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A REPRESENTATION OF THE COMMUTANT OF THE INITIAL VALUE STURM—LIOUVILLE OPERATOR

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An explicit convolutional representation of a class of bounded linear operators $M: C\to C$, commuting with the Sturm — Liouville operator $D=d^2/dt^2-q(t)$ in the initial subspace $C_0^2=\{f\in C^2: f(0)=f'(0)=0\}$ is found. The case when M acts on locally integrable functions is also considered.

Let C denotes the space $C[0, T]$ or $C[-T, T]$ of the continuous functions in $[0, T]$ or $[-T; T]$, respectively, or the spaces $C[0, \infty)$ or $C(-\infty, +\infty)$. The exact meaning of C should be clear from the context. In the cases of finite interval, C is considered as a Banach space with the usual uniform norm. In the cases of infinite interval, C is considered as a Fréchet space with the topology of the almost uniform convergence.

We consider an arbitrary, but fixed Sturm — Liouville operator

(1) $$D=d^2/dt^2-q(t),\quad q\in C.$$

Let $C_0^2=\{f\in C^2: f(0)=f'(0)=0\}$. Our main aim is to find an explicit representation of all bounded linear operators $M: C\to C$ with $M(C_0^2)\subset C_0^2$, which commute with D in C_0^2, i. e.

(2) $$MDf=DMf\quad \text{for}\quad f\in C_0^2.$$

Definition 1. *Let $L_0: C\to C_0^2$ be the initial right inverse operator of D for the point $t=0$, i. e. $DL_0f=f$, $L_0f(0)=(L_0f)'(0)=0$ for $f\in C$.*

Let y_1, y_2 be the fundamental system of D for the point $t=0$, i. e. $Dy_i=0$, $i=1, 2$ and $y_1(0)=1$, $y_1'(0)=0$; $y_2(0)=0$, $y_2'(0)=1$. Then, it is easy to see that

(3) $$L_0Df=f-f(0)y_1-f'(0)y_2,\ \text{for}\ f\in C^2.$$

Lemma 1. *A linear operator $M: C\to C$ with $M(C_0^2)\subset C_0^2$ commutes with D in C_0^2 iff M commutes with L_0 in C.*

Proof. Let $MDf=DMf$ for each $f\in C_0^2$. Since $L_0f\in C_0^2$ for $f\in C$, then $MDL_0f=DML_0f$. Hence $Mf=DML_0f$ and, using (3), we obtain $L_0Mf=L_0DML_0f=ML_0f$. Conversely, let $ML_0f=L_0Mf$ for $f\in C$. Then $DML_0f=Mf$. Now, we put $f=Dg$ for arbitrary $g\in C_0^2$, and by (3), we receive $MDg=DML_0g=DMg$.

The lemma shows that it is enough to find a representation of all bounded linear operators $M: C\to C$ commuting with L_0 in C. We reduce this problem

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to the simpler problem for finding of the commutant of the square l^2 of the integration operator

$$(4) \qquad lf=\int_0^t f(u)du.$$

In [1] is proved the similarity relation $L_0=X^{-1}l^2X$, where X is the well known transmutation operator of Delsarte — Povzner [2, p. 13—26] and [3, p. 140—151]. Since $X:C\to C$ is a bounded operator, and X has continuous inverse X^{-1} then $M:C\to C$ commutes with L_0 in C iff the bounded operator $\widetilde{M}=XMX^{-1}$ commutes with l^2 in C. Hence our problem is reduced to the problem of explicit representation of the commutant of l^2 in C.

1. Explicit representation of the commutant of l^2 in C. A basic role in our considerations is plyed by the convolution

$$(5) \qquad f*g=\int_0^t f(t-u)\,g(u)du.$$

It is well known that (5) is a bilinear, commutative and associative operation in C. The operator l has the representation

$$(6) \qquad lf=1*f.$$

We shall make use of a special case of the associativity relation, viz.

$$(7) \qquad l(f*g)=(lf)*g=f*(lg).$$

Theorem 1. *Let $M:C\to C$ be a bounded linear operator commuting with the integration operator l in C. Then*

$$(8) \qquad M(f*g)=(Mf)*g=f*(Mg) \text{ for all } f,g\in C$$

and

$$(9) \qquad Mf=d/dt\,[M(1)*f] \text{ for each } f\in C.$$

Proof. We use the well known fact that the constant function $\{1\}$ is a cyclic element of the integration operator l (equivalent to Weierstrass' approximation theorem). From the evident identity $M(1)*1=1*M(1)$, by (7) and using the commutation of M and l, we obtain $Ml^n(1)*l^m(1)=l^n(1)*Ml^m(1)$, $n,m=0,1,2,\dots$ Hence the equality

$$(10) \qquad Mf*g=f*Mg$$

holds true for linear combinations of $l^n(1)=t^n/n!$, i. e. for polynomials. Since the polynomials are dense in C, then (10) holds in C. Using (6), (7), (10) and the associativity of (5), we get

$$l[M(f*g)]=1*M(f*g)=M(1)*(f*g)=[M(1)*f]*g=[1*Mf]*g=l[(Mf)*g].$$

Hence $M(f*g)=(Mf)*g$, and (8) is proved. Then (9) follows from (8). In fact, $lMf=1*Mf=M(1)*f$.

In order to have a complete description of the commutant of l in C, we should characterize the class of the functions $m\in C$ with $m*f\in C^1$ for each $f\in C$. A sufficient condition is m to be continuous function with bounded variation in the case $T<+\infty$, or such a function in every finite subinterval in the case $T=+\infty$ (see [4]). Let BV denotes the space of the function with bounded variation in the case $T<+\infty$, or the functions with bounded variation in each finite subinterval, in the noncompact case.

Theorem 2. *Let* $C=C[0,T]$, $T\leqq+\infty$ *and* $M:C\to C$ *be a bounded linear operator commuting with* l *in* C. *Then* $m\overset{\text{def}}{=}M(1)$ *belongs to* $BV\cap C$ *and the equivalent representations*

$$Mf=m(0)f(t)+\int_0^t f(t-u)\,dm(u) \tag{11}$$

$$Mf=d/dt\,(m*f) \tag{11'}$$

hold, where the integral is understood in the Riemann—Stieltjes sense. Conversely, for each $m\in BV\cap C$ *the operator* (11) *or* (11′) *is a bounded linear operator, commuting with* l *in* C.

Proof. Let $f\in C^1$. From the universal formula (9), we have

$$Mf(t)=f(0)\,m(t)+\int_0^t f'(t-u)\,m(u)\,du,$$

hence $Mf(0)=m(0)f(0)$. The last formula is true for $f\in C$ too, since M is a continuous operator in C. Let us denote by M_0 the restriction of M to the subspace $C_0=\{f\in C:f(0)=0\}$. From $Mf(0)=m(0)f(0)$ it follows that $M_0:C_0\to C_0$. Evidently, (8) is true for M_0 and $f,g\in C_0$. Now we shall prove that M_0 commutes with the shift operators

$$A_\lambda f=\begin{cases}0 & t\in[0,\lambda)\\ f(t-\lambda) & t\in[\lambda,T]\end{cases},\quad \lambda\geqq 0$$

in C_0. A_λ are bounded linear operators in C_0, commuting with l. By similar consideraration as those used in the proof of theorem 1, we see that the relation (8), i. e. $A_\lambda\,(f*g)=(A_\lambda f)*g=f*(A_\lambda g)$ holds for f,g in C_0 too. Therefore

$$l\,[M_0A_\lambda f]=1*[M_0A_\lambda f]=M_0(1)*A_\lambda f=A_\lambda\,[M_0(1)*f]=(A_\lambda M_0)(1*f)$$
$$=A_\lambda M_0\,lf=lA_\lambda M_0 f.$$

Hence $A_\lambda M_0 f=M_0A_\lambda f$ for each $f\in C_0$, $\lambda\geqq 0$. From a theorem, due to Weston [5], it follows that there exists a function $\nu\in BV$ with $\nu(0)=0$, and $\nu(t-0)=\nu(t)$, such that

$$M_0 f=\int_0^t f(t-u)\,d\nu(u),\quad f\in C_0. \tag{12}$$

We shall prove that $\nu(t)=m(t)$ for $t>0$. Let $f\in C$. Then $f=f(0)+[f-f(0)]$, where $f-f(0)\in C_0$.

Hence

$$Mf=f(0)M(1)+M_0[f-f(0)]=f(0)\,m(t)+\int_0^t[f(t-u)-f(0)]\,d\nu(u)$$

for each $f\in C$. Then

$$lm=lM(1)=Ml(1)=M(\{t\})=\int_0^t(t-u)\,d\nu(u)=(t-u)\nu(u)\,|_{u=0}^{t}+\int_0^t\nu(u)\,du=l\nu$$

and, therefore, $m(t)=\nu(t)$ in each point of continuity t of the fnnction ν. But ν is normalized by the condition $\nu(t-0)=\nu(t)$ for $t>0$. Hence, $m(t)=\nu(t)$ for $t>0$. Let $\nu_0(t)$ be the jump function, defined as 0 for $t>0$, and as $m(0)$ for $t=0$. Obviously, $m=\nu+\nu_0$. Hence

$$Mf=f(0)\,m(t)+\int_0^t[f(t-u)-f(0)]dm(u)-\int_0^t[f(t-u)-f(0)]\,d\nu_0(u)=f(0)m(t)$$

$$-f(0)[m(t)-m(0)]+\int_0^t f(t-u)dm(u)-[f(t)-f(0)]m(0)=m(0)f(t)+\int_0^t f(t-u)dm(u).$$

Thus, the first part of the theorem is proved. Conversely, if $m\in BV\cap C$ then it is clear that (11) gives a bounded operator in C. It is enough to prove the commutating relation $Mlf=lMf$ in C^1 only. If $f\in C^1$, then

$$Mf=m(0)f(t)+f(t-u)\,m(u)\,|_{u=0}^{t}-\int_0^t m(u)\,d_u f(t-u)$$

$$=m(t)f(0)+\int_0^t f'(t-u)\,m(u)\,du=(m*f)'.$$

Hence $lMf=m*f-(m*f)(0)=m*f=Mlf$.

Remark 1. If $M: C_0[-T,0]\to C_0[-T,0]$ is a bounded operator, which commutes with l in $C_0[-T,0]$, then M can be represented by the Weston's formula (12) too. In this case $\nu(0)=0$, $\nu(t+0)=\nu(t)$ for $t<0$. This follows immediately by the change of the variable $u=-t$.

Remark 2. Theorem 2, without any change, is true for the case $C=C[-T,0]$ too.

Let $\mathcal{L}$ denotes the space of Lebesgue integrable functions in $[0,T]$ or $[-T,T]$ or the space of locally integrable functions in $[0,+\infty)$ or $(-\infty,+\infty)$. In the compact case $\mathcal{L}$ is endowed with the usual Lebesgue norm. In the non-compact case $\mathcal{L}$ is considered as a Fréchet space with the family of semi-norms

$$P_n(f)=\int_0^n|f| \quad \text{or} \quad P_n(f)=\int_{-n}^n|f|,\ n=1,2,\ldots$$

in the cases $[0,+\infty)$ or $(-\infty,+\infty)$, respectively.

Corollary 1. *Let C be $C[0,T]$ or $C[-T,0]$. If $m\in\mathcal{L}$ and $m*f\in C^1$ for each $f\in C$, then m coincides, a. e. with a function $\tilde{m}\in BV\cap C$. Then $(m*f)'$ can be expressed by the right part of (11) with m replaced by $\tilde{m}$. If $m\in C$, then $m\in BV$ and $(m*f)'$ is expressable directly by the right part of (11).*

Proof. Let

$$Mf\stackrel{\text{def}}{=}d/dt\,(m*f).$$

Then $M:C\to C$ and $Ml=lM$. From the closed graph theorem it follows that M is a continuous operator in C. Indeed, let $f_n\xrightarrow{C}f$ and $Mf_n\xrightarrow{C}g$. On one hand, $lMf_n=m*f_n\xrightarrow{C}m*f=lMf$, and on other hand, $lMf_n\xrightarrow{C}lg$. Therefore, $Mf=g$, i. e. M is closed. Hence, M is a bounded operator. Then from theorem 2 and remark 2 it follows $\tilde{m}\stackrel{\text{def}}{=}M(1)\in BV\cap C$. But $M(1)=d/dt\,(lm)$ and hence $M(1)=m$ a. e. If $m\in C$, then $M(1)=m$ everywhere.

Corollary 2. *Let C_0 be $C_0[0, T]$ or $C_0[-T, 0]$. If $m \in \mathcal{L}$ and $m * f \in C^1$ for each $f \in C_0$, then m, coincides a. e. with a function ν from BV. Then*

$$(13) \qquad (m * f)' = \int_0^t f(t-u)\, d\nu(u) + \nu(0) f(t).$$

Proof. If $Mf = d/dt\,(m * f)$, then $M: C_0 \to C$. In the same way as in corollary 1, it can be shown that $Mlf = lMf$ for $f \in C_0$ and that M is a bounded operator from C_0 to C. Now, we shall prove that $M(C_0) \subset C_0$. In fact, if $f \in C_0^1 = \{f \in C^1, f(0) = 0\}$, then $Mf = m(t) f(0) + m * f' = m * f'$ a. e. ([4], th. 6). This equality is true not only a. e., but everywhere, because its both sides are continuous functions ([4], th. 4). It is true especially for the point $t = 0$, i. e. $Mf(0) = 0$ for $f \in C_0^1$. But C_0^1 is dense in C_0 with respect to the uniform norm and M is continuous. Hence $Mf(0) = 0$ for each $f \in C$, i. e. $M: C_0 \to C_0$. In the same way as in theorem 3 and remark 3, it follows that M has the representation (13).

Corollary 3. *If a bounded linear operator $M: C_0 \to C$ commutes with l, then $M(C_0) \subset C_0$. The operators M allow a bounded linear extension from C_0 to C iff $M(\{t\}) \in C^1$, i. e. iff the function ν in representation* (12) *is a continuous one.*

Proof. Let $f \in C^1$, $f(0) = 0$. Then $f = lf'$ and $Mf = Mlf' = lMf'$. Hence $Mf(0) = 0$. But from the continuity of M in C, it follows that $Mf(0) = 0$ holds in C too.

The second part follows from the identity $Mf(t+0) - Mf(t-0) = f(t)[\nu(t+0) - \nu(t-0)]$.

Remark 3. Theorem 2 and corollaries 2, 3 hold true for $C = C[-T, T]$, or even for $C = [a, b]$ with $a \leq 0 \leq b$ too. We omit the evident modifications.

Theorem 3. *Let $M: C \to C$ be an arbitrary bounded linear operator which commutes with the integration operator l in C. Then M can be extended as a bounded linear operator on $\mathcal{L}$, commuting with l in $\mathcal{L}$.*

This follows immediately from the representation (11), provided the integral is understood in Lebesgue—Stieltjes sense. Then (11) gives the desired extension of M to $\mathcal{L}$, which commutes with l in $\mathcal{L}$.

Now, using corollary 3 and the representation formula (12), a similar theorem about operators $M: C_0 \to C$ commuting with l can be proved.

Theorem 4. *A bounded linear operator $M: C \to C$, where $C = C[0, T]$, commutes with the square of the integration operator l^2 in C iff M commutes with l in C. Then M has the representation* (11).

Proof. We make use of the fact that the constant function $\{1\}$ is a cyclic element of l^2, i. e. that the span of the functions $l^{2n}(1) = t^{2n}/(2n)!$, $n = 0, 1, 2, \ldots$ is dense in C. By the same argument as in the proof of theorem 1, it follows that M satisfies the relation (8), i. e. it is a multiplier of the convolution (5).

The restriction $C = C[0, T]$ in above theorem is essential. The statement of the theorem does not hold in the case $C = C[-T, T]$. Instead, it holds the following:

Theorem 5. *If $C = C[-T, T]$, then a bounded linear operator $M: C \to C$ commutes with l^2 iff it admits a representation of the form*

$$(14) \qquad Mf = d/dt\,[m * f_e] + d^2/dt^2\,[n * f_0],$$

where $m \stackrel{\text{def}}{=} M(1) \in BV \cap C$, and $n \stackrel{\text{def}}{=} M(\{t\})$ is absolutely continuous function with

$n(0)=0$ and $n'\in BV$. Here $f_e(t)=(1/2)[f(t)+f(-t)]$ and $f_0(t)=(1/2)[f(t)-f(-t)]$ denote the even and odd part of f, respectively.

Proof. Let $Ml^2=l^2M$ in $C[-T, T]$. It is easy to see that the functions $\{1\}$ and $\{t\}$ are cyclic elements of l^2 in the spaces C^e and C^0 of even or odd functions of C, respectively. As in the proof of theorem 1, we can show that the identity $Mf*g=f*Mg$ holds true when $f, g\in C^e$ or $f, g\in C^0$. Now, let f be even. Then $M(1)*f=1*Mf=lMf$, i. e. $Mf=d/dt[M(1)*f]$. If f is an odd function, then $M(\{t\})*f=\{t\}*Mf=l^2Mf$. Hence $Mf=d^2/dt^2(M(\{t\})*f)$. If f is an arbitrary function from $C[-T, T]$, then by the decomposition $f=f_e+f_0$ we obtain (14) with $m=M(1)\in C$ and $n=M(\{t\})\in C$.

Now we should characterize exactly the continuous functions m and n. Let m^+, m^- and n^+, n^- are the restrictions of m and n to $[0, T]$ and $[-T, 0]$, respectively. Let us introduce the auxilliary operators $M^+f=d/dt[m^+*f]$ and $M^-f=d/dt[m^-*f]$ in $C[0, T]$ or $C[-T, 0]$, correspondingly. It is clear that $M^+f=(M\tilde{f})|_{[0,T]}$, where $\tilde{f}$ is the even continuation of $f\in C[0, T]$ to $[-T, T]$. Analogously, $M^-f=(M\tilde{f})|_{[-T,0]}$ for $f\in C[-T, 0]$. Hence M^+ and M^- are bounded linear operators in $C[0, T]$ and $C[-T, 0]$, respectively. They commute with l in the corresponding spaces $C[0, T]$ and $C[-T, 0]$. From corollary 1 it follows that $m^+\in BV[0, T]$ and $m^-\in BV[-T, 0]$. Hence $m\in BV[0, T]$.

The exact characterization of $n(t)$ is more involved. First of all, we prove that $n(0)=0$. Let f be an arbitrary odd function from $C^2[-T, T]$. Obviously, $f(0)=0$. Then $Mf=d^2/dt^2[n*f]=f'(0)n(t)+n*f''$. Since M is bounded operator, then $|Mf(0)|=|n(0)f'(0)|\leq A \sup_{t\in[-T,T]}|f(t)|$, when $T<+\infty$. For the special choice $f(t)=\operatorname{arctg}(kt)$ we obtain $|n(0)|\leq A\pi/k$ for each natural k. Hence $n(0)=0$. The non-compact case can be settled in a similar way. Since $h(0)=0$, then we see easily that $Mf(0)=0$ when f is odd, and $f\in C^2[-T, T]$. From the continuity of M we conclude at once the desired relation $Mf(0)=0$ for each odd function f from $C[-T,T]$. Let $N^+f\overset{\text{def}}{=}d^2/dt^2(n^+*f)$ and $N^-f\overset{\text{def}}{=}d^2/dt^2(n^-*f)$ are defined for f in $C_0[0, T]$ or $C_0[-T, 0]$, respectively. It is clear that $N^+f=(M\tilde{\tilde{f}})|_{[0,T]}$ and $N^-f=(M\tilde{\tilde{f}})|_{[-T,0]}$, where $\tilde{\tilde{f}}$ is the odd continuation of $f\in C_0[0, T]$ or $f\in C_0[-T, 0]$. Therefore, $N^+:C_0[0, T]\to C_0[0, T]$; $N^-:C_0[-T, 0]\to C_0[-T, 0]$ are bounded operators. If $f\in C^2[0, T]$, $f(0)=0$, then

$$lN^+f=(n^+*f)'-(n^+*f)'(0)=(n^+*f)'-n^+(0)f(0)-(n^+*f')(0)=(n^+*f)'=N^+lf.$$

Then N^+ commutes with l in $C_0[0, T]$, by the continuity of N^+. The same is true for N^-. From (12) and from remark 1, it follows the existence of functions $\nu^+\in BV[0, T]$ and $\nu^-\in BV[-T, 0]$ with $\nu^+(0)=\nu^-(0)=0$, such that

$$N^+f=\int_0^t f(t-u)d\nu^+(u),\ t\in[0, T]$$

and

$$N^-f=\int_0^t f(t-u)d\nu^-(u),\ t\in[-T, 0].$$

Using integration by parts, we get $N^+(\{t\})=l\nu^+$ and $N^-(\{t\})=l\nu^-$ in $[0, T]$ or $[-T, 0]$, respectively. Now, if ν is defined as $\nu^+(t)$ in $[0, T]$, and as $\nu^-(t)$ in $[-T, 0]$, then $\nu\in BV[-T, T]$, $\nu(0)=0$ and $M(\{t\})=l\nu$, since $M(\{t\})|_{[0,T]}$

$=N^+(\{t\})$ and $M(\{t\})\,|_{[-T,\,0]}=N^-(\{t\})$. Hence the function $n(t)=M(\{t\})$ is absolutely continuous, $n(0)=0$, and $n'\in BV$.

Conversely, if M is an operator, defined by (14), then M commutes with l^2 in $C[-T,\,T]$, as it can be shown by a direct check. The boundness of M is easily seen too.

Corollary 4. *A bounded linear operator* $M:C[-T,\,T]\rightarrow C[-T,\,T]$ *commutes with* l^2 *iff it admits a representation of the form*

$$Mf=d/dt\,[m*f_e]+d/dt\,[\nu*f_0], \tag{15}$$

where $m=M(1)\in BV\cap C$ *and* $\nu=M(\{t\})'\in BV$.

This is another formulation of theorem 5. The idea to look for two functions m and n, needed for a representation of the commutant of l^2 in $C[-T,\,T]$, occurs to the authors from the interesting paper of I. Raichinov [10].

A representation of M in the form (11) can be obtained from (15), provided the function $\nu=M(t)'$ be defined in an arbitrary manner in its points of discontinuity. We should note that in these points $M(t)$ has right and left derivatives. Then

$$Mf=m(0)\,f_e(t)+\nu(0)\,f_0(t)+\int_0^t f_e(t-u)\,dm(u)+\int_0^t f_0(t-u)\,d\nu(u). \tag{16}$$

Especially, it is convenient to define $\nu(0)=m(0)$. Then (16) takes the simpler form

$$Mf=m(0)\,f(t)+\int_0^t f_e(t-u)\,dm(u)+\int_0^t f_0(t-u)\,d\nu(u). \tag{17}$$

Corollary 5. *A bounded operator* $M:C[-T,\,T]\rightarrow C[-T,\,T]$, *commuting with* l^2, *commutes with* l *iff* $M(1)=M(t)'$, *that is, when* M *and* l *commute only on the function* $\{1\}$.

Proof. The conditions $M(1)=M(t)'$ and $Ml(1)=lM(1)$ are equivalent since by theorem 5, $M(\{t\})(0)=0$. Now, let $Ml(1)=lM(1)$. It is not difficult to prove that $Ml^n(1)=l^nM(1)$ for $n=0,1,2,\ldots$ Therefore, $Ml^n(1)*l^m(1)=l^n(1)*Ml^m(1)$ for $n,\,m=0,1,2,\ldots$. As in the proof of theorem 1, it follows that M satisfies (8) in $C[-T,\,T]$. The converse statement is obvious.

If we transform (14) and (17) in the forms

$$Mf=d/dt\,(m*f)+d/dt\,(\nu_0*f_0) \tag{18}$$

and

$$Mf=m(0)\,f(t)+\int_0^t f(t-u)\,dm(u)+\int_0^t f_0(t-u)\,d\nu_0(u) \tag{19}$$

with $\nu_0=\nu-m=M(t)'-M(1)$, then it is seen that the operator

$$Mf=d/dt\,[\nu_0*f_0]=\int_0^t f_0(t-u)\,d\nu_0(u), \tag{20}$$

where ν_0 is a function from BV, which does not vanish a. e. is an example of an operator commuting with l^2, which does not commute with l. The representation (20) is characteristic for this type of operators in the sense that each operator, commuting with l^2, but not commuting with l, is a sum of an operator of the type (20) and an operator, commuting with l. This follows from (18) and (19).

2. The commutant of L_0 in C.

L e m m a 2. *Let $X:C\to C$ be an arbitrary continuous and continuously invertible linear similarity from L_0 to l^2, i. e. with $L_0=X^{-1}l^2X$. A bounded linear operator $M:C\to C$ commutes with L_0 in C iff the bounded operator $\widetilde{M}=XMX^{-1}$ commutes with l^2 in C.*

The proof is immediate and we omit it.

We shall use similarities, due to Delsarte — Povzner. First, let us consider the one-sided case $C=C[0, T]$. Let X_i, $i=1, 2$ are the Volterra's second kind operators

$$X_i f=f(t)+\int_0^t K_i(t,\tau)f(\tau)\,d\tau \tag{21}$$

with kernels of the form $K_i(t,\tau)=A_i[(1/2)(t+\tau), (1/2)(t-\tau)]$, where $A_1(u, v)$ satisfies the integral equation [2; 25]

$$A_1(u,v)=-\frac{1}{2}\int_0^u q-\frac{1}{2}\int_0^v q-2\int_0^v d\xi\int_0^\xi q(\xi-\eta)A_1(\xi,\eta)d\eta-\int_v^u d\xi\int_0^v q(\xi-\eta)A_1(\xi,\eta)d\eta \tag{22}$$

and $A_2(u, v)$ satisfies the integral equation

$$A_2(u,v)=-\frac{1}{2}\int_0^u q+\frac{1}{2}\int_0^v q-\int_v^u d\xi\int_0^v q(\xi-\eta)A_2(\xi,\eta)d\eta. \tag{23}$$

If y_1, y_2 is the fundamental system of $D=d^2/dt^2-q(t)$ for the point $t=0$, then $X_1y_1=1$ and $X_2y_2=\{t\}$.

In [1] we had shown that the operation

$$f\,\tilde{*}\,g=X_2^{-1}[X_2f*X_2g];\quad f,g\in C[0, T] \tag{24}$$

is a convolution of the operator L_0 in C, i. e. it is a bilinear, commutative and associative operation in C with

$$L_0f=y_2\,\tilde{*}\,f. \tag{25}$$

The next theorem gives two representation formulas of the commutant of L_0.

T h e o r e m 6. *Let $C=C[0, T]$, $T\leq+\infty$. A linear bounded operator $M:C\to C$ commutes with L_0 iff it has either the form*

$$Mf=X_1^{-1}\left[\int_0^t X_1f(t-u)\,dX_1m_1(u)\right]+m_1(0)f(t) \tag{26}$$

with $m_1\overset{\text{def}}{=}My_1\in BV\cap C$, or

$$Mf=D[m_2\,\tilde{*}\,f] \tag{27}$$

with $m_2\overset{\text{def}}{=}My_2\in C^1$, and $m_2'\in BV\cap C$. All these operators can be extended as bounded linear operators over $\mathfrak{L}$, commuting with L_0 in $\mathfrak{L}$.

Proof. The proof uses some elementary properties of the similarities X_1, X_2 and theorem 3. Let $M:C\to C$ commutes with L_0. Then, by lemma 2, the operators $M_i=X_iMX_i^{-1}$ $i=1, 2$ commute with l^2 in C. Therefore, M_i is represented by (11) or (11′). Hence

$$Mf=X_1^{-1}M_1X_1f=X_1^{-1}[M_1(1)(0)X_1f+\int_0^t X_1f(t-u)dM_1(1)(u)].$$

However, $M_1(1)=X_1MX_1^{-1}(1)=X_1My_1=X_1m_1\in BV\cap C$ and $Xm_1(0)=m_1(0)$. Therefore $m_1\in BV\cap C$ since K_1 is a smooth function, and thus we obtain (26). Using (11′) and the similarity X_2, M could be represented by means of the convolution (24). Now, we can use (11′):

$$M_2f=d/dt\,[M_2(1)*f]=d^2/dt^2\,[lM_2(1)*f]=d^2/dt^2\,[M_2(t)*f],$$

where $M_2(t)=M_2(X_2\,y_2)=X_2My_2=X_2m_2\in C^1$.
From the formula

$$X_2m_2=m_2(t)+\int_0^t K_2(t,u)m_2(u)du$$

it follows that $m_2\in C^1$, since K_2 is smooth and $m_2\in C$. The formula

$$\frac{\partial}{\partial t}K_2(t,\tau)=-\frac{1}{4}q\left(\frac{t+\tau}{2}\right)+\frac{1}{4}q\left(\frac{t-\tau}{2}\right)+F(t,\tau),$$

obtained from integral equation (23) with a smooth function $F(t,\tau)$ shows that $f\in C^1$ implies $(X_2f)'-f'\in C^1$. Indeed

$$(X_2f)_t'-f'=K_2(t,t)f(t)+\int_0^t K_{2t}(t,\tau)f(\tau)\,d\tau=-\frac{1}{2}f(t)\int_0^t q-\frac{1}{4}\int_0^t q\left(\frac{t+\tau}{2}\right)f(\tau)d\tau$$

$$+\frac{1}{4}\int_0^t q\left(\frac{t-\tau}{2}\right)f(\tau)\,d\tau+\int_0^t F(t,\tau)f(\tau)d\tau=-\frac{1}{2}f(t)\int_0^t q(\xi)d\xi$$

$$-\frac{1}{8}\int_{t/2}^t q(\xi)f(2\xi-t)d\xi+\frac{1}{8}\int_0^{t/2} q(\xi)f(t-2\xi)d\xi+\int_0^t F(t,\tau)f(\tau)d\tau.$$

If we take $f=m_2$, we get

$$m_2'=(Xm_2)'+\text{smooth function},$$

i. e. $m_2'\in BV\cap C$ since $(Xm_2)'\in BV\cap C$.

Let us now consider the two-sided case $C=C[-T,T]$, $T\leqq+\infty$. Now we shall use a Delsarte—Povzner similarity of the form

$$Xf=f(t)+\int_{-t}^t K(t,\tau)f(\tau)\,d\tau, \tag{28}$$

where the kernel $K(t,\tau)$ is the solution of the Volterra integral equation

$$K(t,\tau)=-\frac{1}{2}\int_0^{(t+\tau)/2} q(\xi)\,d\xi-\int_0^{(t+\tau)/2} d\xi\int_0^{(t-\tau)/2} q(\xi-\eta)K(\xi+\eta,\xi-\eta)d\eta. \tag{29}$$

The assumption $q\in C$ implies that $K(t,\tau)$ is a smooth function, and $X:C\to C$ is a linear isomorphism in C with $X(C^1)=C^1$. As in the proof of theorem 6, it is easily shown that $f \in C^1$ implies $(Xf)' - f' \in C^1$. Therefore, X maps C^2 on C^2. In [3; 140—151] is shown that in the case $q \in C^1$, the similarity relation

$$XDf = \frac{d^2}{dt^2} Xf,\ f \in C^2$$

is satisfied, and $Xy_1 = 1$, $Xy_2 = \{t\}$. The same is true in the case $q \in C$ too, as it can be shown by approximation.

The operation

$$f \tilde{*} g = X^{-1}[Xf * Xg]$$

is also a convolution for L_0 in $C[-T, T]$, and $L_0 f = y_2 \tilde{*} f$.

Let us denote: $D^{1/2} = X^{-1}\, d/dt X$, $L_0^{1/2} = X^{-1} l X$. Evidently, $D^{1/2} : C^1 \to C$, $L_0^{1/2} : C \to C^1$, and $(D^{1/2})^2 = D$, $(L_0^{1/2})^2 = L_0$, $L_0^{1/2} f = y_1 \tilde{*} f$, i. e. $y_1 \tilde{*} y_1 = y_2$. The operator $L_0^{1/2}$ is a right inverse of the operator $D_0^{1/2}$, and

$$L_0^{1/2} D^{1/2} f = f - f(0)\, y_1.$$

The spaces C^e and C^0 of even and odd functions in $C[-T, T]$ are invariant subspaces of L_0 and $C = C^e \oplus C^0$. Here it is convenient to introduce the invariant subspaces $\widetilde{C^e} = X^{-1}(C^e)$ and $\widetilde{C^0} = X^{-1}(C^0)$ of the operator L_0. It is clear that $C = \widetilde{C}^e \oplus \widetilde{C}^0$.

Let P^e, P^0, $\widetilde{P}^e$ and $\widetilde{P}^0$ are the projection operators of C over the spaces C^e, C^0, $\widetilde{C}^e$ and $\widetilde{C}^0$, respectively. It is not difficult to prove that

$$X\widetilde{P}^e = P^e X \quad \text{and} \quad X\widetilde{P}^0 = P^0 X.$$

Theorem 7. Let $C = C[-T, T]$. A bounded linear operator $M : C \to C$ commutes with L_0 in C iff

(30) $$Mf = D^{1/2}[m_1 \tilde{*} \widetilde{P}^e f] + D[m_2 \tilde{*} \widetilde{P}^0 f]$$

or, equivalently,

(30′) $$Mf = D[L_0^{1/2} m_1 \tilde{*} \widetilde{P}^e f + m_2 \tilde{*} \widetilde{P}^0 f],$$

where $m_1 \overset{\text{def}}{=} My_1 \in BV \cap C$, $m_2 \overset{\text{def}}{=} My_2 \in AC$, $m_2(0) = 0$, and $m_2' \in BV$.

Proof. Let us denote $N = XMX^{-1}$. Obviously, N is a bounded operator, commuting with l^2 in C. According to theorem 5, N has the representation

$$Nf = d/dt\,[N(1) * f_e] + d^2/dt^2\,[N(t) * f_0],$$

where $N(1) = XMX^{-1}(1) = XMy_1 = Xm_1 \in BV \cap C$,

$$N(t) = XMX^{-1}(t) = XMy_2 = Xm_2 \in AC \text{ with } N(t)(0) = 0,\ N(\{t\})' \in BV.$$

As in the proof of theorem 6, it is seen that $m_1 \in BV \cap C$, $m_2 \in AC$, $m_2(0) = 0$, $m_2' \in BV$. Thus, we get

$$Mf = X^{-1}NXf = X^{-1} d/dt\,[Xm_1 * (Xf)_e] + X^{-1} d^2/dt^2\,[Xm_2 * (Xf)_0]$$

$$= D^{1/2} X^{-1}[Xm_1 * X\widetilde{P}^e f] + DX^{-1}[Xm_2 * X\widetilde{P}^0 f] = D^{1/2}[m_1 \tilde{*} f] + D[m_2 \tilde{*} f].$$

Another representations are easily obtainable from (15) or (16). For example, we have

$$Mf = m_1(0)\widetilde{P}^e f + m_2'(0)\widetilde{P}^o f + X^{-1}\int_0^t \widetilde{P}^e f(t-u)\,dXm_1(u)$$
(31)
$$+ X^{-1}\int_0^t \widetilde{P}^o f(t-u)\,d(Xm_2)'(u),$$

where $(Xm_2)' \in BV$ should be defined in an arbitrary manner in the points of discontinuity of m_2.

Thus, the problem for explicit characterization of all bounded linear operators $M: C \to C$ with $M(C_0^2) \subset C_0^2$ and $MDf = DMf$ for $f \in C_0^2$ is solved. The corresponding representation formulas are given by theorems 6 and 7 for $C[0, T]$ and $C[-T, T]$, respectively.

3. Extension of above results for Lebesgue integrable, or locally integrable functions. Let AC denote the space of absolutely continuous functions, and $AC^1 \overset{\text{def}}{=} \{f \in C^1, f' \in AC\}$, $AC_0^1 \overset{\text{def}}{=} \{f \in AC^1; f(0) = f'(0) = 0\}$.

Now we assume that $q \in \mathfrak{L}$. Then the operator $D = d^2/dt^2 - q(t)$ maps AC^1 in $\mathfrak{L}$. The problem could be stated in the following way. We search an explicit representation of all continuous linear operators $M: \mathfrak{L} \to \mathfrak{L}$ with $M(AC_0^1) \subset AC_0^1$, which commute with D in AC_0^1, i. e.

$$MDf = DMf \text{ for } f \in AC_0^1.$$

In the same way, as in the continuous case, let us introduce the right inverse operator L_0 of D, defined by the initial value problem

$$DL_0 f = f, \quad L_0 f(0) = (L_0 f)'(0) = 0; \quad f \in \mathfrak{L}.$$

The existence of L_0 is a well known fact, and $L_0: \mathfrak{L} \to AC_0^1$.

It is easy to see that the problem for an explicit representation of the operators M, we are interested in, is reduced to the problem for finding of the commutant of l^2 in $\mathfrak{L}$. For this purpose we should use similarities of the same kind as those used in section 2. Their kernels now are solutions of the same integral equations (21), (22) and (29), but with $q \in \mathfrak{L}$. In this case the kernels $K_i(t, \tau)$, $i = 1, 2$ and $K(t, \tau)$ are continuous. It happens that the similarities X_i, $i = 1, 2$ and X map C^1 in C^1, and AC^1 in AC^1. Now, the problem for the commutant of l^2 in $\mathfrak{L}$ can be solved in a similar way as in section 1. In the one-side case $[0, T]$ or $[0, \infty)$ we should use Dixmier [6] or Edwards [8] representation formulas.

Theorem 8. (Dixmier, Edwards) *Let $\mathfrak{L} = \mathfrak{L}[0, T]$, $T \leq \infty$. A bounded linear operator $M: \mathfrak{L} \to \mathfrak{L}$ commutes with l in $\mathfrak{L}$ iff it can be represented in the form*

(32) $$Mf \overset{\text{a. e.}}{=} \frac{d}{dt}(M(1) * f)$$

with $M(1) \overset{\text{a. e.}}{=} m \in BV[0, T]$.

Proof. Following the lines of the proof of theorem 1, now we can prove that $M(f * g) \overset{\text{a. e.}}{=} (Mf) * g \overset{\text{a. e.}}{=} f * (Mg)$ for $f, g \in \mathfrak{L}$. Then $lMf \overset{\text{a. e.}}{=} M(1) * f$. Dixmier [6] proved that $M(1) \overset{\text{a. e.}}{=} m \in BV$. Hence $M(1) * f \in C$ (see [4]), and $lMf = M(1) * f$ everywhere. Thus (32) holds.

Let $m \in BV[0, T]$, and let us extend m on $\mathbf{R}^1$ as $m(0)$ in $(-\infty, 0)$, and as $m(T)$ in (T, ∞), when $T<\infty$. Let μ be the complex measure in $\mathbf{R}^1$, defined by the extension of m (for the construction of μ, see [9], ch. 4). In [7] is proved that if $f \in \mathscr{L}[0, T]$, and $m \in BV$, then $m * f \in AC$ and the identity

$$(33)\qquad (m*f)' \overset{\text{a. e.}}{=} m(0) f(t) + \int_0^t f(t-u)\, d\mu(u)$$

in $\mathscr{L}[0, T]$ holds. Thus, we obtain two eqivalent forms of (32):

$$(32')\qquad Mf = m(0) f(t) + \int_0^t f(t-u) d\mu(u)$$

and

$$(32'')\qquad Mf = \int_0^t f(t-u) d\mu_0(u),$$

where μ_0 is the complex measure, defined by the function

$$m_0(t) = \begin{cases} m(t), & t \in (0, T] \\ 0 & t=0 \end{cases}.$$

Obviously, $M[1] \overset{\text{a. e.}}{=} m_0 \in BV$. The Dixmier — Edwards representation is proved in the form (32″).

Let μ be a complex measure in $[-T, T]$. We use the convention

$$\int_0^t f\, d\mu \overset{\text{def}}{=} \begin{cases} \int_{[0,t]} f\, d\mu & \text{for } t \geq 0 \\ -\int_{[t,0]} f d\mu & \text{for } t<0 \end{cases}.$$

Theorem 9. *Let $\mathscr{L} = \mathscr{L}[-T, T]$ and $T \leq +\infty$. A bounded linear operator $M: \mathscr{L} \to \mathscr{L}$ commutes with l in $\mathscr{L}$ iff M admits the representation* (32) *with $M(1) \overset{\text{a. e.}}{=} m \in BV[-T, T]$ or equivalently,*

$$(34)\qquad Mf \overset{\text{a. e.}}{=} H(t) f(t) + \int_0^t f(t-u) d\mu(u),\ t \in [-T, T]$$

in $[-T, T]$ holds. Here μ is the complex measure in $[-T, T]$, determined by the function m, and H is the step-function

$$(35)\qquad H(t) = \begin{cases} m(0+), & t \in [-T, 0) \\ m(0-). & t \in [0, T] \end{cases}.$$

Proof. In the same way as in the proof of theorem 1, we can prove that $M(f*g) \overset{\text{a. e.}}{=} (Mf)*g \overset{\text{a. e.}}{=} f*(Mg)$ holds for $f, g \in \mathscr{L}$. Hence

$$(36)\qquad lMf \overset{\text{a. e.}}{=} M(1)*f,$$

and $l^2 Mf = lM(1)*f = Ml(1)*f = M(t)*f$, everywhere. The above formulas show that $\mathscr{K}[\alpha, \beta] = \{f \in \mathscr{L} : f \overset{\text{a. e.}}{=} 0 \text{ in } [\alpha, \beta]\}$ are invariant subspaces of M when $0 \in [\alpha, \beta] \subset [-T, T]$. Let now $M^+ : \mathscr{L}[0, T] \to \mathscr{L}[0, T]$ be the operator $M^+ f = M\tilde{f}|_{[0,T]}$, where $\tilde{f}(t) \overset{\text{def}}{=} f(t)$, when $t \in [0, T]$ and $\tilde{f}(t) \overset{\text{def}}{=} 0$ when $t \in [-T, 0]$. It is easy to prove that M^+ is a bounded operator, which commutes with l in $\mathscr{L}[0, T]$. According to theorem 8, $M(\{\widetilde{1}\})|_{[0,T]} = M^+(1) \overset{\text{a.e.}}{=} m^+ \in BV[0, T]$, and $M(\{\widetilde{1}\})|_{[-T, 0]} = 0$ since $\mathscr{H}_{[-T, 0]}$ is an invariant subspace of M. Analogously, if $\widetilde{\widetilde{f}}$ is the function $\widetilde{\widetilde{f}}(t) \overset{\text{def}}{=} f(t)$ in $[-T, 0]$, and $\widetilde{\widetilde{f}}(t) \overset{\text{def}}{=} 0$ in $[0, T]$, where $f \in \mathscr{L}([-T, 0]$, then $M(\{\widetilde{\widetilde{1}}\})|_{[-T,0]} \overset{\text{a. e.}}{=} m^{-1} \in BV[-T, 0]$, and $M(\{\widetilde{\widetilde{1}}\})|_{[0, T]} \overset{\text{a. e.}}{=} 0$. Then $M(1) = M(\{\widetilde{1}\}) + M(\{\widetilde{\widetilde{1}}\}) \overset{\text{a. e.}}{=} m \in BV[-T, T]$. It is clear that (36) holds everywhere since the both sides are continuous functions. Thus, we obtained (32). The representation (34) follows from a formula, established in [7]:

$$(m * f)' \overset{\text{a. e.}}{=} H(t) f(t) + \int_0^t f(t-u)\, d\mu(u), \quad t \in [-T, T], \tag{37}$$

where $m \in BV[-T, T]$, $f \in \mathscr{L}[-T, T]$ and H is defined by (35).

Let us note that in the two-side case $\mathscr{L}[-T, T]$ it is impossible to represent the operator $M_0 f(t) = H(t) f(t)$ with a step-function (35) in the form (32″) of Dixmier—Edwards, as it is possible by means of Riemann—Stieltjes integral in the continuous case $C[-T, T]$. Even the identity operator $If(t) = f(t)$ in $\mathscr{L}[-T, T]$ cannot be represented in the form (32″). It is easy to see that M can be represented in the form (32″) iff $m(0-) + m(0+) = 0$.

In this connection, let us note that it is possible to find a formula for M similar to (32″). However, now, in general, M should be represented by means of two measures in the form

$$Mf = \begin{cases} \int_0^t f(t-u)\, d\mu_1(u), & t \geq 0, \\ \int_0^t f(t-u)\, d\mu_2(u), & t < 0 \end{cases}$$

with μ_1 and μ_2, defined by the functions

$$m_1(t) = \begin{cases} m(t), & t \in (0, T] \\ 0, & t \in [-T, 0] \end{cases}, \qquad m_2(t) = \begin{cases} 0\ , & t \in [0, T] \\ m(t), & t \in [-T, 0) \end{cases},$$

correspondingly. In general, it is impossible to consider μ_1 and μ_2 as restrictions of one measure μ in $[-T, T]$ to $[0, T]$ and $[-T, 0]$, respectively.

Theorem 10. *Let $\mathscr{L} = \mathscr{L}[0, T]$, $T \leq +\infty$. A bounded linear operator $M : \mathscr{L} \to \mathscr{L}$ commutes with l^2 in $\mathscr{L}$ iff M commutes with l in $\mathscr{L}$. Then $M(1) \overset{\text{a. e.}}{=} m \in BV$, and the representations* (32)–(32″) *hold.*

The proof is as those of theorem 4.

Theorem 11. *Let $\mathscr{L} = \mathscr{L}[0, T]$, $T \leq +\infty$. A bounded linear operator $M : \mathscr{L} \to \mathscr{L}$ commutes with l^2 in $\mathscr{L}$ iff M is representable in the form*

$$Mf = \frac{d}{dt}[M(1) * f_e] + \frac{d^2}{dt^2}[M(t) * f_0] \tag{38}$$

with $M(1) \overset{\text{a. e.}}{=} m \in BV$, and $M(t) \overset{\text{a. e.}}{=} n \in AC$, $n(0) = 0$, $n' \in BV$.

Proof. Let $\widetilde{f}$ and $\widetilde{\widetilde{f}}$ denote the even and odd extension of $f \in \mathscr{L}[0, T]$ to $[-T, T]$, respectively. It is evident that $\widetilde{l^2 f} = l^2 \widetilde{f}$ and $\widetilde{\widetilde{l^2 f}} = l^2 \widetilde{\widetilde{f}}$ for $f \in \mathscr{L}[0, T]$. Now let $M^+ f \overset{\text{def}}{=} M\widetilde{f}|_{[0, T]}$, $N^+ f \overset{\text{def}}{=} M\widetilde{\widetilde{f}}|_{[0, T]}$, for $f \in \mathscr{L}[0, T]$. Then M^+ and N^+ commute with l^2 in $\mathscr{L}[0, T]$. Indeed, $M^+ l^2 f = M(\widetilde{l^2 f})|_{[0, T]} = M(l^2 \widetilde{f})|_{[0, T]} = l^2 M\widetilde{f}|_{[0, T}$

$=l^2(M\widetilde{f}_{[0,T]})=l^2M^+f$. Since the bounded operators M^+ and N^+ commute with l in $\mathscr{L}[0, T]$ according to theorem 10, then $M^+(1)=M(1)|_{[0,T]} \overset{\text{a. e.}}{=} m' \in BV$, $N^+(1) \overset{\text{a. e.}}{=} \nu^+ \in BV$, and $M(t)|_{[0,T]}=N^+(t)=N^+(l1) \overset{\text{a. e.}}{=} lN^+(1)=l\nu^+$. In a similar way we can prove that $M(1)|_{[-T,0]} \overset{\text{a. e.}}{=} m^- \in BV$; $M(t)|_{[-T,0]} \overset{\text{a. e.}}{=} l\nu^-$, $\nu^- \in BV$. Hence $M(1) \overset{\text{a. e.}}{=} m \in BV$ and $M(t) \overset{\text{a. e.}}{=} l\nu$ with $\nu \in BV$.

Now, as in theorem 5, we can prove that $f*(Mg) \overset{\text{a. e.}}{=} (Mf)*g$ holds for $f, g \in \mathscr{L}_e$ and $f, g \in \mathscr{L}_0$, where $\mathscr{L}_e$ and $\mathscr{L}_0$ denote the subspaces of even or odd functions of $\mathscr{L}[-T, T]$. Hence if $f \in \mathscr{L}_e$, then $lMf=M(1)*f$ holds everywhere, and if $f \in l_0$, the $l^2Mf=M(t)*f$ everywhere. Now, from the decomposition $f=f_e+f_0$ (38) follows.

Another representations can easily be obtained from (38), using (37).

Now, using Delsarte — Povzner transmutation operators, as in section 2 we immediately can obtain a complete solution of the representation problem for the commutant of L_0 in $\mathscr{L}[0, T]$ or $\mathscr{L}[-T, T]$. Thus, the problem for representation of all bounded linear operators $M: \mathscr{L} \to \mathscr{L}$ with $M[AC_0^1] \subset AC_0^1$ and $MDf=DMf$ for $f \in AC_0^1$ is solved.

REFERENCES

1. N. S. Bozhinov, I. H. Dimovski. Operational calculus for general linear differential operator of second order. *C. R. Acad. Bulg. Sci.*, **28**, 1975, 727—730.
2. В. А. Марченко. Спектральная теория операторов Штурма-Лиувилля, Киев, 1972.
3. Б. М. Левитан. Теория операторов обобщенного сдвига. Москва, 1973.
4. J. Mikusinski, Cz. Rill-Nardzewski. Sur le produit de composition. *Stud. Math.*, **12**, 1951, 51—57.
5. J. D. Weston. On the representation of operators by convolution integral. *Pacif. J. Math.*, **10**, 1960, 1453—1468.
6. J. Dixmier. Les operateurs permutables a l'operateur integral. *Portug. Math.*, **8**, 1949, 2, 73—84.
7. N. S. Bozhinov. Differentiation properties and multipliers of Berg-Dimovski convolution for differentiation. *Serdica*, **6**, 1980 (to appear).
8. R. E. Edwards. Representation theorems for certain operators. *Pacif. J. Math.*, **7**, 1957, 1333—1339.
9. P. E. Едвардс. Функциональный анализ. Москва, 1968.
10. И. Райчинов. Линейни оператори, действуващи в пространства от аналитични функции и комутиращи с фиксирана степен на оператора на интегрирането. *Годишник на ВТУЗ. Математика*, **6**, 1970, № 2, 25—32.

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