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DIFFERENTIATION PROPERTIES AND MULTIPLIERS OF BERG-DIMOVSKI CONVOLUTION FOR THE DIFFERENTIATION OPERATOR

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The paper deals with the Berg-Dimovski convolution $f *_0 g$ for the continuous right inverse l_0 of d/dt connected to the problem $y'=f$, $N(y)=0$, where N is an arbitrary continuous linear functional in the space of continuous functions C. This convolution is extended in the spaces L^1 or L^1_{loc} and the properties of $*_0$ in some of their subspaces are studied. On the basis of these properties a representation formula for the continuous operators commuting with l_0 on some subspaces of L^1 or L^1_{loc} is found. It happens that these operators are all multipliers of $*_0$. Also all continuous convolutions for the operator l_0 by means of the Berg-Dimovski convolution are expressed. A Titchmarsh type theorem for all continuous convolutions of l_0 is proved. In particular every continuous convolution for the integration operator $lf=\int_0^t f$ in the spaces $C[0, \infty)$ and $L^1_{\mathrm{loc}}[0, \infty)$ has no divisors of zero.

0. Introduction and notations. Let Δ be an interval which contains the point 0. When Δ is the compact interval $[a, b]$ let L^p, $1\leq p\leq +\infty$ denote the space $L^p\{\Delta, m\}$ (m is the Lebesgue measure in R^1) provided with the usual norm $\| \ \|_p$. With C, C^k, $1\leq k<+\infty$ we denote the Banach spaces $C(\Delta)$, $C^k(\Delta)$, with their usual norms $\| \ \|_C$ and $\| \ \|_{C^k}$. Let also BV, AC denote the spaces of function with bounded variation and the space of absolutely continuous functions provided with the norms $\|f\|_{BV}=\|f\|_\infty+\overset{b}{\underset{a}{V}}f$, $\|f\|_{AC}=\|f\|_\infty+\|f'\|_1$, respectively. When Δ is noncompact, L^1_{loc} denotes the space of locally integrable functions. L^1_{loc} is a Frechet space with respect to the inductive topology which is given by the countable family of seminorms $P_n(f)=\|f\|_{\Delta\cap[-n, n]}$, $n=1, 2, \ldots$. Also L^p_{loc} denotes the space of integrable functions with locally integrable p-power. Now the spaces C, C^k, $1\leq k\leq+\infty$ are Frechet spaces relative to the usual inductive topology. Especially C is provided with the topology of the almost uniform convergence, and BV denotes the space of functions with bounded variation in every compact subinterval of Δ.

Definition (Dimovski [15]). *Let X be a linear space, and let M: $X\to X$ be a linear operator. A bilinear, commutative and associative operator $*$ in X is said to be a convolution of M iff*

$$M(f*g)=Mf*g=f*Mg \text{ holds for all } f, g\in X. \tag{0.1}$$

When an operator M satisfies (0.1) *we say that M is a multiplier of $*$*

Let l be the integration operator

$$(0.2)\qquad lf=\int_0^t f(u)du, \quad t\in\Delta, \quad f\in L^1$$

and let

$$(0.3)\qquad f*g=\int_0^t f(t-u)\,g(u)du; \quad f,\ g\in L^1$$

be its convolution which represents l by the function $\{1\}$. Mikusinski and Rill-Nardzewski [1] have studied this convolution in details. The operator l is a right inverse of the operator d/dt such that the function $y=lf$ is the solution of the Cauchy problem $y'=f$, $y(0)=0$ when $f\in L^1$.

Let now N be an arbitrary continuous linear functional in C, i. e. N belongs to the adjoint space C^*. It is known (see [8, 284]) that now N have compact support including in Δ. Let also l_0 denotes the right inverse of d/dt so that the function $y=l_0 f$ is the solution of the more general problem

$$(0.4)\qquad y'=f, \quad N(y)=0 \quad \text{for} \quad f\in L^1.$$

It is clear that this problem has unique solution for every $f\in L^1$ or L^1_{loc} iff $N(1)\neq 0$, and without loss of generality we assume that $N(1)=-1$. Now l_0 has the representation

$$(0.5)\qquad l_0 f=lf+N(lf).$$

The problem for existence of convolutions for an arbitrary continuous right inverse operator of d/dt is considered independently from L. Berg [2] and I. Dimovski [3; 4]. L. Berg has considered a convolution of the operator (0.5) in C. His convolution is a continuous bilinear operator $C\times C\to C$, which represents by

$$(0.6)\qquad f*_0 g=N_x\{\int_t^x f(x+t-u)g(u)du\}$$

for all f, $g\in C$. In the above formula the index x of N shows that the functional N is applied to the variable x. I. Dimovski has considered a convolution for the general continuous right inverse of $d/dt: lf+\Phi(f)$, $\Phi\in C^*$. His convolution exists in C^1, and it is identical to (0.6) when $\Phi(f)=N(lf)$, $N(1)=-1$. In this special case (0.6) is a continuation of Dimovski's convolution in C. We consider the general Dimovski's convolution in section 5.

In the present paper $f*_0 g$ denotes the operation (0.6) with an arbitrary continuous linear functional N in C, especially it is possible that $N(1)=0$. In the present paper is proved that the Berg-Dimovski convolution (0.3) can be continuated as continuous convolution of l_0 in L^1 or L^1_{loc} in the compact or noncompact case for Δ respectively. In section 1, 2 (where the main properties of $*_0$ are proved) for simplicity we consider the compact case $\Delta=[a, b]$. The noncompact case follows from these results if we consider the interval $[a, b]$ as an arbitrary compact subinterval of noncompact interval Δ. In this case in the formulations of the statements of sections 1, 2 L^p must be replaced by L^p_{loc}. The properties of $f*_0 g$ in some subspaces of L^1 are studied in section 1. Section 2 is devoted to the question of differentiability of $f*_0 g$. The main properties of $f*_0 g$ are collected in tables 1, 2. An analogous table for the convolution $f*g$ (a special case of (0.6) when $N(f)=-f(0)$) is obtained by Mikusinski and Rill-Nardzewki in [1]. Based on the properties of $f*_0 g$ in section 3 representation formulas for classes of $*_0$-multipliers and commuting operators with l_0 are found. In section 4 a representation formula for an arbitrary continuous convolution of l_0 by means of the convolution (0.6) is obtained and some statements about description of their divisors and nondivisors of zero are proved. In particular, a Titchmarsh type theorem for an arbitrary continuous convolution of integration operator l in $C[0, \infty)$ or $L^1_{loc}[0, \infty)$ is obtained. It is proved that every continuous convolution of l in $C[0, \infty)$ or $L^1_{loc}[0, \infty)$ has not divisors of zero. The isomorphism is also proved between the Mikusinski's rings for arbitrary continuous convolutions of l_0 in classes of subspaces of L^1 or L^1_{loc}.

The general Dimovski convolution for the operator $lf+\Phi(f)$, $\Phi\in C^*$ is extended to the space BV in section 5.

The symbol $=_{a.e.}$ denotes equality almost everywhere.

1. Continuation of the convolution $f*_0 g$ in L^1. L^p properties. Let us consider the bilinear expression

$$f\circ g(x, t)=\int_t^x f(x+t-u)g(u)du. \tag{1.1}$$

Obviously it exists if $f, g\in C$ and then $f\circ g(x, t)$ is a continuous function of two variables. However, it is not sure if $f\circ g(x, t)$ exists when $f, g\in L^1$. This problem will be solved in theorem 1.7. The question about the continuation of $f*_0 g$ in L^1 depends on the possibility to apply the functional N on the variable x in $f\circ g(x, t)$, since now it is not always a continuous function of x for a fixed t. Let now $\Delta=[a, b]$ be a compact segment.

Theorem 1.1. *If $f\in L^p$, $g\in L^q$ with $1\leq p\leq +\infty$, $1/p+1/q=1$ then $f\circ g(x, t)$ belongs to $C(\Delta^2)$ and*

$$\sup_{(x, t)\in\Delta^2} |f\circ g(x, t)|\leq\|f\|_p\|g\|_q. \tag{1.2}$$

Proof. If (x, t) is a fixed point of Δ^2 it is clear that the function of u, $r_{x,t}(u)=f(x+t-u)$ belongs to L^p in the segment with endpoints x, t which is contained in Δ. Then the function $r_{x,t}(u)g(u)$ is Lebesgue integrable in the same segment, i. e. $f\circ g(x, t)$ exists. Let h, k be such numbers that $(x+h, t+k)\in\Delta^2$. From Hölder's inequality one has $|f\circ g(x+h, t+k)-f\circ g(x, t)|$

$$\leq\|g\|_q\{|\int_{x+k}^{x+h+k}|f|^p|^{1/p}+|\int_{t+k}^{t+h+k}|f|^p|^{1/p}+|\int_t^x|f(u+h+k)-f(u)|^p|^{1/p}du\}$$

and we obtain the continuity of $f\circ g(x, t)$ by the fact that if $f\in L^1[c, d]$ then $\int_c^d|f(u+\theta)-f(u)|du$ tends to 0 when $\theta\to 0$. Thus (1.2) becomes evident.

The above theorem shows that if $f\in L^p$, $g\in L^q$ then $f*_0g(t)$ exists for each $t\in\Delta$ and it is a continuous function of t.

Theorem 1.2. a) *$f*_0g$ is a continuous bilinear operation $L^p\times L^q\to C$ when $1\leq p\leq +\infty$, $1/p+1/q=1$ and*

$$\|f*_0g\|_C\leq\|N\|\ \|f\|_p\|g\|_q, \tag{1.3}$$

b) $f*_0 g$ *is a continuous bilinear operation* $L^p \times L^p \to C$ *when* $2 \leq p \leq +\infty$ *and* $\| f*_0 g \|_C \leq m(\Delta)^{(p-2)/p} \| N \| \| f \|_p \| g \|_p$.

P r o o f. a) is evident. b) follows from the fact that $L^p \subset L^q$ when $p \geq q$ since Δ is compact. Then $\| g \|_q \leq m(\Delta)^{(p-2)/p} \| g \|_p$ for $g \in L^p$.

For the following considerations we need some facts from complex measure theory. It is known by the Riesz theorem that the functional N with compact support can be represented by the Riemann-Stieltjes integral

$$N(f) = (R.S)\int_a^b f(x)dn(x), \quad n \in BV. \tag{1.4}$$

We shall consider the complex-valued function $n(t)$ extended as $n(a)$ in $(-\infty, a)$ and as $n(b)$ in $(b, +\infty)$. The function n defines a complex regular Borel measure ν which support is compact, containing in Δ. The construction of ν may be different with respect to the approach of the integration theory. For our considerations the H e w i t t and R o s s aproach [5, ch. 3] or E d w a r d s approach [8, ch. 4] are most convenient. In their approach the measure ν is defined by a continuation of the functional $N(f) = (R.S)\int_{-\infty}^{+\infty} f(x)dn(x)$, where $f \in C(R^1)$. Let $|\nu|$ denote the total variation of the complex measure ν. The functional N extends to the space $L^1\{\Delta, |\nu|\}$ as Lebesgue-Stieltjes integral $\int_\Delta f d\nu$ and $\|N\| = |\nu|(\Delta)$. For this continuation we shall use also the notation $N(f)$. If λ and μ are complex measures, then the $\lambda \times \mu$ denotes their product. It is known that $|\lambda \times \mu| = |\lambda| \times |\mu|$ [5, 236]. We shall often use the notation $\int_c^d d\mu$ defined as $\int_{[c,d]} f d\mu$ when $a \leq c \leq d \leq b$ and by $-\int_{[d,c]} f d\mu$ when $a \leq d < C \leq b$.

L e m m a 1.3. *Let ν and μ are arbitrary full regular complex Borel maesures in R^1, let m be the Lebesgue measure, and let $\lambda = \nu \times \mu \times m$. Then the function $F(x, t, u) = f(x+t-u)$ is $|\lambda|$-measurable when $f \in L^1$.*

The lemma can be proved in a similar way as the analogous statements about the $\mu \times m$-measurability of the function $f(x-u)$, see [5, 367] and [6, 17].

L e m m a 1.4. *Let $f \in L^p$, $h \in L^q$ where $1 \leq p \leq +\infty$ $1/p + 1/q = 1$. Then the functions $\int_a^u f(x+t-u)h(t)dt$ and $\int_u^b f(x+t-u)h(t)dt$ are continuous in the sets $\{(x, u): x \in [a, b], u \in [a, x]\}$ and $\{(x, u): x \in [a, b], u \in [x, b]\}$ respectively.*

The proof is as in theorem 1.1 and will be omitted.

L e m m a 1.5. *Let $f \in L^1$ and $|\int_\Delta fh| \leq K \| h \|_q$ for each $h \in L^q$, $1 \leq q < +\infty$. Then $f \in L^p$ with $1/p + 1/q = 1$ and $\| f \|_p \leq k$.*

For a simple proof see [6, 24].

T h e o r e m 1.6. *Let μ be an arbitrary regular complex Borel measure in R^1. Then for each $f \in L^1$ the function*

$$Mf(t) = N_x\{\int_t^x f(x+t-u)d\mu(u)\} \tag{1.5}$$

exists m-almost everywhere in Δ and belongs to L^1. The operator M is a continuous linear operator from L^p to L^p, $1 \leq p \leq +\infty$ from BV to BV and

$$\| Mf \|_p \leq \| N \| \, |\mu|(\Delta) \| f \|_p, \quad f \in L^p, \tag{1.6}$$

$$\| Mf \|_{BV} \leq 2 \| N \| \, |\mu|(\Delta) \| f \|_{BV}, \quad f \in BV. \tag{1.7}$$

P r o o f. There are some differences between the cases $p=1$ and $1 < p \leq +\infty$, but we shall prove the theorem in both cases simultaneousy. Let us introduce the function h where $h\equiv 1$ when $p=1$ and h is an arbitrary function of L^q, $1/p+1/q=1$ *when* $1<p\leqq+\infty$ i. e. $1\leqq q<+\infty$. Let

$$D_1=\{(x,t):x\in[a,b],\ t\in[a,x]\},\quad D_2=\{(x,t):x\in[a,b],\quad t\in(x,b]\}.$$

Let us consider the integrals

$$I_1=\int_\Delta d|\nu|(x)\int_{[a,x]}dt\int_{[t,x]}|f(x+t-u)h(t)|\,d|\mu|(u)=\iint_{D_1}A_1(x,u)d|\nu|\times|\mu|,$$

$$I_2=\int_\Delta d|\nu|(x)\int_{(x,b]}dt\int_{[x,t]}|f(x+t-u)h(t)|\,d|\mu|(u)=\iint_{D_2}A_2(x,u)d|\nu|\times|\mu|,$$

where $A_1(x,u)=\int_a^u|f(x+t-u)h(t)|\,dt$, $A_2(x,u)=\int_u^b|f(x+t-u)h(t)|\,dt$ by lemma 1.4 are continuous functions in the compact sets D_1, $\bar{D}_2$. Hence the last integrals in the formulas for I_1, I_2 exist. Tonneli's theorem shows that $f(x+t-u)h(t)$ is $|\nu|\times m\times|\mu|$-integrable in the Borel sets $S_1=\{(x,t,u):(x,t)\in D_1,\ u\in[x,t]\}$, $S_2=\{(x,t,u):(x,t)\in D_2,\ u\in[t,x]\}$. Now from the complex measure Fubini's theorem [5, 237] we obtain that the functions

$$F_1(x,t)=h(t)\int_{[t,x]}f(x+t-u)d\mu(u),\quad F_2(x,t)=-h(t)\int_{[x,t]}f(x+t-u)d\mu(u)$$

are $|\nu|\times m$-integrable in D_1, D_2 respectively. Hence the function

$$F(x,t)=h(t)\int_t^x f(x+t-u)d\mu(u)=\begin{cases}F_1(x,t), & (x,t)\in D_1\\ F_2(x,t), & (x,t)\in D_2\end{cases}$$

is $|\nu|\times m$-integrable in Δ^2. It follows now that the function $h(t)Mf(t)=\int_\Delta F(x,t)d\nu(x)$ is m-integrable. Since $h\equiv 1$ when $p=1$ then $Mf\in L^1$ holds. Then it is easy to see that $\|Mf\|_1\leqq I_1+I_2\leqq\|N\|\,|\mu|(\Delta)\|f\|_1$. When $1<p\leqq+\infty$ we have that $|A_i(x,t)|\leqq\|f\|_p\|h\|_q$ and $|\int_\Delta h(t)Mf(t)dt|\leqq I_1+I_2\leqq\|N\|\,|\mu|(\Delta)\|f\|_p\|h\|_q$ hold for each $h\in L^q$, hence by lemma 1.5 $Mf\in L^p$ and (1.6) hold.

Let now $g\in BV$. Then the function $F(x,t)=\int_t^x f(x+t-u)d\mu(u)$ exists for each $(x,t)\in\Delta^2$ (see remark 1 after theorem 2.2). Let x_0 be an arbitrary point in $[a,b]$ and $\alpha=(t_0,\ldots,t_n)$ be an arbitrary subdivision of $[a,b]$. Let $\overset{b}{\underset{a}{S}}_\alpha F(x_0,t)=_{\text{def}}\sum_{k=0}^{n-1}|F(x_0,t_{k+1})-F(x_0,t_k)|$. We shall prove that $\overset{b}{\underset{a}{S}}_\alpha F(x_0,t)\leqq\|f\|_{BV}\|N\|\,|\mu|(\Delta)$ for each $x_0\in\Delta$. Let α' be the subdivision of $[a,b]$ obtained from α by addition of the point x_0. It is clear that

$$\overset{b}{\underset{a}{S}}_\alpha F(x_0,t)\leqq\overset{b}{\underset{a}{S}}_{\alpha'}F(x_0,t)\leqq\overset{b}{\underset{a}{S}}_\beta F(x_0,t)+\overset{b}{\underset{a}{S}}_\gamma F(x_0,t),$$

where $\beta=(\xi_0,\ldots,\xi_r)$, $a=\xi_0<\ldots<\xi_r=x_0$ and $\gamma=(\eta_0,\ldots,\eta_s)$, $x_0=\eta_0<\ldots<\eta_s=b$. When $x_0=a$ or $x_0=b$ we consider the corresponding β or γ empty and their sums equal to 0. By usual but tedious considerations it can be proved that $\overset{x_0}{\underset{a}{S}}_\beta F(x_0,t)\leqq V_a^{x_0}f\,.\,|\mu|(\Delta)+\|f\|_\infty|\mu|\,[a,x_0)$ and $\overset{b}{\underset{x_0}{S}}_\gamma F(x_0,t)\leqq V_{x_0}^b f\,.\,|\mu|(\Delta)+\|f\|_\infty|\mu|(x_0,b]$. Now it is clear that $\sum_{k=0}^{n-1}|Mf(t_{k+1})-Mf(t_k)|\leqq\|N\|\,|\mu|(\Delta)\|f\|_{BV}$ hence $Mf\in BV$ and $V_a^b Mf\leqq\|f\|_{BV}\|N\|\,|\mu|(\Delta)$. The theorem is proved.

It is true also that $M:C\to C$ when μ is a continuous measure but we shall not need this fact.

Theorem 1.7. a) *Let* $f, g \in L^1$. *Then* $fog(x,t) \in L^1\{\Delta^2, |\nu| \times m\}$, $f*_0 g$ *exists m-almost everywhere in* Δ *and*

b) *If* $f \in L^p$, $1 \leq p \leq +\infty$ *and* $g \in L^1$ *then* $f*_0 g \in L^p$.

c) *If* $f, g \in L^p$, $1 \leq p \leq +\infty$ *then* $f*_0 g \in L^p$.

d) *If* $f \in L^p, g \in L^q$ *and* $1 \leq p, q, r \leq +\infty$ *with* $1/p + 1/q = 1 + 1/r$, *then* $f*_0 g \in L^r$ *and*

$$\|f*_0 g\|_r \leq \|N\| \, \|f\|_p \, \|g\|_q. \tag{1.8}$$

e) $f*_0 g$ *is a continuous bilinear operation* $L^1 \times L^p \to L^p$, $L^p \times L^p \to L^p$ *and* $L^p \times L^q \to L^r$, *when* $1/p + 1/q = 1 + 1/r$.

Proof. Let μ_α be the complex measure defined by the absolutely continuous function $\alpha(t) = \int_0^t g$. Let us apply theorem 1.6 to the operator $Mf = N_x\{\int_t^x f(x+t-u)d\mu_\alpha(u)\} = N_x\{\int_t^x f(x+t-u)g(u)du\}$. Then we obtain a), b), c). By (1.6) it is not difficult to prove the inequalities $\|f*_0 g\|_p \leq \|N\| \, \|f\|_p \, \|g\|_1$ for $f \in L^p$, $g \in L^1$ and $\|f*_0 g\|_p \leq m(\Delta)^{1-1/p} \|N\| \, \|f\|_p \, \|g\|_p$ for $f, g \in L^p$. On the basis of the first inequality d) can be proved by the Riesz — Thorin theorem as for the usual convolutions in [10, 142]; e) is evident.

Theorem 1.8. *The operation* $f*_0 g$ *is bilinear, commutative and associative in* L^1. *If* $fog(x,t) \in L^1\{\Delta^2, |\nu| \times |\nu|\}$ (*this is true for example when* $fog \in C(\Delta^2)$ *as in theorem* 1.1 *or when* N *is as in theorem* 2.7), *then*

$$N(f*_0 g) = 0. \tag{1.9}$$

If $N(1) = -1$ *then a* $f*_0 g$ *is a convolution for* l_0 (0.2) *and*

$$l_0 f = 1*_0 f. \tag{1.10}$$

Proof. The commutativity can be easily obtained by change of variable. Now we shall prove that for all $f, g, h \in L^1$ a. e. holds

$$(f*_0 g)*_0 h = f*_0(g*_0 h). \tag{1.11}$$

Let α, β are different complex numbers. It is easy to see that $e^{\alpha t} *_0 e^{\beta t} = (\alpha-\beta)^{-1} N_x\{e^{\alpha x+\beta t} - e^{\beta x+\alpha t}\}$ and $[e^{i\xi t} *_0 e^{i\eta t}] *_0 e^{i\zeta t} = e^{i\zeta t} *_0 [e^{i\eta t} *_0 e^{i\zeta t}]$ for all pairwise different real numbers ξ, η, ζ. From the continuity of the operation $*_0 : L^1 \times L^\infty \to C$ it follows that the above formula holds everywhere when certain pairs of ξ, η, ζ are equaltoo. Since the linear span of the set $\{e^{i\xi t} : \xi \in R^1\}$ is dense in L^2 and $*_0 : L^2 \times L^2 \to C$ is continuous bilinear operation then it follows that (1.11) holds everywhere for functions of L^2. Now if we use the density of L^2 in L^1 and the continuity $*_0 : L^1 \times L^1 \to L^1$ we obtain that (1.11) holds a. e. for $f, g, h \in L^1$.

When $fog(x, t) \in L^1\{\Delta^2, |\nu| \times |\nu|\}$, from the obvious equality $fog(x, t) = -fog(t, x)$ and Fubini's theorem, it follows that

$$N(f*_0 g) = N_t N_x\{fog(x, t)\} = -N_x N_t\{fog(t, x)\} = -N(f*_0 g).$$

Hence $N(f*_0 g) = 0$. The idea of the last proof is due to I. Dimovski. Now when $N(1) = -1$ we have

$$1*_0 f = N_x\{\int_t^x f\} = N_x[lf(x) - lf(t)] = -N(1)lf(t) + N(lf) = l_0 f.$$

Remark. The equality (1.11) holds everywhere when $f \in L^p$, $g \in L^q$, $1/p + 1/q = 1$, $h \in L^1$.

2. **Differential properties of** $f*_0g$.

Theorem 2.1. *If $f\in AC$, $g\in L^1$ then $f*_0g\in AC$ and*

$$(f*_0g)'=_{\text{a.e.}}f'*_0g-N(f)g, \tag{2.1}$$

$$\|f*_0g\|_{AC}\leq K\|f\|_{AC}\,\|g\|_1. \tag{2.2}$$

The operation $*_0$ is continuous $AC\times L^1\to AC$.

Proof. Let $f\in AC$ and $N(1)=-1$. Then $f=lf'+f(0)=l_0f'-N(f)$ and from (1.10) it follows that $f*_0g=l_0[f'*_0g-N(f)g]$, i. e. $f*_0g\in AC$ and (2.1) holds. For the special case $N(f)=-f(0)$ it follows that (2.1) is true for the convolution $f*g$. When N is an arbitrary continuous linear functional then $N(f)=\widetilde{N}(f)+[1+N(1)]f(0)$, where $\widetilde{N}(f)=N(f)-[1+N(1)]f(0)$ and $\widetilde{N}(1)=-1$. Therefore $f*_0g=f\widetilde{*_0}g-[1+N(1)]f*g$, where $f\widetilde{*_0}g=\widetilde{N}_x\{f\circ g(x,t)\}$. Now we can apply (2.1) on $f\widetilde{*_0}g$ and $f*g$. Hence (2.1) holds and in the general case; (2.2) can be proved directly by (2.1) and the definition of $\|\cdot\|_{AC}$.

It is also true that if $f\in L^1$ and $g\in BV$ the convolution $f*_0g\in AC$, but this fact is not evident. It is useful for our considerations to define Riemann-Stieltjes integral when its upper limit is less or equal to the lower limit. Let f, g be two functions so that $(R.S)\int_c^d fdg$ exists (for the definition see [7]). We use the notations

$$(R.S)\int_c^c fdg=_{\text{def}}(R\cdot S)\int_c^{c+0}fdg \quad\text{and}\quad (R.S)\int_c^d fdg=-(R.S)\int_d^c fdg \text{ when } d<c.$$

By this definition the formula for integration by parts holds when $d\neq c$. It is a well-known fact that if $f\in C$, $g\in BV$ then

$$(R.S)\int_c^c gdf=0 \quad\text{and}\quad (R.S)\int_c^c fdg=f(c)[g(c+0)-g(c)].$$

From (2.1) it follows that if $f\in AC$ and $g\in BV$ then $f*_0g=\int_0^t(f'*_0g)(v)dv-N(t)\int_0^t g+N(f*g)$. The last formula becomes

$$f*_0g=\int_0^t N_x\{(R.S)\int_v^x g(x+v-u)df(u)\}-N(f)\int_0^t g+N(f*g). \tag{2.3}$$

Theorem 2.2. *Let $f\in C$ and $g\in BV$, then $f*_0g$ is represented by the formula (2.3) and $f*_0g\in AC$. The function $N_x\{(R.S)\int_t^x g(x+t-u)df(u)\}$ belongs to C and $(f*_0g)'$ exists except may be in the countable set of the points of discontinuity of the function g; further on we have*

$$(f*_0g)'=N_x\{(R.S)\int_t^x g(x+t-u)df(u)\}-N(f)g(t). \tag{2.4}$$

The derivative $(f*_0g)'$ belongs to $C+BV$.

Proof. Let $F(x,t)=\int_t^x g(x+t-u)df(u)$. If $f\in C^1$ it is clear that $F=f'\circ g$ and by theorem 1.1 $F\in C(\Delta^2)$. Let now $f\in C$, $f_n\in C^1$, $f_n\xrightarrow{c}f$ and $F_n(x,t)=\int_t^x g(x+t-u)df_n(u)$. Then for a fixed $(x,t)\in\Delta^2$, $x\neq t$ the relations

$$F_n(x,t)=g(t)f_n(x)-g(x)f_n(t)+\int_t^x f_n(x+t-u)dg(u) \quad \text{and}$$

$$F(x,t)=g(t)f(x)-g(x)f(t)+\int_t^x f(x+t-u)dg(u)$$

hold.

However $F_n(x,x)=F(x,x)=0$, hence for each $(x,t)\in\Delta^2$:

$$|F_n(x,t)-F(x,t)|\leq 2\|g\|_\infty\|f_n-f\|_C+\bigvee_a^b g\,.\,\|f_n-f\|_C\,.$$

Therefore, F_n tends uniformly to F, i. e. $F\in C(\Delta^2)$ and (2.3) holds for $f\in C$. We note that the right and the left derivative of $\int_0^t g$ exist at a point of discontinuity t_0 of g and they are equal to $g(t_0-0)$ and $g(t_0+0)$, respectively. The theorem is proved.

Remark 1. It is easy to see that if $g\in BV$ then $g\in L^1\{\Delta,|\nu|\}$ for each complex regular Borel measure, and the functions of t: $\int_{[a,t)}gd\nu$ and $\int_{[t,b]}gd\nu$ belong to BV. We note that the first statement can be easily proved for the jump functions.

Remark 2. We need also the connection between Riemann-Stieltjes and Lebesgue-Stieltjes integral. Let $f\in C$, $g\in BV$ and let μ_g be the complex measure in R^1 defined by g. It is clear that $(R.S)\int_a^b fdg=\int_{[a,b]}fd\mu_g$ since g is extended as $g(a)$ in $(-\infty,a)$ and as $g(b)$ in $(b,+\infty)$ (for details see [8, 261—263, 275—278]). The above formula is not true in general when g is discontinuous and $[a,b]$ is replaced by a proper subsegment $[c,d]$. We shall note that now the formula

$$(2.5)\qquad (R.S)\int_c^d fdg=\int_{[c,d]}fd\mu_g-[g(d+0)-g(d)]f(d)-[g(c)-g(c-0)]f(c)$$

is true.

Lemma 2.3. *If $f\in C$ and $g\in BV$ then for each $(x,t)\in\Delta^2$*

$$(2.6)\qquad (R.S)\int_t^x g(x+t-u)df(u)=\int_t^x f(x+t-u)d\mu_g(u)+G(x,t)$$

holds, where

$$G(x,t)=\begin{cases} g(t-0)f(x)-g(x+0)f(t), & a\leq t\leq x\leq b.\\ g(t+0)f(x)-g(x-0)f(t), & a\leq x<t\leq b.\end{cases}$$

If $f\in L^1$, $g\in BV$ then $G(x,t)\in L^1\{\Delta^2,|\nu|\times m\}$.

Proof. Through integration by parts when $x\neq t$ we obtain

$$\int_t^x g(x+t-u)df(u)=g(t)f(x)-g(x)f(t)+\int_t^x f(x+t-u)dg(u),$$

which together with (2.5) settles the cases $x<t$ and $t<x$. If $t=x$ then the left hand side of (2.6) is 0. The right hand side is also 0, since now $\int_t^t f(2t-u)d\mu_g(u)=f(t)[g(t+0)-g(t-0)]$. Now it is easy to see that if $f\in L^1$, $g\in BV$ then the functions $F_1(x,t)=g(t-0)f(x)-g(x+0)f(t)$, $F_2(x,t)=g(t+0)f(x)$

$g(x-0)f(t)$ belong to $L^1\{\Delta^2, |\nu|\times m\}$. For example $g(x+0)\in L^1\{\Delta, |\nu|\}$ by Remark 1, $f(t)\in L^1\{\Delta, m\}$, hence $g(x+0)f(t)\in L^1\{\Delta^2, |\nu|\times m\}$. Then the function $G(x,t) = F_1(x, t)$ for $(x, t)\in D_1$, $G(x, t)=F_2(x, t)$ for $(x, t)\in D_2$ is of $L^1\{\Delta^2, |\nu|\times m\}$ since F_1 and F_2 are $|\nu|\times m$-integrable in D_1, D_2, respectively (for the definition of D_1, D_2 see theorem 1.6).

Theorem 2.4. *If $f\in L^1$, $g\in BV$, then $f*_0g\in AC$ and*

$$f*_0g=\int_0^t N_x\{\int_v^x f(x+v-u)d\mu_g(u)\}\,dv \tag{2.7}$$

$$-\int_0^t f(v)\{\int_{[a,v)} g(x-0)d\nu(x)+\int_{[v,b]} g(x+0)d\nu(x)\}dv+N(f*g).$$

The formula

$$(f*_0g)'=N_x\{\int_t^x f(x+t-u)d\mu_g(u)\}-f(t)\{\int_{[a,t)} g(x-0)d\nu(x)+\int_{[t,b]} g(x+0)d\nu(x)\} \tag{2.8}$$

holds almost everywhere. The convolution $f*_0g$ is a continuous bilinear operation $L^1\times BV\to AC$ and $\|f*_0g\|_{AC}\leq K\|N\|\,\|f\|_1\,\|g\|_{BV}$. When the functions g and n (in the representation of N (1.4)) have no common points of discontinuity then

$$N(g)=\int_\Delta g\,d\nu=\int_{[a,t)} g(x-0)d\nu(x)+\int_{[t,b]} g(x+0)d\nu(x). \tag{2.9}$$

Proof. First let $f\in C$. From (2.3) and (2.6) it follows:

$$f*_0g=\int_0^t N_x\{\int_v^x f(x+v-u)\,d\mu_g(u)-G(x,v)\}dv-N(f)\int_0^t g+N(f*g)$$

$$=\int_0^t N_x\{\int_v^x f(x+v-u)d\mu_g(u)\}dv+\int_v^t F(v)dv+N(f*g),$$

where

$$F(t)=N_x\{G(x,t)\}-g(t)N(f)=\int_{[a,t)}[g(t+0)f(x)-g(x-0)f(t)]d\nu(x)$$

$$+\int_{[t,b]}[g(t-0)f(x)-g(x+0)f(t)]d\nu(x)-g(t)\int_{[a,t)} f(x)d\nu(x)-g(t)\int_{[t,b]} f(x)d\nu(x)$$

$$=[g(t+0)-g(t)]\int_{[a,t)} f\,d\nu+[g(t-0)-g(t)]\int_{[t,b]} f\,d\nu$$

$$+f(t)[\int_{[a,t)} g(x-0)d\nu(x)+\int_{[t,b]} g(x+0)d\nu(x)].$$

However, $g(t-0)=g(t)=g(t+0)$ m-almost everywhere, hence (2.7) holds for $f\in C$. Since the operators of the type (1.5) are continuous in L^1 we obtain that (2.7) holds for $f\in L^1$. If g and n have no common points of discontinuity, then the countable set of discontinuity points of g has $|\nu|$-measure 0. Hence $g(t-0)=g(t)=g(t+0)$, $|\nu|$-almost everywhere and (2.9) holds.

Remark. It is clear that theorems 2.2, 2.4 are also true when $g\in L^1$ and g is identical a. e. to $\tilde{g}\in BV$. Indeed now $f*_0g=f*_0\tilde{g}$. However, in the formulas (2.4), (2.8) we must replace g with $\tilde{g}$.

As immediate consequence of theorem 2.4 we obtain a result for the usual convolution $f*g$. It happens that this result is a generalization of the theorem for differentiation of $f*g$ proved by Mikusinski and Rill-Nardzewski [1] when $f\in C$, $g\in BV$.

Corollary 2.5. *Let* $f\in L^1$, $g\in BV$. *Then*:

a) *If* $\Delta=[a, b]$, $a\leq 0<b$ *then*

$$f*g=\int_0^t\{\int_0^v f(v-u)d\mu_g(u)\}dv+\int_0^t f(v)H(v)dv, \tag{2.10}$$

where $H(t)=g(0+)$ in $[a, 0)$ *and* $H(t)=g(0-)$ in $[0, b]$.

b) *If* $\Delta=[0, b]$ *then*

$$(f*g)'=_{\text{a.e.}}\int_0^t f(t-u)d\mu_g(u)+g(0)f(t). \tag{2.11}$$

Proof. It is clear that $f*g=f*_0g$ with $N(f)=-f(0)=-\delta_0(f)$ hence now $N(f*g)=0$. Using (2.7) and the obvious equality

$$\widetilde{H}(t)=_{\text{def}}-\int_{[a,t)}g(x-0)d\delta_0(x)-\int_{[t,b]}g(x+0)d\delta_0(x)=\begin{cases}g(0+), & a\leq t\leq 0,\\ g(0-) & 0<t\leq b,\end{cases}$$

we obtain (2.10), where H is replaced by $\widetilde{H}$. Since $H=_{\text{a.e.}}\widetilde{H}$ (2.10) holds.

From the theorems 1.1—1.8 and 2.1—2.4 many consequences can be obtained. In Table 1 (on p. 229) we formulate the properties of $f*_0g$, when N is an arbitrary continuous linear functional in C. In this table Lip denotes the space of Lipschitz functions.

When some of the spaces C, BV, AC, Lip has index N it means, that the functions f of the corresponding space satisfy the condition $N(f)=0$. We denote $C_N^k=_{\text{def}}\{f\in C^k: N(f)=\ldots=N(f^{(k-1)})=0\}$ and $C_{N,p}^k=_{\text{def}}\{f\in C^k: N(f)=\ldots=N(f^{(p-1)})\}=0$, $0\leq p\leq k$. Obviously $C_N^k=C_{N,k}^k$.

The conditions imposed on the functions f, g in theorem 2.1—2.4 are sufficient, but not necessary for the differentiability of $f*_0g$ (see theorem 2.7). We can give some necessary conditions for the differentiability of $f*_0g$ when the functional N is of the form $N(f)=kf(x_0)+\int_a^b f(x)\alpha(x)dx$ with an arbitrary $\alpha\in BV$, $k\neq 0$, $x_0\in\Delta$, (theorem 2.9). There exist functionals N for which $f*_0g$ is differentiable always when f, $g\in L^1$. In the following considerations a class of such functionals is given.

Lemma 2.6. *The functional* N *is of the type* $N(f)=\int_a^b f(x)\alpha(x)dx$ *with* $\alpha\in BV$ *iff* $N(f)=F(lf)$ *where* F *is a continuous linear functional in* C. *A functional representing* N *is*

$$F_0(f)=\alpha(b)f(b)-\alpha(a)f(a)-\int_a^b f d\alpha,\quad F_0(1)=0. \tag{2.12}$$

Both functionals F_1, F_2 *for which* $F_1(lf)=F_2(lf)$ *are connected with the identity* $F_1(f)=f(0)[F_1(1)-F_2(1)]+F_2(f)$ *for each* $f\in C$.

The proof of the lemma is trivial and will be omitted.

Theorem 2.7. *Let* N *be of the type* $N(f)=\int_a^b f(x)\alpha(x)dx$, $\alpha\in BV$, *i. e.* $N(f)=F(lf)$ *with* F *given by Lemma* 2.6. *Now if* f, $g\in L^1$ *then* $f*_0g\in AC$, *and*

Table 1

N is an arbitrary continuous linear functional C							
f	g	$f *_0 g$	$(f *_0 g)'$	f	g	$f *_0 g$	$(f *_0 g)'$
L^1	L^1	L^1		L^1	AC	AC	L^1
L^p	L^1	L^p		L^∞	AC_N	C^1	C
L^p	L^p	L^p		AC	AC	C^1	AC
L^p	L^q	C		AC_N	AC_N	C^2	C^1
$1/p + 1/q = 1$				L^1	Lip_N	C^1	C_N
L^p	L^p	C		L^1	C^k_N	C^k_N	
	$p \geq 2$			$1 \leq k \leq +\infty$			
L^p	L^q	L^r		C^k	C^k	C^k	
$1/p + 1/q = 1 + 1/r$				$1 \leq k \leq +\infty$			
L^1	BV	AC	L^1	$C^m_{N,p}$	$C^n_{N,q}$	C^r	
L^p	BV	AC	L^p	$r = \min(m+n, m+q+1, m+p+1)$			
C	BV	AC	$C+BV$				
C_N	BV	C^1	C_N				
C	$BV \cap C$	C^1	C				
BV	BV	AC	BV				
AC	BV	AC	BV				

(2.13) $$(f*_0 g)' =_{a.e.} F_x\{f \circ g(x,t)\} + F(1) f*g - f.N(g) - g.N(f),$$

where F denotes also the Lebesgue continuation applied on the variable x

Proof. Let $f \in C^1$, $g \in C$. Now $\partial/\partial t\,[f \circ g(x,t)] = -f(t)g(x) - f(x)g(t) + \partial/\partial x[f \circ g(x,t)]$. By an elementary theorem for differentiation under the integral sign [11, p. 663]) and lemma 2.6 we obtain

$$(f*_0 g)' = \frac{\partial}{\partial t}\int_a^b f \circ g(x,t)\alpha(x)dx = \int_a^b \alpha(x)\frac{\partial}{\partial t} f \circ g(x,t)dx$$

$$= \int_a^b \alpha(x)[-f(t)g(x) - g(t)f(x) + \frac{\partial}{\partial x} f \circ g(x,t)]dx = A(f,g),$$

where $A(f,g) =_{def} -f(t)N(g) - g(t)N(f) + F_x\{f \circ g(x,t)\} + F(1) f*g$, hence

(2.14) $$f*_0 g = \int_0^t A(f,g)(u)du + N(f*g).$$

Now if F denotes the Lebesgue continuation, then by theorem 1.7 $A(f,g)$. exists for $f, g \in L^1$, and it is continuous bilinear operation $L^1 \times L^1 \to L^1$. By an approximation we conclude that (2.14) holds in L^1. The theorem is proved.

Corollary 2.8. *Let the functional N be as in the previous theorem. Now if $f, g \in C$, then $f*_0 g \in C^1$ and $(f*_0 g)'$ is given by* (2.13).

Many conclusions from (2.13) are given in table 2.

Theorem 2.9. *Let $N(f) = kf(x_0) + \int_a^b f(x)\alpha(x)dx$ with $\alpha \in BV$, $x_0 \in I$ and a constant $k \neq 0$. Then*

a) *If $g \in L^1$ and $f*_0 g \in AC$ for each $f \in L^1$ then $g =_{a.e.} \tilde{g} \in BV$.*

b) *If $g \in L^1$ and $f*_0 g \in L^1$ for each $f \in C$ then $g =_{a.e.} \tilde{g} \in BV \cap C$.*

c) *If $\alpha \equiv 0$ and if $g \in L^1$ and $f*_0 g \in C^1$ for each $f \in C_N$, then $g =_{a.e.} \tilde{g} \in BV$ and $N[(f*_0 g)'] = 0$.*

Proof. In the case $N(f) = -f(0)$ the proof can be found in [9]. In the more general case, when $N(f) = -f(x_0)$ it is easy to see that $f*_0 g = T_{x_0}^{-1}(T_{x_0} f * T_{x_0} g)$, where $T_{x_0}: L^1[a,b] \to L^1[a-x_0, b-x_0]$ is the shift operator $T_{x_0} f(t) = f(x_0+t)$, and the problem can be reduced to the case $x_0 = 0$. Thus the theorem is established for the functionals of Dirak type $N = \delta_{x_0}$. Now to obtain a) in the case when $N(f) = f(x_0) + F(lf)$ we note that now $f*_0 g = \bar{f*_0} g + \bar{\bar{f*_0}} g$, where $\bar{\bar{f*_0}} g = F_x\{l_x f \circ g(x,t)\} \in AC$ by theorem 2.7., and hence $\bar{f*_0} g =_{def} \delta_{x_0,x}\{f \circ g(x,t)\} \in AC$ for each $f \in L^1$. Therefore $g =_{a.e.} \tilde{g} \in BV$.

The case b) is similar one, but now corollary 2.8 can be used.

Theorem 2.10. a) *If N is an arbitrary continuous linear functional in C, then the operation*

(2.15) $$f \tilde{*}_0 g = \frac{d}{dt}(f*_0 g)$$

is a continuous convolution with a unit element—the function {1} for the operator l_0 in the spaces BV, AC, C^k, $1 \leq k \leq \infty$.

b) *If N is of the type $N(f) = \int_a^b f(x)\alpha(x)dx$, with $\alpha \in BV$, i.e. if $N(f) = F(lf)$, $F \in C^*$, then* (2.15) *is a continuous convolution for l_0 with a unit element—the function {1} in the spaces L^1, L^p, $1 \leq p \leq \infty$, C and BV, and it can be expressed by* (2.13).

3. Classes of multipliers of $*_0$ and operators commuting with l_0. In this section we find classes continuous operators, commuting with l_0 in some subspaces of L^1 or L^1_{loc} in the compact or noncompact case for Δ, respectively.

Lemma 3.1. a) *The linear span of the set $S=\{l_0^k(1)\}_{k=0}^{\infty}$ is dense in C.*

b) *Let $\psi_k=l_0^k(1)$ for a fixed $1\leq k<+\infty$. Then the linear span of the set $S_k=\{l_0^n(\psi_k)\}_{n=0}^{\infty}$ is dense in C_N^k.*

c) *The space C_N^1 is dense in C_N relative the topology of C.*

d) *If $M:C_N\to C$ is a continuous operator commuting with l_0 in C_N then $M(C_N)\subset C_N$.*

Proof. a) holds since the linear span of S is the set of all polynomials. To prove b) we note that f is of the linear span of S_k iff $f=l_0^k(p)$, where p is a polynomial. Also the functions of C_N^k are in the form $l_0^k(g)$, $g\in C$. The proof of the last statement can be obtained by induction. Let now $f\in C_N$. There exists a sequence of polynomials $P_n\overset{C}{\to}f$. Then $Q_n=p_n-N(p_n)\overset{C}{\to}f-N(f)=f$ and $Q_n\in C_N^1$ since $N(Q_n)=0$ and c) is proved; d) follows from the density of C_N^1 in C_N, and from the fact that $Mf=M\,l_0g=l_0Mg$ for $f\in C_N^1$, i. e. $N(Mf)=0$ for $f\in C_N^1$.

The main purpose of the next considerations is to find representation formulas for classes of operators commuting with l_0. The main theorem is proved by many assumptions, but it is an useful instrument to find a series representation theorem in some subspaces of L^1 and L^1_{loc}. We shall consider only the compact case. The noncompact case can be delt with in the same way.

Let $X\subset Y$ be subalgebras of $L^1(+,*_0,.)$ and let X and Y be Banach or Fréchet spaces relative to the linear operations. The identity of the functions in these spaces is understood in the sense of identity almost everywhere. Let X be an ideal in Y, and let $f*_0g$ be a continuous bilinear operation $X\times X\to X$ and $X\times Y\to X$ relative to the topologies of X and Y. We shall assume for simplicity that $N(1)=-1$.

Theorem 3.1. *Let X contain all polynomials and let they be dense in X. Then $M:X\to Y$ is a continuous linear operator commuting with l_0 in X iff for all $f,g\in X$:*

$$M(f*_0g)=_{\text{a.e.}}Mf*_0g=_{\text{a.e.}}f*_0Mg. \tag{3.1}$$

Let $m_0=M(1)$. Then for each $f\in X$ we have $m_0*_0f\in AC_N$, $(m_0*_0f)'\in Y$ and

$$Mf=_{\text{a.e.}}d/dt\,[m_0*_0f]. \tag{3.2}$$

Conversely, each $m_0\in Y$ with the properties $m_0*_0f\in AC_N$, $(m_0*_0f)'\in Y$ for each $f\in X$, defines by (3.2) a continuous linear operator $M:X\to Y$, commuting with l_0.

An equivalent form of the representation (3.2) *is*

$$Mf=_{\text{a.e.}}d^{k+1}/dt^{k+1}[m_k*_0f], \tag{3.3}$$

where $m_k=Ml_0^k(1)$, $k=1,2,\ldots$. If Y is subalgebra of C then (3.1)—(3.3) *hold everywhere. If Y is subalgebra of BV then* (3.1)—(3.3) *hold everywhere except may be on countable set.*

Table 2

N is of the type $N(f)=F(lf)$, $F \in C^*$.									
f	g	$f *_0 g$	$(f *_0 g)'$	$(f *_0 g)''$	f	g	$f *_0 g$	$(f *_0 g)'$	$(f * g)''$
L^1	L^1	AC	L^1		C	C	C^1	C	
L^p	L^p	AC	L^p		C	BV	AC	$BV+C$	
L^p_N	L^1	AC	L^p		C_N	BV	C^1	C	
L^p_N	L^p_N	C^1	C		C	BV_N	AC	BV	
$1/p+1/q=1$					C_N	BV_N	C^1	AC	$BV+C$
L^p_N	L^p_N	C^1	C		C_N	$BV_N \cap C$	C^2	C^1	C
	$p \geqq 2$				C	AC	C^1	C	
L^1	C_N	C^1	C		C	AC_N	C^1	AC	L^1
L^1	BV_N	AC	BV	L^1	BV	AC	C^1	BV	L^1
L^1	AC_N	C^1	AC	L^1	BV	AC_N	C^1	AC	BV
L^1_N	BV_N	C^1	AC	L^1	AC	AC	C^1	AC	L^1
L^p_N	BV_N	C^1	AC	L^p	AC_N	AC_N	C^2	C^1	AC
BV	BV	AC	BV	L^1	BV_N	BV_N	C^1	AC	BV

First we shall make the following remarks: The algebras L^1, X, Y do not have unit elements relative to the convolution $*_0$, since if $l *_0 f = f$ for each $f \in X$ then $l(e) = 1 *_0 l = 1$, hence $e = \{1\}' = 0$. Now the operator $l_0 f = 1 *_0 f$ is a continuous one from X to X and from Y to X. Also $l_0(X) \subset X$.

As usual we consider the spaces $X, Y, C, BV, AC, \ldots$ as subspaces of L^1 consisting of the classes of functions which are identical a. e. to the functions of $X, Y, C, BV, AC, \ldots$, respectively. When we use the notation f' for a class of integrable functions $f \in L^1$ it means that f has a differentiable element $\tilde{f}$ with integrable $\tilde{f}'$, determining the class $f' \in L^1$. This is the sense in which we use the derivative in (3.2), since as we shall see from the proof, $m_0 *_0 f$ is equal a. e. to the absolutely continuous function $l_0 Mf$. We can avoid this complication if we use the representation (3.3) for $k=1$ since now $l_0^2Mf=l_0m_0*_0f=m_1*_0f$ everywhere. As we shall see also from the proof if $X*_0Y\subset C$ then $l_0Mf=m_0*_0f$ everywhere.

Proof. Let M be continuous linear operator commuting with l_0 in X. From the obvious identity $M1*_01=1*_0M1$ it follows that $Ml_0^n(1)*_0l_0^m(1) = l_0^n(1)*_0Ml_0^m(1)$ for $n, m=0, 1, \ldots$, hence

$$Mf*_0g=f*_0Mg, \tag{3.4}$$

when f, g are polynomials. By approximation we obtain that (3.4) holds in X. Let $\psi=M(f*_0g)-f*_0Mg$, then by the remark after theorem 1.8 it follows that $l_0^2\psi = 1*_0M(l_0f*_0g)-1*_0l_0(Mf*_0g)=0$ is true everywhere since l_0f is continuous function. Hence $\psi=_{\text{a.e.}}0$. Conversely if M is an operator from X to Y which satisfies (3.4) from the above formula it follows that M satisfies (3.1). In a similar way as in [12, p. 20—21] it can be proved that M is a linear closed operator from X to Y. Hence M is continuous.

From (3.1) we obtain that $l_0Mf=1*_0Mf=_{\text{a.e.}}M(1)*_0f$ and (3.2) holds in the sense of the remark to the proof of the present theorem. The last equality holds everywhere when $X*_0Y\subset C$. Obviously if $Y\subset C$ then (3.2) holds everywhere. Conversely if $m_0\in Y$ so that $m_0*_0f\in AC_N$, $(m_0*_0f)'\in Y$ for each $f\in A$ and M is defined by (3.2), then we can obtain directly that $l_0Mf=m_0*_0f=Ml_0f$ and by the closed graph theorem we can conclude that $M: X\to Y$ is continuous.

The above theorem is convenient for applications in the cases when X and Y are some of the spaces L^1, L^p, C, BV, AC, or when $X=Y=C^k$, $1\leq k\leq+\infty$. By theorems 2.6 and 2.8 we can obtain a complete description of commuting operators, for example in the cases $X=Y=L^1$, C, C_N when $N(f) =kf(x_0)+\int_b^a f.\alpha$ with $\alpha\in BV, x_0\in\Delta, k\neq 0$, or in the cases $X=Y=L^1$, C when $N(f) =\int_a^b f.\alpha$, $\alpha\in BV$.

For the general case with an arbitrary functional $N\in C^*$ we obtain by theorem 2.4 a complete description of the commuting operators and $*_0$-multipliers, for example in the cases $X=Y=BV$; $X=BV$, $Y=L^p$, $1\leq p\leq+\infty$; $X=Y=L^1$, $M(X)\subset BV$. Also other results can be obtained. In all above cases of complete description, the commuting operators can be represented by (3.2) and (3.3) in the forms (2.1), (2.4) or (2.8). See also tables 3, 4.

Representation formulas can also be obtained for the commuting operators in the case when it is not sure that X contains all polynomials, but $C_N^k\subset C$ for some $1\leq k<+\infty$. We note that there are spaces (for example C^∞) which contain all polynomials but do not contain C_N^k. Now it is clear that the operator $l_0^{k+1}f=l_0^k(1)*_0f$ is continuous from X to X and from Y to X.

Theorem 3.2. *Let $C_N^k\subset X$ for a fixed k and $l_0(X)\subset X$. Then $M: X\to Y$ is a continuous linear operator commuting with l_0 in X iff* (3.1) *holds. Let $m_k=Ml_0^k(1)$. Then for each $f\in X$ we have $m_k*_0f\in C_N^k$, $(m_k*_0f)^{(k)}\in AC_N$, $(m_k*_0f)^{(k+1)}\in Y$ and* (3.3) *holds. Conversely, each $m_k\in Y$ satisfying the above properties defines by* (3.3) *a continuous linear operator from X to Y commuting with l_0.*

The proof is similar as in theorem 3.1. First using lemma 3.1 b) from the obvious identity $M\psi_k*_0\psi_k=\psi_k*_0M\psi_k$ we prove that (3.4) holds in C_N^k. Then we form the function $\psi=Mf*_0g-f*_0Mg$ and prove that $l_0^{2k+2}\psi=0$, hence (3.4) holds. The other part of the theorem we prove as in theorem 3.2, but now we must use l_0^k instead of l_0.

Table 3

X	Y	To be $M : X \to V$ (defined by (3.2)) continuous operator, commuting with l_0, it is		Now (3.2) can be expressed by
N is an arbitrary functional of C^*				
		enough $m_0 = M(1)$ to be in	necessary and sufficient $m_0 = M(1)$ to be in	
L^1	L^1	BV	unknown	
L^p	L^1	BV	unknown	
L^p	L^p	BV	unknown	
C	L^1	BV	unknown	
C	C	$BV \cap C$	unknown	
C_N	C	BV	unknown	
C_N	C_N	BV	unknown	
BV	L^1	L^1	L^1	(2.8)
BV	L^p	L^p	L^p	(2.8)
BV	BV	BV	BV	(2.8)
AC	L^1	L^1	L^1	(2.1)
AC	C	C	C	(2.1)
AC	BV	BV	BV	(2.1)
C^k	C^k	C^k	C^k	(2.1)

The idea to represent the multipliers in this way is due to I. Dimovski [12]. The results of the present section can be formulated in the form of the representation of the operators, commuting with d/dt in invariant a hyperplane $N(f)=0$ as it is done in [14].

Remark. If $1 \in X$ and X is as in theorem 3.2 or 3.3, then l_0 is continuous operator from X to X, from Y to X and $\{f \in AC : f' \in Y\} \subset X$ since $f = l_0 f' + N(f)$. In the case $Y = L^1$ when $1 \in X$ then $AC \subset X$, hence $C^1_N \subset X$ and X satisfies at least the condition of theorem 3.3. We note that now X contains all polynomials, but their density relative the topology of X is not sure. When $Y = L^1$ and $l_0^k(1) \in X$ for some $k : 1 \leq k < \infty$ then it is easy to see that $C_N^{k+1} \subset X$ hence X satisfies also the condition of theorem 3.3.

By theorems 3.2, 3.3 using table 1, 2 and the results in sections 1, 2 we can obtain many facts about the representation of the multipliers of $f *_0 g$ for series subspaces of L^1 or L^1_{loc}.

4. Representation of the continuous convolutions for the operator l_0. We shall consider simultaneously the compact and noncompact case for Δ. Let X be a Banach or Frechet subalgebra of $L^1(+, *_0, \cdot)$ or $L^1_{loc}(+, *_0, \cdot)$. Let

Table 4

N is of the form $N(f)=\int_a^b f(x)\,\alpha(x)\,dx$ with $\alpha\in BV$, i. e. $N(f)=F(lf), F\in C^*$

X	Y	$M: X\to Y$ is a continuous linear operator commuting with l_0, iff $m_0=M(1)$ belongs to	Now (3.2) transforms by
L^1	L^1	L^1	(2.13)
L^p	L^p	L^p	(2.13)
C	C	C	(2.13)

N is of the form $N(f)=kf(x_0)+\int_a^b f(x)\,\alpha(x)\,dx$ with $\alpha\in BV$, $x_0\in\Delta$ $k\neq0$, i.e.

$N(f)=kf(x_0)+F(lf),\ F\in C^*$

X	Y	$M: X\to Y$ is a continuous linear operator commuting with l_0, iff $m_0=M(1)$ belongs to	Now (3.2) transforms by
L^1	L^1	BV	(2.8) and (2.13)
C	C	$BV\cap C$	(2.4) and (2.13)
C_N when $\alpha\equiv0$	C	BV	(2.4)
C_N wehen $\alpha\equiv0$	C_N	BV	(2.4)

the space of the polynomials be dense in X relative to the topology of X, or let $C_N^k\subset X$ and $l_0(x)\subset X$. Let also the convolution $*_0$ be a continuous operation $X\times X\to X$.

Lemma 4.1. *Let $\tilde{*}$ be another continuous convolution of l_0 in X. Then for all $f, g, h\in X$ is true*

$$f\tilde{*}(g*_0h)=(f\tilde{*}g)*_0h=f*_0(g\tilde{*}h). \tag{4.1}$$

Proof. Let the operator T_f be defined by $T_f(g)=f\tilde{*}g$ for a fixed $f\in X$. Since $\tilde{*}$ is convolution of l_0 we have $l_0(f\tilde{*}g)=f\tilde{*}l_0g$, i. e. $l_0T_f=T_fl_0$ then T_f is a multiplier of $\tilde{*}$ by theorems 3.2, 3.3. Hence $T_f(g*_0h)=(T_fg)*_0h$, i. e. the first identity of (4.1) is established. This identity and the commutativity of $*_0$ imply for arbitrary $a, b, c\in X$: $a\tilde{*}(b*_0c)=(a\tilde{*}b)*_0c=(b\tilde{*}a)*_0c=b\tilde{*}(a*_0c)=b\tilde{*}(a*_0c)=b\tilde{*}(c*_0a)=(b\tilde{*}c)*_0a=a*_0(b\tilde{*}c)$.

Definition. *We say that the element $f\in X$, $f\neq 0$ is an anihilator for the operation $\tilde{*}$ iff $f\tilde{*}g=0$ for each $g\in X$. The fact that $\tilde{*}$ has no anihilators we denote shortly as $\tilde{*}$ is w. a. (without anihilators).*

It is clear that $*_0$ is w. a. in X since neither $\{1\}$ nor $l_0^k(1)$ are divisors of 0 in the cases when X contains the set of all polynomials or $C_N^k\subset X$, respectively. We shall use the notations $H_0(X)$ for the nonempty set of nontrivial nondivisors of 0 for $*_0$ in X.

Theorem 4.2. *Every continuous convolution w. a. $\tilde{*}$ for l_0 in X has the same set of nontrivial nondivisors of zero as $*_0$. The convolutions $*_0$ and $\tilde{*}$ have one and the same set of multipliers.*

Proof. Let $\tilde{H}(x)$ be the set of nontrivial nondivisors of 0 for $\tilde{*}$ in X, and let $f\in H_0(X)$ and $f\tilde{*}g=0$ for some $g\in X$. Then $(\tilde{f*g})*_0h=\tilde{f*}(g*_0h)=f*_0(\tilde{g*_0h})=0$ holds for each $h\in X$. Hence $g\tilde{*}h=0$ for each $h\in X$ and $g=0$ since $\tilde{*}$ is w.a., i.e $f\in\tilde{H}(X)$. This means $H_0(X)\subset\tilde{H}(X)$, hence $\tilde{H}(X)\neq\emptyset$. Now we can proceed in the same way and prove that $\tilde{H}(X)\subset H_0(X)$. Let now M be a multiplier for $*_0$, and let h be an arbitrary element of X. Then

$$[M(f\tilde{*}g)-Mf\tilde{*}g]*_0h=M[(f\tilde{*}g)*_0h]-[Mf\tilde{*}g]*_0h=M[f*_0(g\tilde{*}h)]=[Mf\tilde{*}g]*_0h=0,$$

hence $M(f\tilde{*}g)=Mf\tilde{*}g$, i. e. M is a multiplier for $\tilde{*}$. The converse statement is evident.

As corollary we can formulate a theorem proved by I. Dimovski [13] when $X=C$.

Theorem 4.3. *The rings of Mikusiński for all continuous convolutions w. a. for the operator l_0 in X are isomorphical.*

Proof. Let the first convolution be $*_0$, and the second $\tilde{*}$ is an arbitrary continuous convolution of l_0 in X. The ring of Mikusiński $\mathfrak{M}_0(X)$ for $*_0$ consists of the fractions f/g, where $f\in X$, $g\in H_0(X)$ and $f_1/g_1=f_2/g_2$ iff $f_1*_0g_2=f_2*_0g_1$. Since $\tilde{*}$ is w. a., then $H_0(X)=\tilde{H}(X)$, therefore the ring of Mikusiński $\tilde{\mathfrak{M}}(X)$ for $\tilde{*}$ consists also of the fractions f/g, $f\in X$, $g\in\tilde{H}(X)=H_0(X)$ and $f_1/g_1=f_2/g_2$ iff $f_1\tilde{*}g_2=f_2\tilde{*}g_1$. It is easy to see using (4.1) that $f_1/g_1=f_2/g_2$ in $\mathfrak{M}_0(X)$ iff $f_1/g_1=f_2/g_2$ in $\tilde{\mathfrak{M}}(X)$. Then the sets $\mathfrak{M}_0(X)$ and $\tilde{\mathfrak{M}}(X)$ coincide, and it is easy to see that the identical operator is an algebraic isomorphism between $\mathfrak{M}_0(X)$ and $\tilde{\mathfrak{M}}(X)$.

Theorem 4.4. *Let there exist $a\in H(L^1)$ so that $a*_0L^1\subset X$. Then the rings of Mikusiński $\mathfrak{M}_0(L^1)$ and $\mathfrak{M}_0(X)$ are isomophical. The assertion is true in particular when X contains one of the spaces C, BV, AC, L^1, C_N^k, $1\leq k<+\infty$.*

Proof. By the assumption of the existence of the element a it is easy to prove the relations $H_0(X)=H_0(L^1)\cap X$, $a*_0H_0(L^1)\subset H_0(X)$. Now it is clear that, if $f/g\in\mathfrak{M}_0(X)$, (i. e. $f\in X$, $g\in H_0(X)\subset H_0(L^1)$), then $f/g\in\mathfrak{M}_0(L^1)$ and if $f_1/g_1=f_2/g_2$ in $\mathfrak{M}_0(X)$, (i. e. $f_1*_0g_2=f_2*_0g_1$) then $f_1/g_1=f_2/g_2$ in $\mathfrak{M}_0(L^1)$, therefore $\mathfrak{M}_0(X)\subset\mathfrak{M}_0(L^1)$. Now it is clear that the identical embedding $I:\mathfrak{M}_0(X)\rightarrow\mathfrak{M}_0(L^1)$ is an injective homomorphism. However I is surjective since if $f/g\in\mathfrak{M}_0(L^1)$ then $f/g=a*_0f/a*_0g$, where $a*_0f\in X$ and $a*_0g\in H(X)$. Hence I is an isomorphism between $\mathfrak{M}_0(L^1)$ and $\mathfrak{M}_0(X)$.

We note that such an element $a \in H_0(L^1)$, $a *_0 L^1 \subset X$ exists always when X is an ideal in L^1 and $H_0(X) \neq \emptyset$. The existence of a is possible also in the case when X is not an ideal, but X contains an ideal in L^1 which has at least one nontrivial nondivisor of zero. For example $X = C^k$ is not an ideal in L^1 but $a = l_0^k(1)$ has the above properties.

Theorem 4.5. *Let $1 \in X$. Every continuous convolution $\tilde{*}$ for l_0 in X can be expressed by means of $*_0$ with the formula*

$$f \tilde{*} g = d^2/dt^2[\varrho *_0 f *_0 g]. \tag{4.2}$$

Proof. Let $f, g \in X$. Then using (4.1) we obtain: $\varrho *_0 f *_0 g = (1 \tilde{*} 1) *_0 f *_0 g = 1 *_0 1 *_0 (f \tilde{*} g) = l_0^2[f \tilde{*} g]$. Hence (4.2) holds.

We can prove also the formula of the type

$$f \tilde{*} g = d^{2k+2}/dt^{2k+2}[\varrho *_0 f *_0 g] \tag{4.3}$$

with $\varrho = l_0^k[1] *_0 l_0^k[1]$ when $l_0^k[1] \in X$, i. e. (4.3) holds always when $C_N^k \subset X$.

Theorem 4.6. *Let $1 \in X$. A nontrivial continuous convolution $\tilde{*}$ for l_0 is w. a. in X iff the function $\varrho = 1 \tilde{*} 1$ is nontrivial nondivisor of 0 for $*_0$, or equivalently iff $\{1\}$ is a nondivisor of 0 for $\tilde{*}$.*

Proof. It is clear from (4.2) that $\varrho = 1 \tilde{*} 1 = 0$ iff $\tilde{*}$ is the trivial convolution $f \tilde{*} g \equiv 0$. Let now $\tilde{*}$ be w. a. and let $\varrho *_0 f = 0$. Then for each $g \in X$ we have $\varrho *_0 f *_0 g = 1 *_0 1 *_0 (f \tilde{*} g) = 0$ and $f \tilde{*} g = 0$ for each $g \in X$, i. e. $f = 0$. That means $\varrho \in H_0(X)$. Conversely, let $\varrho = 1 \tilde{*} 1 \in H_0(X)$ and let $f \tilde{*} g = 0$ for each $g \in X$. Then $\varrho *_0 f *_0 g = 1 *_0 1 *_0 (f \tilde{*} g) = 0$ for each $g \in X$. Hence $\varrho *_0 f = 0$, i. e. $f = 0$. That means $\tilde{*}$ is w. a.

Now if $\varrho \in H_0(X)$ then $\tilde{*}$ is w. a. and $\{1\} \in \tilde{H}(X)$ by theorem 4.3. Conversely, if $1 \in \tilde{H}(X)$ and $\varrho *_0 f = 0$ we have $1 *_0 (1 \tilde{*} f) = (1 \tilde{*} 1) *_0 f = \varrho *_0 f = 0$. Hence $1 \tilde{*} f = 0$, i. e. $f = 0$. That means $\varrho \in H_0(X)$.

An analogous statement relative to the functions $\varrho = l_0^k[1] \tilde{*} l_0^k[1]$, $\{1\}$ in the case $l_0^k[1] \in X$ can be formulated.

The above statements show that if ϱ is divisor of zero then $\tilde{*}$ has annihilators. All continuous convolutions with anihilators are expressed by (4.2) or (4.3), where ϱ is a divisor of zero. The anihilators for $\tilde{*}$ are all elements f of X with $1 \tilde{*} f = 0$ when $\{1\} \in X$ or $l_0^k[1] \tilde{*} f = 0$ when $l_0^k[1] \in X$. Now $\tilde{H}(X) = \emptyset$.

A theorem of Titchmarsh type follows.

Theorem 4.7. *If $*_0$ has no divisors of 0 in X, then every nontrivial continuous convolution for l_0 in X has no divisors of 0. In particular every continuous convolution for the integration operator $lf = \int_0^t f$ in the space $C[0, +\infty)$ and $L^1_{\text{loc}}[0, +\infty)$ has no divisors of 0.*

5. The general Dimovski's convolution in BV. From the theorems in sections 1, 2 some results about the general Dimovski's convolution for the operator $l_0 f =_{\text{def}} lf + \Phi(f)$ $\Phi \in C^*$ can be obtained. It is known [3], [4] that

$$f *_0 g =_{\text{def}} f * g + \Phi_x\{\frac{\partial}{\partial x} \int_t^x f(x+t-u) g(u) du\} \tag{5.1}$$

is a convolution for l_0 in C^1, so that $l_0 f = 1 *_0 f$. The convolution which we have considered in the previous sections is a special case of (5.1) when $\Phi(f)$

$=N(lf)$, $N(1)=-1$. We aim to extend (5.1) in BV. From theorem 2.3 applied to the variable x of $f\circ g(x, t)=-f\circ g(x, t)$ for a fixed t we obtain that if $f\in C$, $g\in BV\cap C$ then for each $(x, t)\in\Delta^2$ the partial derivative $\partial/\partial x[f\circ g]$ exists, belongs to $C(\Delta^2)$ and

$$(5.2)\qquad \frac{\partial}{\partial x} f\circ g(x, t)=(R.\ S)\int_t^x g(x+t-u)d f(u)+f(t)g(x).$$

Hence the expression (5.1) exists for $f\in C$, $g\in BV\cap C$ and the operation

$$(5.3)\qquad f*_0 g=_{\text{def}} f*g+\Phi_x\{(R.S)\int_t^x g(x+t-u)df(u)\}+f(t)\Phi(g)$$

is an extension of (5.1), so that $*_0: C\times BV\cap C\to C$. If $f\in C$, $g\in BV$ it is clear by theorem 2.2 that (5.3) exists too, since $\int_t^x g(x+t-u)df(u)\in C(\Delta^2)$ and $g\in L^1\{\Delta, |\varphi|\}$, where φ is the complex measure defined by Φ. Hence (5.3) defines an operation $C\times BV\to C$. However by lemma 2.3 we obtain for $f\in C$, $g\in BV$ that (R. S) $\int_t^x g(x+t-u)df(u)=\int_t^x f(x+t-u)d\mu_g(u)+K(t, x)$, where

$$K(t, x)=\begin{cases} g(t-0)f(x)+[g(x)-g(x+0)]f(t), & a\le t\le x\le b\\ g(t+0)f(x)+[g(x)-g(x-0)]f(t), & a\le x<t\le b.\end{cases}$$

Let now $f\in L^1$, $g\in BV$ then $\int_t^x f(x+t-u)d\mu_g(u)\in L^1\{\Delta^2, m\times|\varphi|\}$ by theorem 1.6. As in lemma 2.3 it can be shown that $K\in L^1\{\Delta^2, m\times|\varphi|\}$, when $f\in L^1\{\Delta, |\varphi|\}$, $g\in BV$. Therefore the operation

$$(5.4)\qquad f*_0 g=_{\text{def}} f*g+\Phi_x\{\int_t^x f(x+t-u)d\mu_g(u)\}+\Phi_x\{K(t, x)\}$$

exists for $f\in L^1\cap L^1(|\varphi|)$, $g\in BV$ and $f*_0 g\in L^1$, i. e. (5.4) is a bilinear operation $L^1\cap L^1(|\varphi|)\times BV\to L^1$. It can be verified that

$$(5.5)\qquad \Phi_x\{K(t, x\}=g(t+0)\int_{[a,t)} f(x)d\varphi(x)+g(t-0)\int_{[t,b]} f(x)d\varphi(x)$$

$$+f(t)\{\int_{[a,t)}[g(x)-g(x-0)]d\varphi(x)+\int_{[t,b]}[g(x)-g(x+0)]d\varphi(x)\}.$$

In particular if g and the function of BV, representing by the Riesz theorem the functional Φ, have no common points of discontinuity then

$$(5.6)\qquad \Phi_x\{K(t, x)\}=g(t+0)\int_{[a,t)} f(x)d\varphi(x)+g(t-0)\int_{[t,b]} f(x)d\varphi(x).$$

If f, $g\in BV$ then from (5.5) it is clear that $f*_0 g\in BV$ and by theorem 1.6 we obtain the following

Theorem 5.1. *For each continuous linear functional Φ in C the general Dimovski's convolution* (5.1) *extends by the formula* (5.4) *as continuous bilinear operation $BV\times BV\to BV$ and $l_0 f=1*_0 f$ too.*

The proof follows immediately from theorem 1.6.

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