alpha\in A$. For $X=\underleftarrow{\lim}\, S$ there are given conditions under which $\Phi(X)\leqq\tau$. If $\mathrm{nw}(X)>\aleph_0$ but $\aleph_1\notin G_{\mathrm{nw}}(X)$ then one of E. Michael's problems (1971) is answered in the affirmative sense.

The aim of this paper is to study the behaviour of different cardinal functions on the well ordered inverse spectra.

Later on considering the spectrum $\{X_\alpha, \omega^\beta_\alpha, \alpha<\tau\}$ mentioned above, we shall always assume that τ is a regular cardinal. Results will not lose the generality because of the homeomorphism of limits of cofinal spectra.

Speaking on continuous spectrum we shall always assume that all non-limit projections are mappings "onto".

Cardinals are identified with the corresponding ordinals. An ordinal is considered being a set of all the preceding ordinals.

In general our notations coincide with these in [2]. Namely, the weight, net weight, π-weight, Souslin number, Lindelöf number and density of a space X will be denoted by $w(X)$, $\mathrm{nw}(X)$, $\pi w(X)$, $c(X)$, $\mathrm{ic}(X)$ and $s(X)$, respectively.

For any cardinal function Φ we put $\bar{\Phi}(X)=\sup\{\Phi(M): M\subseteq X\}$.

The function, which corresponds to the symbol R, was introduced by the author in [3] and named a weak extensibility.

Definition 0. *A subspace $Y\subseteq X$ is called weakly extended if there are a family γ of open subsets of X and a bijection $\varphi: Y\to\gamma$ such that*

(x) $x\in\varphi(x)$ *for each* $x\in Y$;

(xx) *if $x, y\in Y$ and $x\neq y$, then either $x\notin\varphi(y)$ or $y\notin\varphi(x)$. We put* $R(X)=\sup\{|Y|: Y\subseteq X$ and Y is weakly extended$\}$.

1. Continuity of cardinal functious. Let $\mathcal{O}$ be any class of spaces and Φ be a cardinal function defined on this class. Let $\mathcal{O}$ contain the limits of all inverse spectra consisting of the spaces belonging to $\mathcal{O}$.

Definition 1. *A function Φ is called spectral continuous in the class $\mathcal{O}$ if for every well ordered inverse spectrum $S=\{X_\alpha, \omega^\beta_\alpha, \alpha<\tau\}$ the conditions $X_\alpha\in\mathcal{O}$ and $\Phi(X_\alpha)<\lambda$ for every $\alpha<\tau$ imply that $\Phi(\underleftarrow{\lim}\, S)\leqq\lambda$.*

Let $\mathcal{O}$ be a hereditary class, i. e. the conditions $X\in\mathcal{O}$ and $Y\subset X$ imply that $Y\in\mathcal{O}$.

Definition 2. *A function Φ is called continuous in a class $\mathcal{O}$ if for every $X\in\mathcal{O}$ and for every chain C of subspaces of X such that $X=\cup C$, the condition $\Phi(M)<\lambda$ for every $M\in C$ implies that $\Phi(X)\leqq\lambda$.*

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Theorem 1. *The functions $\bar{c}, \bar{s}, \overline{ic}, R$ are continuous and spectral continuous in the class of all topological spaces.*

Proof. The continuity of the functions $\bar{c}$, $\bar{s}$ and $\overline{ic}$ is proved in [3]. The proof of this fact for the function R is similar to the proof just mentioned and hence it is omitted. The proofs of the spectral continuity of these four functions are also alike. Therefore, we only give the proof for the function $\bar{c}$.

Let $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$, $X=\underleftarrow{\lim} S$. and $\omega_\alpha: X \to X_\alpha$ be a limit projection for every $\alpha<\tau$. We assume now that $\bar{c}(X_\alpha)<\lambda$ for every $\alpha<\tau$. The sets of the form $\omega_\alpha^{-1}(V)$, where $\alpha<\tau$ and V is open in X_α, will always be called a standard open sets in $X=\underleftarrow{\lim} S$. This agreement acts all over this paper.

Let $M\subset X$ and γ be a system of open sets in X such that $U\cap M\neq\Lambda$ and $U\cap M\cap V=\Lambda$ for every $U, V\in\gamma$, $U\neq V$. Without any loss of generality one can consider a system γ consisting of standard open sets in X.

For every $U\in\gamma$ over $\alpha(U)$ we consider the minimal element $\alpha<\tau$ such that $U=\omega_\alpha^{-1}(V)$ for some set V, which is open in X_α.

Put $\gamma_\alpha=\{U\in\gamma: \alpha(U)\leq\alpha\}$. Let us consider the set $\omega_\alpha(M)$ and the system $\{(\omega_{\alpha(U)}^\alpha)^{-1}\varphi(U): U\in\gamma_\alpha\}$ of open sets in X_α. It is easy to verify that

1) $((\omega_{\alpha(U)}^\alpha)^{-1}\varphi(U))\cap\omega_\alpha(M)\neq\Lambda$ for every $U\in\gamma_\alpha$ and

2) $((\omega_{\alpha(U)}^\alpha)^{-1}\varphi(U))\cap\omega_\alpha(M)\cap((\omega_{\alpha(V)}^\alpha)^{-1}\varphi(V))=\Lambda$ for all $U, V\in\gamma_\alpha$, where $U\neq V$ — it follows from the fact that $\omega_\alpha^{-1}((\omega_{\alpha(U)}^\alpha)^{-1}\varphi(U))=U$ for every $U\in\gamma_\alpha$.

As $\bar{c}(X_\alpha)<\lambda$ for every $\alpha<\tau$, one has $|\gamma_\alpha|<\lambda$ for every $\alpha<\tau$. Further, $\gamma=\bigcup\{\gamma_\alpha: \alpha<\tau\}$ and $\gamma_\alpha\subset\gamma_\beta$ for every $\alpha, \beta<\tau$ such that $\alpha<\beta$. Hence the condition $\tau\leq\lambda$ implies $|\gamma|\leq\lambda$.

Let $\lambda<\tau$. We state that $\bar{c}(X)<\lambda$. Indeed if $|\gamma|\geq\lambda$ there exists a subsystem $\mu\subset\gamma$ such that $|\mu|=\lambda$. The regularity of a cardinal τ, an inclusion $\gamma_\alpha\subset\gamma_\beta$ under the condition $\alpha<\beta<\tau$ and the inequality $\lambda<\tau$ imply that there exists an ordinal $\alpha^*<\tau$ such that $\mu\subset\gamma_{\alpha^*}$. Consequently, $\lambda=|\mu|\leq|\gamma_{\alpha^*}|$. But we have already established that $|\gamma_\alpha|<\lambda$ for every $\alpha<\tau$. This contradiction completes the proof of the theorem.

Remark 1. When proving the previous theorem we established the following fact.

If $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ is a well ordered inverse spectrum, $\Phi\in\{\bar{c}, \bar{s}, \overline{ic}, R\}$, $\Phi(X_\alpha)<\lambda$ for every $\alpha<\lambda$, then the condition $\lambda<\tau$ implies $\Phi(\underleftarrow{\lim} S)<\lambda$ (the regularity of τ was assumed above).

Remark 2. The function c is also continuous but it is not a spectral continuous one. This corrects a false statement about the function c in theorem 2.3 of [3].

But the following result holds. Let $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ be a well ordered inverse spectrum such that for every $\alpha<\tau$ a limit projection $\omega_\alpha: X\to X_\alpha$ is a mapping onto X_α, where $X=\underleftarrow{\lim} S$. If $c(X_\alpha)<\lambda$ for every $\alpha<\tau$ then $c(X)\leq\lambda$.

One can prove it by the same method as it was done in the proof of the theorem 1.

Remark 3. It is impossible to prove a spectral continuity of the function w, because in [4] V. I. Malihin, assuming AM, constructed a well ordered inverse spectrum (all projections in which are continuous bijections), consisting of regular spaces of countable weight, such that the π-weight of a limit of this spectrum is equal to $2^{\aleph_0}$.

So, if we assume that both AM and not-CH are satisfied, we get that the functions w and πw are not spectral continuous ones. Moreover, the functions w and πw can "suffer a gap" even on continuous spectrums (under additional assumptions).

The existence of a counterexample follows from the result of Malihin, which was mentioned above.

Indeed, assuming AM he constructed a strongly increasing well ordered chain $C=\{\mathcal{S}_\alpha: \alpha<2^{\aleph_0}\}$ of regular topologies on a countable set N such that $w(N, \mathcal{S}_\alpha)=\aleph_0$ for every $\alpha<2^{\aleph_0}$. One can assume that the following condition is satisfied

(*) $\mathcal{S}_\alpha$ is not a π-base for a space $(N, \mathcal{S}_{\alpha+1})$ for every $\alpha<2^{\aleph_0}$.

Now we assume that AM is satisfied (hence the desired chain $\{\mathcal{S}_\alpha: \alpha<2^{\aleph_0}\}$ exists) and there exists an infinite singular cardinal which is less than $2^{\aleph_0}$. The last assumption is equivalent to the inequality $\aleph_\omega<2^{\aleph_0}$. Then we enumerate the elements of a chain C by means of all nonlimit ordinals which are less then $2^{\aleph_0}$; $C=\{\mathcal{S}_\beta: \beta<2^{\aleph_0}, \beta$ is a nonlimit ordinal$\}$. Of course, we can do it in such a way that $\mathcal{S}_{\beta'}\subset\mathcal{S}_{\beta''}$ for $\beta'<\beta''<2^{\aleph_0}$.

For every limit ordinal $\alpha<2^{\aleph_0}$ we put $\mathcal{S}_\alpha=\cup\{\mathcal{S}_\beta: \beta<2^{\aleph_0}, \beta$ is a nonlimit ordinal$\}$. Now we can define a chain $\widetilde{C}=\{\mathcal{S}_\alpha: \alpha<\aleph_\omega^+\}$ Then the chain $\widetilde{C}$ is continuous and $w(N, \mathcal{S}_\alpha)<\aleph_\omega$ for every $\alpha<\aleph_\omega^+$ because of $\mathrm{cf}(\alpha)<\aleph_\omega$ for every $\alpha<\aleph_\omega^+$. At the same time $\pi w(N, \mathcal{S}_{\aleph_\omega^+})=\aleph_\omega^+$. Let us assume the contrary, i. e. $\pi w(N, \mathcal{S}_{\aleph_\omega^+})\leqq\aleph_\omega$.

Let $\gamma\subset\mathcal{S}_{\aleph_\omega^+}$ be a π-base for a space $(N, \mathcal{S}_{\aleph_\omega^+})$ such that $|\gamma|\leqq\aleph_\omega$. As $\mathcal{S}_{\aleph_\omega^+}=\cup\{\mathcal{S}_\alpha: \alpha<\aleph_\omega^+\}$, so for every $U\in\gamma$ there are an ordinal $\alpha(U)<\aleph_\omega^+$ and a set $V(U)\in\mathcal{S}_{\alpha(U)}$ such that $V(U)\subset U$, $V(U)\neq\Lambda$.

Put $A=\{\alpha(U): U\in\gamma\}$. Then $|A|\leqq|\gamma|\leqq\aleph_\omega$.

The regularity of the cardinal $\aleph_\omega^+$ implies that there exists an ordinal $\alpha<\aleph_\omega^+$ such that $\alpha(U)\leqq\alpha$ for every $U\in\gamma$. Then $V(U)\in\mathcal{S}_\alpha$ for every $U\in\gamma$. Consequently the system $\mathcal{S}_\alpha$ is a π-base in a space $(N, \mathcal{S}_{\aleph_\omega^+})$, which contradicts the property (*) of a chain C.

Now we put $\omega_\alpha^\beta=\mathrm{id}_N$ for every $\alpha, \beta<\aleph_\omega^+$, $\alpha<\beta$, where ω_α^β maps a space $X_\beta=(N, \mathcal{S}_\beta)$ onto a space $X_\alpha=(N, \mathcal{S}_\alpha)$ (then ω_α^β is a continuous trivial mapping).

Then we immediately get the desired continuous well odered inverse spectrum $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\aleph_\omega^+\}$, on which both functions w and πw "suffer a gap".

So, the point (c) of the theorem 2.2 from [3] is not true in its "naive" form. But it can be corrected by means of adding some set-theoretic assumptions.

Theorems 2 and 3 show the way of this correction.

However, we first need

Lemma 1. *Let* $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ *be an inverse well ordered spectrum consisting of Hausdorff spaces and* $R(X_\alpha)\leqq\lambda$ *for every* $\alpha<\tau$. *Then,* $w(X)\leqq\exp(\lambda)$, *where* $X=\underleftarrow{\lim} S$.

Proof. First we note that an inequality $w(Y)\leqq\exp(R(Y))$ takes place for every Hausdorff space Y. It follows from proposition 2 of [2] because $\bar{s}(Y).\bar{\mathrm{ic}}(Y)\leqq R(Y)$ (the last inequality is proved in [3]).

Therefore, $w(X_\alpha)\leqq\exp(\lambda)$ for every $\alpha<\tau$.

Consequently, if $\tau\leqq\lambda^+$, then $w(X)\leqq\exp(\lambda)$. Let $\lambda^+<\tau$. Then the remark 1 (after the theorem 1) implies that $R(X)\leqq\lambda$ and hence $w(X)\leqq\exp(\lambda)$.

Theorem 2 [GCH]. *The functions* w *and* nw *are spectral continuous in the class of all Hausdorff spaces.*

Proof. Let $\Phi\in\{w, \mathrm{nw}\}$ and $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ be an inverse spectrum, consisting of Hausdorff spaces, such that $\Phi(X_\alpha)<\lambda$ for every $\alpha<\tau$. If $\tau\leqq\lambda$ then $\Phi(X)\leqq\lambda$, where $X=\underleftarrow{\lim} S$, which is obvious.

Let $\lambda<\tau$. We consider two cases.

I. $\lambda=\mu^+$ for some cardinal μ. Then $\Phi(X_\alpha)\leqq\mu$ for every $\alpha<\tau$ and lemma 1 implies that $\Phi(X)\leqq w(X)\leqq\exp(\mu)=\mu^+=\lambda$.

II. λ is a limit cardinal. Then there exists a strongly increasing transfinite sequence $C=\{\lambda_\alpha: \alpha<\mathrm{cf}(\lambda)\}$ consisting of cardinals, such that $\lambda_\alpha<\lambda$ for every $\alpha<\mathrm{cf}(\lambda)$ and $\lambda=\sup C$.

For every $\alpha<\mathrm{cf}(\lambda)$ we put $A_\alpha=\{\beta<\tau: \Phi(X_\beta)\leqq\lambda_\alpha\}$. Then $\tau=\cup\{A_\alpha: \alpha<\mathrm{cf}(\lambda)\}$. The regularity of a cardinal τ and the inequality $\mathrm{cf}(\lambda)\leqq\lambda<\tau$ imply that there exists an ordinal $\alpha<\mathrm{cf}(\lambda)$ such that $|A_\alpha|=\tau$, consequently A_α is a cofinal subset in τ. Hence $X=\underleftarrow{\lim}\widetilde{S}$, where $\widetilde{S}=\{X_\beta, \omega_\beta^\gamma, \beta\in A_\alpha\}$. We know that $\Phi(X_\beta)\leqq\lambda_\alpha$ for every $\beta\in A_\alpha$. Applying lemma 1, we get the chain of inequalities $\Phi(X)\leqq w(X)\leqq\exp(\lambda_\alpha)=\lambda_\alpha^+<\lambda$, i. e. $\Phi(X)<\lambda$.

It is necessary to note that we can apply the lemma 1 because $R(X)\leqq\mathrm{nw}(X)\leqq w(X)$ for every space X. The theorem is proved.

Let FH mean that for every cardinal τ there exists only finitely many cardinals λ such that $\tau<\lambda<\exp(\tau)$.

Theorem 3 [FH]. *Let* $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ *be a continuous spectrum consisting of Hausdorff spaces,* $\Phi\in\{w, \pi w\}$ *and* $\Phi(X_\alpha)<\lambda$ *for every* $\alpha<\tau$. *Then* $\Phi(X)\leqq\lambda$, *where* $X=\underleftarrow{\lim} S$

Proof. Let $\Phi\in\{w, \pi w\}$. If $\tau\leqq\lambda$ then it is easy to see that $\Phi(X)\leqq\lambda$. Hence we assume that $\lambda<\tau$.

Now we note the following obvious fact: if there exists an ordinal $\alpha<\tau$ such that for some (and then for any) base (π-base) γ in X_α the system $(\omega_\alpha^\beta)^{-1}\gamma=\{(\omega_\alpha^\beta)^{-1}U: U\in\gamma\}$ is a base (π-base) in X_β for every β, where $\alpha<\beta<\tau$, then $\omega_\alpha^{-1}\gamma=\{\omega_\alpha^{-1}(U): U\in\gamma\}$ is a base (π-base) in X.

So, if there exists an ordinal $\alpha<\tau$, which has the property mentioned above, then $\Phi(X)<\lambda$. Therefore, we assume that there is no such an ordinal $\alpha<\tau$. This last assumption we designate over (*).

For every $\alpha<\tau$ we fix a base (π-base) γ_α in X_α such that $|\gamma_\alpha|=\Phi(X_\alpha)<\lambda$. The assumption (*) implies that for every $\alpha<\tau$ there exists an ordinal $\beta(\alpha)$ such that $\alpha<\beta(\alpha)<\tau$ and $(\omega_\alpha^{\beta(\alpha)})^{-1}\gamma_\alpha$ is not a base (π-base) in a space $X_{\beta(\alpha)}$.

Put $\beta_0=0$. Let $\alpha<\tau$ and for every $\theta<\alpha$ an ordinal β_θ has already been defined. We put $\beta_\alpha=\sup\{\beta_\theta: \theta<\alpha\}$ if α is a limit ordinal and $\beta_\alpha=\beta(\beta_{\alpha'})$ if $\alpha=\alpha'+1$.

Let the set $A=\{\beta_\alpha:\ \alpha<\tau\}$ be already defined.

Then, firstly, a set A is closed and cofinal in τ with order topology and, secondly, if $\alpha, \beta\in A$ and $\alpha<\beta$ then $(\omega^\beta_\alpha)^{-1}\gamma_\alpha$ is not a base (π-base) in a space X_β.

We consider two cases.

I. λ is a regular cardinal. Put $B=A\cap\beta_\lambda$ and $\widetilde{S}=\{X_\alpha, \omega^\beta_\alpha, \alpha\in B\}$. As a spectrum S is continuous and $\beta_\lambda=\sup\{\beta_\alpha:\ \alpha<\lambda\}$, so $X_{\beta_\lambda}=\underleftarrow{\lim}\,\widetilde{S}$. Without any loss of generality one can consider a sysstem γ_{β_λ} consisting of standard open sets in X_{β_λ}, i. e. $\gamma_{\beta_\lambda}=\{(\omega^{\beta_\lambda}_{\alpha(U)})^{-1}\varphi(U):\ U\in\gamma_{\beta_\lambda}\}$, where $\alpha(U)\in B$ and $\varphi(U)$ is open in a space $X_{\alpha(U)}$ for every $U\in\gamma_{\beta_\lambda}$. We know that $|\gamma_{\beta_\lambda}|<\lambda$ and λ is a regular cardinal, hence there exists an ordinal $\alpha^*\in B$ such that $\alpha(U)\leqq\alpha^*$ for every $U\in\gamma_{\beta_\lambda}$. But then the system $(\omega^{\beta_\lambda}_{\alpha^*})^{-1}\gamma_{\alpha^*}$ is a base (π-base) in a space X_{β_λ}, which contradicts the assumption (*).

II. λ is a singular cardinal. Then there exists a strongly increasing transfinite sequence $C=\{\lambda_\alpha:\ \alpha<\mathrm{cf}(\lambda)\}$, consisting of regular cardinals, such that $\lambda_\alpha<\lambda$ for every $\alpha<\mathrm{cf}(\lambda)$ and $\lambda=\sup C$.

Put $A_\alpha=\{\beta\in A:\ \Phi(X_\beta)\leqq\lambda_\alpha\}$. Then $A=\cup\{A_\alpha:\ \alpha<\mathrm{cf}(\lambda)\}$. The inequality $\lambda<\tau$ implies that there exists an ordinal $\varkappa<\mathrm{cf}(\lambda)$ such that $|A_\varkappa|=\tau$. Therefore, $A_\varkappa$ is a cofinal subset in τ. The cofinality $A_\varkappa$ in τ implies that $X=\underleftarrow{\lim}\,\widetilde{S}$, where $\widetilde{S}=\{X_\alpha, \omega^\beta_\alpha, \alpha\in A_\varkappa\}$. Then for every $\alpha\in A_\varkappa$, $\Phi(X_\alpha)\leqq\lambda_\varkappa$ — it follows from the definition of a set $A_\varkappa$.

Let $\Phi=w$. Applying lemma 1, we get the inequality $w(X)\leqq\exp(\lambda_\varkappa)$. However, **FH** implies that there exist only finitely many cardinals which lie between $\lambda_\varkappa$ and $\exp(\lambda_\varkappa)$. Further, $\lambda_\varkappa<\lambda$ and λ is a singular cardinal, hence $\exp(\lambda_\varkappa)<\lambda$. So, $w(X)<\lambda$ in the case of a singular cardinal λ.

If $\Phi=\pi w$ then lemma 1 implies that $w(X)\leqq\exp^4(\lambda_\varkappa)$ because a space X is a Hausdorff one and $w(X_\alpha)\leqq\exp^3(s(X_\alpha))\leqq\exp^3(\pi w(X_\alpha))$ for every $\alpha<\tau$. Applying FH as above we get the chain of inequalities $\pi w(X)\leqq w(X)\leqq\exp^4(\lambda_\varkappa)<\lambda$. Thus the theorem is proved.

Definiton 3. *A system γ of subsets of a set X is called T_2-separable for X if for $x, y\in X$, where $x\neq y$, there exist $U, V\in\gamma$* such that $x\in U, y\in V$ *and $U\cap V=\Lambda$.*

Further we need

Proposition 1. *Let X be a Hausdorff space and γ be T_2-separable system for X, consisting of open sets in X. Then there exists a subsystem $\mu\subset\gamma$ such that μ is T_2-separabte for X and $|\mu|\leqq\mathrm{ic}(X^2\setminus\Delta)$, where $\Delta=\{(x, x):\ x\in X\}\subset X^2$.*

The proof of this proposition is omitted because of its simplicity.

Definition 4. *A mapping $f: X\to Y$ is called nontrivial if there exists a point $y\in Y$ such that $|f^{-1}(y)|>1$.*

Lemma 2. *Let τ be a regular cardinal, $S=\{X_\alpha, \omega^\beta_\alpha, \alpha<\tau\}$ be a well odered inverse spectrum, consisting of Hausdorff spaces, in which all limit projections $\omega_\alpha: X\to X_\alpha$ are mappings onto X_α, where $X=\underleftarrow{\lim}\,S$. If $\mathrm{nw}(X)<\tau$, then there exists an ordinal $\alpha^*<\tau$ such that a mapping ω_{α^*} is a trivial one.*

Proof. The system $\gamma=\{\omega^{-1}_\alpha(O):\ \alpha<\tau,\ O$ is an open set in $X_\alpha\}$ of open sets in X is T_2-separable for X.

Proposition 1 implies that there exists a subsystem $\mu \subset \gamma$ such that μ is T_2-separable for X and $|\mu| \leq \mathrm{ic}(X^2 \setminus \Delta)$. However, $\mathrm{ic}(X^2 \setminus \Delta) \leq \overline{\mathrm{ic}}(X^2) \leq \mathrm{nw}(X) < \tau$, i. e. $|\mu| < \tau$. This last inequality and a regularity of τ imply that there exists an ordinal $\alpha^* < \tau$ such that $\mu \subset \{\omega_{\alpha^*}^{-1}(O) : O$ is an open set in a space $X_{\alpha^*}\}$. Consequently, w_{α^*} is a trivial mapping.

Lemma 3. *Let $S = \{X_\alpha, \omega_\alpha^\beta, \alpha < \tau\}$ be a well ordered inverse spectrum consisting of Hausdorff spaces and for all $\alpha, \beta < \tau$, where $\alpha < \beta$, a mapping ω_α^β is nontrivial and onto one. Then for every cardinal $\lambda < \tau$ there exists an ordinal $\alpha < \tau$ such that $\lambda \leq \mathrm{nw}(X_\alpha)$.*

Proof. Let $\lambda < \tau$ and $\mathrm{nw}(X_\alpha) < \lambda$ for every $\alpha < \tau$.

We consider two cases.

a) $\lambda = \mu^+$ for some cardinal μ;

b) λ is a limit cardinal.

For every set $A \subset \tau$ and every ordinal $\alpha < \tau$ we put $A^\alpha = \{\beta \in A : \beta < \alpha\}$. In the case (a) we put $B = \lambda$ and $\widetilde{S} = \{X_\alpha, \omega_\alpha^\beta, \alpha < \lambda\}$. Now we consider the case (b).

Let $C = \{\lambda_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ be a well ordered strongly increasing transfinite sequence consisting of cardinals, which are less then λ, such that $\lambda = \sup C$. For every $\alpha < \mathrm{cf}(\lambda)$ we put $A_\alpha = \{\beta < \tau : \mathrm{nw}(X_\beta) \leq \lambda_\alpha\}$.

Then $\tau = \cup \{A_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ because $\mathrm{nw}(X_\beta) < \lambda$ for every $\beta < \tau$. Further, so as $\mathrm{cf}(\lambda) \leq \lambda < \tau$, there is $\alpha^* < \mathrm{cf}(\lambda)$ such that $|A_{\alpha^*}| = \tau$. We put now $\varrho^* = \min\{\varrho \in A_{\alpha^*} : |(A_{\alpha^*})^\varrho| > \lambda_{\alpha^*}\}$, $B = (A_{\alpha^*})^{\varrho^*}$ and $\widetilde{S} = \{X_\alpha, \omega_\alpha^\beta, \alpha \in B\}$.

So, a subspectrum $\widetilde{S}$ of a spectrum S is defined in both cases. We put $Y = \varprojlim \widetilde{S}$. For every $\alpha < \tau$ such that $\alpha^* \leq \alpha$, in a natural way a continuous mapping $f_\alpha : X_\alpha \to Y$ arises, namely $f_\alpha = \varprojlim \{\omega_\beta^\alpha : \beta \in B\}$ or, which is the same, $f_\alpha = \Delta\{\omega_\beta^\alpha : \beta \in B\}$ is a diagonal product of the family of mappings $\{\omega_\beta^\alpha : \beta \in B\}$.

Let us define a cardinal as

$$\lambda_* = \begin{cases} \mu, & \text{in the case (a);} \\ \lambda_{\alpha^*}, & \text{in the case } (b). \end{cases}$$

The central point of our proof is the equality $f_\alpha(X_\alpha) = Y$ for every $\alpha < \tau$ such that $\alpha^* < \alpha$ and $\mathrm{nw}(X_\alpha) \leq \lambda_*$.

It is obvious that $f_\alpha(X_\alpha) \subset Y$. We will show that f_α maps X_α onto Y. Let $y \in Y$. For every $\beta \in B$ let π_β be a limit projection from Y into X_β. It is easy to see that the family $\{(\omega_\beta^\alpha)^{-1}(\pi_\beta(y)) : \beta \in B\}$ is a decreasing sequence of nonempty closed subsets in X_α of the length λ_*^+ (the fact that these sets are nonempty follows from the equality $\omega_\beta^\alpha(X_\alpha) = X_\beta$ for all α, β, where $\alpha < \beta$). However, $\mathrm{nw}(X_\alpha) \leq \lambda_*$, therefore, there exists an ordinal $\gamma \in B$, such that $\Lambda \neq (\omega_\gamma^\alpha)^{-1}(\pi_\gamma(y)) = (\omega_\beta^\alpha)^{-1}(\pi_\beta(y))$ for every $\beta \in B$, where $\gamma \leq \beta$.

Hence for every point $x \in (\omega_\gamma^\alpha)^{-1}(\pi_\gamma(y))$ we get $\omega_\beta^\alpha(x) = \pi_\beta(y)$, so $f_\alpha(x) = y$. In such a way $f_\alpha(X_\alpha) = Y$. However, f_α is a continuous mapping and $\mathrm{nw}(X_\alpha) \leq \lambda_*$, therefore, $\mathrm{nw}(Y) \leq \lambda_*$. We must note that $\pi_\beta(Y) = X_\beta$ for every $\beta \in B$ because $\omega_\beta^\alpha = \pi_\beta \circ f_\alpha$, i. e. $X_\beta = \omega_\beta^\alpha(X_\alpha) = \pi_\beta(f_\alpha(X_\alpha)) = \pi_\beta(Y)$.

Applying lemma 3 to the spectrum $\widetilde{S}$ we get $\mathrm{nw}(Y) \geq \lambda_*^+$ (if $\mathrm{nw}(Y) \leq \lambda_*$ then the spectrum $\widetilde{S}$ would have a trivial projections). This contradiction completes our proof.

The next theorem is the reformulation of lemma 3.

Theorem 4. *Let* $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ *be a well ordered inverse spectrum consisting of Hausdorff and for all* $\alpha, \beta<\tau, \alpha<\beta$, *a mapping* ω_α^β *be nontrivial and onto one. Let* $\Phi\in\{w, \mathrm{nw}\}$ *and* $\Phi(X_\alpha)<\lambda$ *for every* $\alpha<\tau$. *Then* $\Phi(\varprojlim S)\leq\lambda$.

Corollary 1. *Let* $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ *be an inverse spectrum consisting of Hausdorff spaces and for all* $\alpha, \beta<\tau$, *where* $\alpha<\beta$, ω_α^β *be quotient mapping onto* X_α. *Let* $\Phi\in\{w, \mathrm{nw}\}$ *and* $\Phi(X_\alpha)<\lambda$ *for every* $\alpha<\tau$. *Then* $\Phi(X)\leq\lambda$, *where* $X=\varprojlim S$.

Proof. It is obvious that a quotient one-to-one mapping is a homeomorhism. Therefore, either there exists an ordinal $\alpha<\tau$ such that a mapping ω_α^β is a homeomorphism for every $\beta>\alpha$ (then X is homeomorphic to X_α and consequently $\Phi(X)=\Phi(X_\alpha)<\lambda$) or there exists a cofinal subspectrum $\widetilde{S}=\{X_\alpha, \omega_\alpha^\beta, \alpha\in B\}$ of a spectrum S, where $B\subset\tau$, such that ω_α^β is a nontrivial mapping for all $\alpha, \beta\in B, \alpha<\beta$. Then theorem 4 implies an inequality $\Phi(X)\leq\lambda$.

Corollary 2. *Let* $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ *be an inverse spectrum and all projections* ω_α^β *are either* a) *open or* b) *closed. Further, let* $\Phi\in\{w, \mathrm{nw}\}$ *and* $\Phi(X_\alpha)<\lambda$ *for every* $\alpha<\tau$.

Then $\Phi(X)\leq\lambda$, *where* $X=\varprojlim S$.

Proof. The case $\tau\leq\lambda$ is a trivial one. Let $\lambda<\tau$. a) Let the projections ω_α^β be open. For every $\alpha<\tau$ we put $X_\alpha^*=\cap\{\omega_\alpha^\beta(X_\beta):\alpha<\beta<\tau\}$. However, for every $\alpha<\tau$ a decreasing sequence $\{\omega_\alpha^\beta(X_\beta):\alpha<\beta<\tau\}$ of open in X_α sets is stabilized because $\bar{s}(X_\alpha)\leq\mathrm{nw}(X_\alpha)<\tau$ and τ is a regular cardinal. Hence X_α^* is an open set in X_α.

We state that $\omega_\alpha^\beta(X_\beta^*)=X_\alpha^*$ for all $\alpha, \beta<\tau$, $\alpha<\beta$. Really, the inclusion $\omega_\alpha^\beta(X_\beta^*)\subset X_\alpha^*$ follows from the fact that $f(\cap\gamma)\subset\cap\{f(M):M\in\gamma\}$ for every mapping $f:A\to B$ and every system γ of subsets of A. The inverse inclusion follows from the fact that there exists an ordinal $\gamma>\beta$ such that $\omega_\beta^\gamma(X_\gamma)=X_\beta^*$ and, therefore, $\omega_\alpha^\beta(X_\beta^*)=\omega_\alpha^\beta\omega_\beta^\gamma(X_\gamma)=\omega_\alpha^\gamma(X_\gamma)\supset X_\alpha^*$. The equality $\omega_\alpha^\beta(X_\beta^*)=X_\alpha^*$ is proved. Put $\widetilde{S}=\{X_\alpha^*, \omega_\alpha^\beta|X_\beta^*, \alpha<\tau\}$. Then $\widetilde{S}$ is a spectrum with open projections, which are mappings "onto" and $\varprojlim\widetilde{S}=\varprojlim S$. The proof is completed by applying corollary 1.

The case of closed projections is completely analogous to the case of open projections.

Proposition 2. *Let* $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ *be a well ordered inverse spectrum and* $\omega_\alpha^\beta(X_\beta)=X_\alpha$ *for all* $\alpha, \beta<\tau$ *such that* $\alpha<\beta$. *Let* τ *be a regular cardinal and* $\bigtriangledown\mathrm{ic}(X)\leq\tau$, *where* $X=\varprojlim S$.

a) *If the mappings* ω_α^β *are quotient, hereditarily quotient, open or closed, then the limit projections* $\omega_\alpha:X\to X_\alpha$ *are the same*;

b) *if* $f:X\to Y$ *is a continuous mapping of* X *onto a space* Y *and* $w(Y)<\tau$, *then there exist an ordinal* $\alpha<\tau$ *and a mapping* $g:X_\alpha\xrightarrow{\text{onto}}Y$ (*not necessarily continuous*) *such that* $f=g\circ\omega_\alpha$. *If* ω_α *is a quotient mapping then, of course,* g *is also continuous.*

L e m m a 4 [GCH]. *Let X be a Hausdorff space and γ be a chain of subspaces of X such that $X=\cup\gamma$ and* $\mathrm{nw}(M)<\tau$ *for every* $M\in\gamma$. *Then* $\mathrm{nw}(X)\leq\tau$,

i. e. the function nw *is continuous in the class of all Hausdorff spaces.*

P r o o f. Primarily we separate the canonical (see [2]) chain μ of a chain γ, $\mu\subset\gamma$. The case $|\mu|\leq\tau$ is trivial. Let $\tau<|\mu|$. The point (b) of the theorem 3 from [2] implies that $\mathrm{ic}(X)<\tau$, thetefore $\mathrm{nw}(X)\leq|X|\leq\exp(\overline{\mathrm{ic}}(X))\leq\tau$. Thus, the Lemma is proved.

Before formulating the further results we deduce some consequences from the facts which were already proved.

I. Comparing remark 3 with theorem 2 we conclude that the assertion about a spectral continuity of the function w in the class of all Hausdorff spaces does not depend on the usual axioms of set theory.

II. Comparing remark 3 with theorem 3 we conclude that the assertion about a spectral continuity of the functions w and πw on continuous spectra, which consist of Hausdorff (and also even of completely regular) spaces does not depend on the system of axioms ZFC.

III. Assuming GCH in the theorem 2 we proved the spectral continuity of the function nw in the class of all Hausdorff spaces. Under the same assumption in lemma 4 we proved the continuity of this function in the same class.

Moreover, the following implication takes place: if the assertion about a spectral continuity of net weight does not depend on ZFC then the assertion about a continuity of net weight does not depend on ZFC also (both continuities are in the class of all completely regular spaces).

Indeed, we will prove the following.

A s s e r t i o n 1. *If the function* nw *is not spectral continuous in the class of all completely regular spaces then this function is not continuous in the same class.*

P r o o f. Theorem 4 implies that if the function nw "suffers a gap" on some spectrum $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$, consisting of completely regular spaces (that is $\mathrm{nw}(X_\alpha)<\lambda$ for every $\alpha<\tau$, but $\mathrm{nw}(X)>\lambda$, where $X=\underleftarrow{\lim} S$ for some cardinal λ), then there exists an ordinal $\alpha^*<\tau$ such that a projection ω_α^β is a trivial mapping for all α, β, such that $\alpha^*<\alpha<\beta<\tau$.

Let S be namely such a spectrum. Without any loss of generality one can consider all the projections of this spectrum to be trivial mappings. As above we suppose that τ is a regular cardinal. It is obvious that $\tau>\lambda$ (if $\tau\leq\lambda$ then the condition $\mathrm{nw}(X_\alpha)<\lambda$ for every $\alpha<\tau$ implies that $\mathrm{nw}(X)\leq\lambda$, which contradicts the condition $\mathrm{nw}(X)>\lambda$). The existence of such a spectrum is exactly equivalent to the existence of a completely regular space $(X, \mathfrak{S})$ and a chain $\{\mathfrak{S}_\alpha:\alpha<\tau\}$ of strongly increasing completely regular topologies such that $\mathfrak{S}=\bigcup\{\mathfrak{S}_\alpha:\alpha<\tau\}$, $\mathrm{nw}(X, \mathfrak{S}_\alpha)<\lambda$ for every $\alpha<\tau$ and $\mathrm{nw}(X, \mathfrak{S})>\lambda$.

Now we construct a completely regular space Y and a strongly increasing chain γ, consisting of subspaces of a space Y such that $\gamma=\{Y_\alpha:\alpha<\tau\}$, $\mathrm{nw}(Y_\alpha)<\lambda$ for every $\alpha<\tau$ and $\mathrm{nw}(Y)>\lambda$, where $Y=\cup\gamma$.

To do it we introduce the following agreement.

If (Z, ϱ) is a topological space then $C_p((Z, \varrho), \mathbb{R})$ is a space of all continuous functions of X to $\mathbb{R}$ with topology of point-wise convergence.

Put $Y_\alpha = C_p((X, \mathfrak{S}_\alpha), \mathsf{R})$ for every $\alpha<\tau$ and $Y=C_p((X, \mathfrak{S}), \mathsf{R})$. Then $\mathrm{nw}(Y_\alpha)=\mathrm{nw}(X, \mathfrak{S}_\alpha)<\lambda$ for every $\alpha<\tau$ and $\mathrm{nw}(Y)=\mathrm{nw}(X, \mathfrak{S})>\lambda$ (see [1, § 1, theorem 3]).

All spaces Y_α and Y are completely regular ones. It is necessary to show only that $Y=\cup\{Y_\alpha : \alpha<\tau\}$ (all spaces Y_α and Y are considered to be subspaces of R^X).

Firstly we mention that $\overline{\mathrm{ic}}(X, \mathfrak{S})\leq\lambda$; this inequality follows from theorem 1 because $\overline{\mathrm{ic}}(X, \mathfrak{S}_\alpha)\leq\mathrm{nw}(X, \mathfrak{S}_\alpha)<\lambda$ for every $\alpha<\tau$. Let V be an open set in $(X, \mathfrak{S})$. Then for every point $x\in V$ there exists an ordinal $\alpha(x)<\tau$ and an open set U_x in a space $(X, \mathfrak{S}_{\alpha(x)})$ such that $x\in U_x\subset V$. We can choose a subfamily $\{U_x : x\in M\}$ of the family $\{U_x : x\in V\}$ of open in $(X, \mathfrak{S})$ sets such that $V = \cup\{U_x : x\in M\}$ and $|M|\leq\overline{\mathrm{ic}}(X, \mathfrak{S})$, where $M\subset V$.

Let B be a set $\{\alpha(x) : x\in M\}$. Then $|B|\leq|M|\leq\lambda$, therefore there exists an ordinal $\alpha(V)<\tau$ such that $\alpha(x)<\alpha(V)$ for every $x\in M$. In fact, $|B|\leq\lambda$, $B\subset\tau$, $\lambda<\tau$ and τ is a regular cardinal. So, a set U_x is open in a space $(X, \mathfrak{S}_{\alpha(V)})$ for every $x\in M$, therefore V is open in a space $(X, \mathfrak{S}_{\alpha(V)})$ also.

In such a way we proved that for every open set V in a space $(X, \mathfrak{S})$ there exists an ordinal $\alpha(V)<\tau$ such that a set V is open in a space $(X, \mathfrak{S}_{\alpha(V)})$.

Let us fix a countable base $\mathcal{B}$ for the usual topology on R. Let $f\in Y= C_p((X, \mathfrak{S}), \mathsf{R})$. Put $\theta=\{f^{-1}(U) : U\in\mathcal{B}\}$. The set $\{\alpha(V) : V\in\theta\}$ is countable, hence there exists an ordinal $\alpha<\tau$ such that $\alpha(V)<\alpha$ for every $V\in\theta$.

Therefore every element of a system θ is open in a space $(X, \mathfrak{S}_\alpha)$, so $f\in Y_\alpha = C_p((X, \mathfrak{S}_\alpha), \mathsf{R})$. Thus the proof is complete.

2. A note on Michael's problem. Let Φ be a cardinal function. We write $\Phi^*(X)\leq\tau$ if $\Phi(X^n)\leq\tau$ for every $n\in\omega$. Further, $\bigtriangledown c(X)\leq\tau$ means that any disjoint system of open sets in X has cardinality less than τ. Analogously, $\bigtriangledown\mathrm{ic}(X)\leq\tau$ means that for every open cover γ of X there exists a subcover $\mu\subset\gamma$ such that $|\mu|<\tau$.

These agreements allow us to define functions $\bigtriangledown c^*$ and $\bigtriangledown\mathrm{ic}^*$. For example, inequality $\bigtriangledown\mathrm{ic}^*(X)\leq\tau$ means that $\bigtriangledown\mathrm{ic}(X^n)\leq\tau$ for every $n\in\omega$.

L e m m a 5. *Let X be T_1-space and $\Phi\in\{c, \overline{\mathrm{ic}}, s, R, \mathrm{nw}, \psi\}$. Then the family $G_\Phi(X)$ of cardinals τ such that there exists a set $M_\tau\subset X$ with the property $|M_\tau|=\Phi(M_\tau)=\tau$, is closed in the order topology of the class Card of all cardinals.*

P r o o f. The conclusion of this lemma is the statement that for every set $\mathcal{L}\subset G_\Phi(X)$ a cardinal $\sup\mathcal{L}$ belongs to $G_\Phi(X)$. Really, let $\{\tau_\alpha : \alpha\in C\}$ be a strongly increasing transfinite sequence of cardinals, belonging to $\mathcal{L}$, which is cofinal in $\mathcal{L}$. Then $|C|\leq|\mathcal{L}|$.

As $\tau_\alpha\in G_\Phi(X)$ for every $\alpha\in C$, so there exists a set $M_\alpha\subset X$ such that $|M_\alpha|=\Phi(M_\alpha)=\tau_\alpha$. Put $M=\cup\{M_\alpha : \alpha\in C\}$. Then $|M|=\Phi(M)=\sup\mathcal{L}$ and hence $\sup\mathcal{L}\in G_\Phi(X)$.

Indeed it is necessary to verify only the equality $\Phi(M)=\sup\mathcal{L}$. Firstly, all functions in the list of this lemma are monotone, hence $\Phi(M)\geq\tau_\alpha$ for every $\alpha\in C$, that is $\Phi(M)\geq\sup\mathcal{L}$. Secondly, the inequality $\Phi(M)\leq\sup\mathcal{L}$ follows from the fact that if a space Y is represented in the form of a union of a chain γ of its subspaces and Φ is a function from the list of this lemma, then $\Phi(Y)\leq|\gamma|\cdot\sup\{\Phi(M) : M\in\gamma\}$. Thus, the lemma is proved.

Let $\mathcal{O}$ be a hereditary class of spaces.

A s s e r t i o n 2. *Let* $\Phi\in\{\mathrm{nw},\psi\}$ *and* $G_\Phi(X)=\{\tau\in\mathrm{Card}:\tau\leq\Phi(X)\}$ *for every* $X\in\mathcal{O}$. *Then the functian* Φ *is continuous in a class* $\mathcal{O}$.

P r o o f. Let $X\in\mathcal{O}$ and γ be a canonical chain of subsets of X. We assume that there exists a cardinal λ such that $\Phi(M)<\lambda$ for every $M\in\gamma$, but $\Phi(X)>\lambda$. Then it is easy to see that $|\gamma|>\lambda$. Let $M_\lambda\subset X$ and $|M_\lambda|=\Phi(M_\lambda)=\lambda$. The fact that λ is a canonical chain of a length greater than λ implies that there exists a set $M\in\gamma$ with the property $M_\lambda\subset M$. A monotony of a function Φ implies the inequality $\Phi(M_\lambda)\leq\Phi(M)<\lambda$. This contradiction completes the proof.

R e m a r k 4. Lemma 4 states that the function nw is continuous in the class of all Hausdorff spaces (assuming GCH). As a matter of fact when proving lemma 4 we show something more. Namely, $G_{\mathrm{nw}}(X)=\{\tau\in\mathrm{Card}:\tau\leq\mathrm{nw}(X)\}$ for every Hausdorff space X.

We assume now that the function nw is not continuous in the class of all regular spaces. Then assertion 2 implies that there exists a regular space X with the property $\mathcal{F}(X)=\{\tau\in\mathrm{Card}:\tau\leq\mathrm{nw}(X)\}\setminus G_{\mathrm{nw}}(X)\neq\Lambda$. We put $\tau^*=\min\mathcal{F}(X)$.

Lemma 5 implies that $G_{\mathrm{nw}}(X)$ is closed in the class Card, hence there exists a cardinal λ such that $\tau^*=\lambda^+$. Further, $|M|=\mathrm{nw}(M)=R(M)$ for every weakly extended space M, therefore $R(X)\leq\lambda$ (because any subspace of a weakly extended space is weakly extended itself, so $[\aleph_0, R(X)]=\{\tau\in\mathrm{Card}:\aleph_0\leq\tau\leq R(X)\}\subset G_{\mathrm{nw}}(X)$.

Moreover, we will show that $R(X^\lambda)\leq\lambda$ here.

Let us assume that $R(X^\lambda)>\lambda$. Then, there exists a weakly extended subspace $M\subset X^\lambda$ such that $|M|=\lambda^+=\tau^*$. It is obvious that $M\subset\Pi\{\pi_\alpha(M):\alpha<\lambda\}$ and $|\pi_\alpha(M)|\leq|M|=\tau^*$ for every $\alpha<\lambda$, where $X^\lambda=\Pi\{X_\alpha:\alpha<\lambda\}$, X_α is homeomorphic to X for every $\alpha<\lambda$ and $\pi_\alpha:X^\lambda\to X_\alpha$ is a natural projection. Therefore, $\mathrm{nw}(\pi_\alpha(M))\leq\lambda$ for every $\alpha<\lambda$. Then, $\mathrm{nw}(\Pi\{\pi_\alpha(M):\alpha<\lambda\})\leq\lambda$, hence $\mathrm{nw}(M)\leq\lambda$ and $R(M)\leq\lambda$, which is a contradiction.

So, $\overline{\mathrm{ic}}(X^\lambda)\cdot s(X^\lambda)\leq R(X^\lambda)\leq\lambda$.

Particularly, if there exists a regular space X such that $\mathrm{nw}(X)\geq\aleph_1$, but $\mathrm{nw}(M)\leq\aleph_0$ for every subspace M of X with the property $|M|\leq\aleph_1$ (or, which is the same, $\aleph_1\notin G_{\mathrm{nw}}(X)$), then $s(X^{\aleph_0})\cdot\overline{\mathrm{ic}}(X^{\aleph_0})\leq\aleph_0$.

This reasoning leads to a interrelation between the concept of a "gapness" of the function nw and the question about the existence of a regular space X such that $\overline{\mathrm{ic}}(X^{\aleph_0})\leq\aleph_0$ and $\mathrm{nw}(X)>\aleph_0$.

The question was put and partially answered by E. M i c h a e l in [5]. We used here the class of all regular spaces because there are examples of Hausdorff spaces without a countable net, a countable power of which is hereditarily Lindelof.

Now we quote some simple proposition about the function G_{nw}, where $G_{\mathrm{nw}}(X)\overset{\mathrm{def}}{=}\{\tau\in\mathrm{Card}:$ there exists a set $M\subset X$ such that $|M|=\mathrm{nw}(M)=\tau\}$ for every space X.

P r o p o s i t i o n 3. *Let* $f:X\xrightarrow{\mathrm{onto}}Y$ *be a continuous mapping. Then* $G_{\mathrm{nw}}(Y)\subset G_{\mathrm{nw}}(X)$.

P r o p o s i t i o n 4. a) *Let space* X *be a union of a family* γ *of its subsets and* $|\gamma|\leq\aleph_0$. *If* $G_{\mathrm{nw}}(M)=\{\tau\in\mathrm{Card}:\tau\leq\mathrm{nw}(M)\}$ *for every* $M\in\gamma$, *then* $G_{\mathrm{nw}}(X)=\{\tau\in\mathrm{Card}:\tau\leq\mathrm{nw}(X)\}$.

b) *If* $G_{nw}(X)=\{\tau\in\text{Card}: \tau\leqq nw(X)\}$ *then* $G_{nw}(M)=\{\tau\in\text{Card}: \tau\leqq nw(M)\}$ *for every subspace* $M\subset X$ *with the property* $|X\setminus M|\leqq\aleph_0$.

Proposition 5. *If a space X is*

a) *metrizable*;

b) *locally compact*,

then the equality $G_{nw}(X)=\{\tau\in\text{Card}: \tau\leqq nw(X)\}$ *takes place.*

The point (b) of the previous proposition follows from the theorem 1.5 of [3] and the point (b) of proposition 4.

It is necessary to mention that $G_\Phi(X)=\{\tau\in\text{Card}: \tau\leqq\Phi(X)\}$ for every space X and for every function $\Phi\in\{\bar{c}, s, \overline{ic}, R\}$.

3. Finite powers of limit spaces. Let F be a functor defined in the category Top of all topological spaces and continuous mappings of them. Let $S=\{X_\alpha, \omega^\beta_\alpha, \alpha\in A\}$ be an inverse spectrum consisting of topological spaces and continuous mappings of them, where A is a directed set of indexes.

We put $F(S)=\{F(X_\alpha), F(\omega^\beta_\alpha), \alpha\in A\}$.

Definition 5. *A functor F is called spectral if* $F(\varprojlim S)=\varprojlim F(S)$ *for every spectrum S, which was mentioned above.*

For the further we need the following simple lemma.

Lemma 6. *The functor* F_n *of a raising to n-th power is spectral for every* $n\in\omega$.

Theorem 5. *Let* λ *be a regular cardinal and* $S=\{X_\alpha, \omega^\beta_\alpha, \alpha<\tau\}$ *be a continuous inverse spectrum with open projections such that* $\nabla c^*(X_\alpha)\leqq\lambda$ *for every* $\alpha<\tau$.

Then $\nabla c^*(X)\leqq\lambda$ *where* $X=\varprojlim S$. *Moreover,* $\nabla c(X^\mu)\leqq\lambda$ *for every cardinal* $\mu\geqq 1$.

Proof. Let $n\in\omega$. We will consider the spectrum $F_n(S)$. Evidently, this spectrum has the following properties:

1) $F_n(S)$ is a continuous inverse spectrum with open projections;

2) $\nabla c(X^n_\alpha)\leqq\lambda$ for every $\alpha<\tau$.

Lemma 6 implies that $X^n=\varprojlim F_n(S)$. Using properties (1) and (2) we will show that $\nabla c(X^n)\leqq\lambda$.

Let γ be a disjoint system of open sets in X^n. We must show that $|\gamma|<\lambda$. Let us prove it.

E. V. Ščepin showed (Lemma 2 from [6]) that open sets of finite type form a base for a limit of any continuous spectrum with open projections.

Hence, without any loss of generality one can assume that a system γ consists of the sets of finite type (with respect to a spectrum $F_n(S)$). For every $V\in\gamma$ we define a set $\mathcal{C}(V)$ as a set of all ordinals $\alpha<\tau$ such that $(\omega^{\alpha+1}_\alpha)^{-1}\omega_\alpha(V)\neq\omega_{\alpha+1}(V)$, where $\omega_\alpha: X^n\to X^n_\alpha$ is a limit projection.

Then $|\mathcal{C}(V)|<\aleph_0$ for every $V\in\gamma$. Put $\mathcal{L}=\{\mathcal{C}(V): V\in\gamma\}$.

Now we assume that $|\gamma|\geqq\lambda$. The theorem A2.2 from [7] implies that there exists a subsystem $\gamma_1\subset\gamma$ such that a subfamily $\mathcal{L}_1=\{\mathcal{C}(V): V\in\gamma_1\}$ is quasidisjoint and $|\gamma_1|=\lambda$. A quasidisjointness of a family $\mathcal{L}_1$ means that there exists a finite set $I\subset\tau$ with the property $M\cap N=I$ for all $M, N\in\mathcal{L}_1$, $M\neq N$. Let $\alpha>\max I$. We state the $\omega_\alpha(V_1)\cap\omega_\alpha(V_2)=\Lambda$ for all $V_1, V_2\in\gamma_1$, $V_1\neq V_2$.

Assuming the contrary we find a point $x_\alpha\in X^n_\alpha$, such that $x_\alpha\in\omega_\alpha(V_1)\cap\omega_\alpha(V_2)$. Put $B_1=\mathcal{C}(V_1)\setminus I$, $B_2=\mathcal{C}(V_2)\setminus I$ and $B=(B_1\cup B_2)\setminus\alpha$. If $B=\Lambda$, then $\Lambda\neq\omega^{-1}_\alpha(x_\alpha)\subset V_1\cap V_2$ that is a contradiction.

Let $B=\{\alpha(0), \ldots, \alpha(m)\}$ be a numeration of B such that $\alpha(i)<\alpha(j)$ for $i<j\leq m$, $m\in\omega$. Let φ be a mapping of the set $\{0, \ldots, m\}$ to $\{1, 2\}$,

$$\varphi(i)=\begin{cases} 1, & \text{if } \alpha(i)\in B_1; \\ 2, & \text{if } \alpha(i)\in B_2, \end{cases} \quad \text{where } i\leq m.$$

The function φ is correctly defined because $B_1\cap B_2=\Lambda$.

Then $(\omega^{\alpha(0)}_{\alpha})^{-1}x_{\alpha}\subset\omega_{\alpha(0)}(V_1)\cap\omega_{\alpha(0)}(V_2)$, because there are no elements from B, which belong to the segment $[\alpha, \alpha(0))=\{\beta<\tau:\alpha\leq\beta<\alpha(0)\}$.

For the sake of simplicity of writing we put $\beta(i)=\alpha(i)+1$ for every $i\leq m$. Evidently, there exists a point $x_{\beta(0)}\in X^n_{\beta(0)}$ such that $\omega^{\beta(0)}_{\alpha(0)}(x_{\beta(0)})=x_{\alpha(0)}$ and $x_{\beta(0)}\in\omega_{\beta(0)}(V_{\varphi(0)})$.

We assert that $x_{\beta(0)}\in\omega_{\beta(0)}(V_1)\cap\omega_{\beta(0)}(V_2)$. Really, for example, if $\varphi(0)=1$ then $\alpha(0)\notin B_2$, hence, $\omega_{\beta(0)}(V_2)=(\omega^{\beta(0)}_{\alpha(0)})^{-1}\omega_{\alpha(0)}(V_2)\supset(\omega^{\beta(0)}_{\alpha(0)})^{-1}x_{\alpha(0)}\ni x_{\beta(0)}$. We note again that $(\omega^{\alpha(1)}_{\beta(0)})^{-1}x_{\beta(0)}\subset\omega_{\alpha(1)}(V_1)\cap\omega_{\alpha(1)}(V_2)$, then we choose a point $x_{\alpha(1)}\in(\omega^{\alpha(1)}_{\beta(0)})^{-1}x_{\beta(0)}$, then $x_{\beta(1)}$ and so on.

Finally, there exists a point $x_{\beta(m)}\in X^n_{\beta(m)}$ such that $x_{\beta(m)}\in\omega_{\beta(m)}(V_1)\cap\omega_{\beta(m)}(V_2)$. The equality $I\cup B_1\cup B_2=\mathcal{C}(V_1)\cup\mathcal{C}(V_2)\subset\beta(m)$ implies that $\omega^{-1}_{\beta(m)}(x_{\beta(m)})\subset V_1\cap V_2$.

This contradiction means that the system $\mathcal{O}=\{\omega_{\alpha}(V): V\in\gamma_1\}$ of open sets in X^n_{α} is disjoint. However, the cardinality of this system is not less than $|\gamma_1|$ (we have already established that the system $\mathcal{O}$ is disjoint) and $|\gamma_1|\geq\lambda$. So, $|\mathcal{O}|\geq\lambda$. It contradicts the inequality $\nabla c(X^n_{\alpha})\leq\lambda$. Consequently, $\nabla c(X^n)\leq\lambda$ for every $n\in\omega$, that is $\nabla c^*(X)\leq\lambda$. The last inequality implies that $\nabla c(X^{\mu})\leq\lambda$ for every cardinal $\mu\geq1$, which is a simple generalization of the theorem of Noble and Ulmer (which asserts that $c^*(X)\leq\lambda$ implies $c(X^{\mu})\leq\lambda$ for every cardinal $\mu\geq1$, where λ is any infinite cardinal).

If follows from the fact, which we have just proved, because in a trivial way a space X^{μ} is represented as a limit of a continuous inverse spectrum with open projections. Transfinite induction on μ completes the proof.

Remark 5. It is useful to note that the theorem of Erdos and Tarski (see, for example, theorem 3.1 from [7]) is equivalent to the following statement: $\nabla c(X)\leq\lambda$ implies that $c(X)<\lambda$ for every space X, where λ is a singular cardinal.

Lemma 7. *Let $f:X\to Y$ be a continuous closed mapping of a normal space X into T_1-space Y and a natural projection $\pi_n: X\times X^n\to X^n$ is closed for every $n\in\omega$. Then a mapping $f^n: X^n\to Y^n$ is closed and continuous for every $n\in\omega$.*

Proof. It is sufficient to verify that a mapping f^n is closed for every $n\in\omega$. We shall prove at first the following

Statement A. *If $f_i: X_i\to Y_i$ is a continuous closed mapping and a natural projection $\pi_i: X_1\times X_2\to X_i$ is closed, where $i\in\{1, 2\}$ and X_1 is a normal space, then a mapping $f=f_1\times f_2: X_1\times X_2\to Y_1\times Y_2$ is continuous and closed too.*

For every point $y_1=(y_1, y_2)\in Y_1\times Y_2$ we have: $f^{-1}(y)=f_1^{-1}(y_1)\times f_2^{-1}(y_2)$. It is necessary to prove that for every open set O in $X_1\times X_2$, which contains a set $f^{-1}(y)$, there exists an open set V in $Y_1\times Y_2$ such that $y\in V$ and $f^{-1}(V)\subset O$.

However, for every open set $V\ni y$ in $Y_1\times Y_2$ there exist open sets V_1 and V_2 in Y_1 and Y_2 respectively, such that $y_i\in V_i$, $i=1, 2$ and $V_1\times V_2\subset V$.

Further, $f^{-1}(V_1\times V_2)=f_1^{-1}(V_1)\times f_2^{-1}(V_2)$ and the fact that mappings f_1 and f_2 are closed implies that the question whether the mapping f is closed is reduced to the question whether there exist two sets O_1 and O_2 such that O_i is open in X_i, $f_i^{-1}(y_i)\subset O_i$, $i=1, 2$ and $O_1\times O_2\subset O$.

We put $F_i=f_i^{-1}(y_i)$, $i=1, 2$. A mapping $q_1=\pi_1|\pi_2^{-1}(F_2)$ is closed, $O\cap\pi_2^{-1}(F_2)$ is open in $\pi_2^{-1}(F_2)$ and $q_1^{-1}(F_1)=\pi_1^{-1}(F_1)\cap\pi_2^{-1}(F_2)=F_1\times F_2\subset O\cap\pi_2^{-1}(F_2)$, hence there exists an open set W in X_1 such that $F_1\subset W$ and $q_1^{-1}(W)\subset O\cap\pi_2^{-1}(F_2)$.

As a space X_1 is a normal one, so there exists an open set O_1 in X_1 such that $F_1\subset O_1\subset[O_1]\subset W$. Put $\Phi=[O_1]$. Evidently, $q_1^{-1}(\Phi)\subset O\cap\pi_2^{-1}(F_2)$. We put $q_2=\pi_2|\pi_1^{-1}(\Phi)$. Then q_2 is a closed mapping. We have: $q_2^{-1}(F_2)=\pi_1^{-1}(\Phi)\cap\pi_2^{-1}(F_2)=\Phi\times F_2\subset\pi_1^{-1}(\Phi)\cap O$, therefore there exists an open set O_2 in X_2 such that $F_2\subset O_2$ and $q_2^{-1}(O_2)\subset\pi_1^{-1}(\Phi)\cap O$. Then $F_1\times F_2\subset O_1\times O_2\subset\Phi\times O_2\subset O$. So, the statement A is proved.

Now we are beginning to prove our lemma by induction along ω. The conclusion of the lemma is obvious for $n=1$. Let us assume that we have already proved the fact that a mapping f^n is closed for $n=m$. Then we will prove that a mapping f^n is closed for $n=m+1$.

Firstly we note that a natural projection $\omega_n: X\times X^n\to X$ is closed for every $n\in\omega$, it follows from the equality $\omega_{n+1}=\omega_n\circ\pi_{n+1}$, where π_{n+1} is a closed mapping (lemma's condition).

So, let f^m be a closed mapping. We have $f^{m+1}: X\times X^m\to Y\times Y^m$. We put $X_1=X$, $X_2=X^m$, $Y_1=Y$, $Y_2=Y^m$, $f_1=f$ and $f_2=f^m$. Now we can apply the statement A, which implies that f^{m+1} is a closed mapping. Lemma is proved.

Theorem 6. *Let τ be a regular cardinal and $S=\{X_\alpha, \omega_\alpha^\beta, \alpha<\tau\}$ be a continuous inverse spectrum such that X_α is a normal space, ω_α^β is a closed mapping for all $\alpha, \beta<\tau$, $\alpha<\beta$ and for every $n\in\omega$ a natural projection of $X_\alpha\times X_\alpha^n$ onto X_α^n is a closed mapping too. Then the condition $\triangledown\mathrm{ic}(X_\alpha)\leqq\tau$ for every $\alpha<\tau$ implies that $\triangledown\mathrm{ic}^*(X)\leq\tau$, where $X=\overleftarrow{\lim}\, S$.*

Proof. Lemma 5 implies that for every $n\in\omega$ and for all $\alpha, \beta<\tau$, $\alpha<\beta$, a mapping $(\omega_\alpha^\beta)^n$ is closed. Further, the conditions of the theorem, namely that a projection $X_\alpha\times X_\alpha^n\to X_\alpha^n$ is closed for every $n\in\omega$ and $\triangledown\mathrm{ic}(X_\alpha)\leqq\tau$ imply that $\triangledown\mathrm{ic}(X_\alpha^n)\leq\tau$ for every $n\in\omega$ (one can prove this statement by induction on n), therefore, $\triangledown\mathrm{ic}^*(X_\alpha)\leq\tau$ for every $\alpha<\tau$.

Now we consider a spectrum $F_n(S)$. It is a continuous inverse spectrum (with closed projections) of a regular length τ and $\triangledown$ — index of compactness of every non-limit space is not greater than τ. In [8] B. A. Pasinkov proved that this properties of a spectrum imply that $\triangledown\mathrm{ic}(\overleftarrow{\lim}\, F_n(S))\leq\tau$ too. So, we get $\overleftarrow{\lim}\, F_n(S)=F_n(\overleftarrow{\lim}\, S)=(\overleftarrow{\lim}\, S)^n=X^n$, hence $\triangledown\mathrm{ic}^*(X)\leq\tau$.

Corollary 3. *If the conditions of the previous theorem are satisfied then $\triangledown t(C_p(X, \mathbb{R}))\leq\tau$. Therefore, if $\tau=\lambda^+$ then $t(C_p(X, \mathbb{R}))\leq\xi$.*

(We write $\triangledown t(Y)\leq\tau$ if conditions $A\subset Y$ and $y\in[A]_Y$ imply that there exists a set $B\subset A$ such that $y\in[B]_Y$ and $|B|<\tau$).

The previous theorem implies that $\triangledown\mathrm{ic}^*(X)\leq\tau$, hence $\triangledown t(C_p(X, \mathbb{R}))\leq\triangledown\mathrm{ic}^*(X)\leq\tau$ (for the special cace of the inequality $\triangledown t(C_o(X, \mathbb{R}))\leq\triangledown\mathrm{ic}^*(X)$ see theorems 2 and 2′ from § 1 of [1]).

Q u e s t i o n 1. Is it possible to prove a special continuity of the function nw without any additional supposition (in the class of all completely regular or Hausdoff spaces, for example)?

Q u e s t i o n 2. Is it possible to prove a negation of a spectral continuity of the function πw without any additional supposition (in the class of all completely regular spaces)?

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