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ON A MAZURKIEWICZ THEOREM

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In the first part for each metric space X an example of zero-dimensional subset M of X is constructed, such that $X\setminus M$ does not contain an arc. In the second part the structure is studied of the sets of type $I^n\setminus M$, where $I=[-1,1]$ and M is a subset of I^n with dimension $k\leqq n-2$. It is shown that every two points of $I^n\setminus M$ are connected in $I^n\setminus M$ by a strengthened Cantor $(n-k-1)$-dimensional manifold in the sense of Alexandroff.

A classical theorem of Mazurkiewicz states, that if M is a subset of n-dimensional cube $I^n=[-1,1]^n$, such that $\dim M\leqq n-2$, then the set $I^n\setminus M$ is a semicontinuum (see [4]). This result is completed by an example due to Vitushkin [3] of such an 0-dimensional set $M\subset I^3$, that for every two points $p_i=(x_i, y_i, z_i)$; $i=1, 2$ of $I^3\setminus M$ with $z_1\neq z_2$, there is not an arc, containing p_1 and p_2 and lying in $I^3\setminus M$ (an arc is a set, which is homeomorphic to a closed interval).

Section I of this paper is a generalization of Vitushkin's example (see theorem 1). There, a 0-dimensional subset M of the Hilbert cube $I^{\aleph_0}$ is constructed, such that $I^{\aleph_0}\setminus M$ does not contain arcs. We shall note, that if M is a F_σ-subset of I^n and $\dim M\leqq n-2$ that $I^n\setminus M$ is a linearly connected (see [7]).

Section III is a reinforcement of the result of Hadjiivanov [6]. It is shown, that if $M\subset I^n$ and $\dim M\leq k$, where $k\leq n-2$ then any two points of $I^n\setminus M$ can be connected with a (V_{n-k-1})-continuum (see Definition 2.), which lies in $I^n\setminus M$.

1. We shall use the following theorem of Bing [2, Theorem 3].

If F and H are two disjoint continua in $I^{\aleph_0}$, there is a hereditarily indecomposable continuum which has exactly two complementary domains and which is irreducible with respect to separating F from H.

The main result of this section is

Theorem 1. *There exists a zero-dimensional subset M of $I^{\aleph_0}$, the complement $I^{\aleph_0}\setminus M$ of which does not contain an arc.*

Proof. Let $\mathcal{B}=\{U_i \mid i=1, 2,\dots\}$ be a countable base of $I^{\aleph_0}$ consisting of open balls. We shall say that the pair $\gamma=(U_i, U_j)$, where U_i and U_j belong to $\mathcal{B}$ is normal, if $[U_i]\subset U_j$ and $I^{\aleph_0}\setminus U_j$ is connected. Denote with $\mathcal{A}$ the set of all normal pairs. Apparently $|\mathcal{A}|=\aleph_0$. From the theorem of Bing it follows, that for every $\gamma\in\mathcal{A}$, $\gamma=(U_i, U_j)$, there exists a hereditarily indecomposable continuum K_γ, which separates $I^{\aleph_0}\setminus U_j$ from U_i. Let $I^{\aleph_0}\setminus K_\gamma=V_\gamma\cup W_\gamma$

and $[U_i]\subset V_\gamma$. We shall show that the set $\{V_\gamma \mid \gamma \in \mathcal{A}\}$ is a base of $I^{\aleph_0}$. Let G be an open set of $I^{\aleph_0}$ and $x \in G$. Since $\mathcal{B}$ is a base of $I^{\aleph_0}$, there is a $U_j \in \mathcal{B}$, such that $x \in U_j \subset G$. Let O be an open set, for which $x \in O \subset [O] \subset U_j$. We may choose a member U_i of $\mathcal{B}$ such that $x \in U_i \subset O$. It is clear that the pair $\gamma = (U_i, U_j)$ is normal. But $x \in U_i \subset V_\gamma \subset I^{\aleph_0} \setminus W_\gamma \subset U_j \subset G$ and therefore $\{V_\gamma \mid \gamma \in \Gamma\}$ is a base of $I^{\aleph_0}$. Hence, for the set $M = I^{\aleph_0} \setminus \cup \{K_\gamma \mid \gamma \in \mathcal{A}\}$ it is true, that ind $M = 0$ because $F \subset V_\gamma \subset K_\gamma$ and, therefore, $\dim M = 0$. Suppose that $I^{\aleph_0} \setminus M$ contains the arc Γ, and let $\varphi : [0, 1] \to \Gamma \subset I^{\aleph_0} \setminus M$ is a homeomorphism. Then $[0, 1] \subset \cup \{\varphi^{-1}(K_\gamma \cap \Gamma) \mid \gamma \in \mathcal{A}\}$ and from the inequality $|\mathcal{A}| \leqq \aleph_0$ and the theorem of Baire, it follows that there exists a member γ_0 of $\mathcal{A}$ for which Int $(\varphi^{-1}(K_{\gamma_0} \cap \Gamma)) \neq \Phi$. Let, for example, $[a, b] \subset \varphi^{-1}(K_{\gamma_0} \cap \Gamma)$. Then $\varphi([a\, b]) \subset K_{\gamma_0}$, which is a contradiction to the fact that K_{γ_0} is hereditarily indecomposable.

It is easy to check, that from Theorem 1 follows

Corollary 1. *Every metric space X contains a zero-dimensional subset M_X, the complement $X \setminus M_X$ of which does not contain an arc.*

Proof. Let $\varphi : X \to I^{\aleph_0}$ be a uniformly zero-dimensional map form X to $I^{\aleph_0}$ (such a map exisis; see for example [5]). Then if $M_X = f^{-1}(M)$ we obtain that $\dim M_X = \dim M = 0$. If $\Gamma \subset X \setminus M_X$ is an arc then $\dim f(\Gamma) = 1$ and therefore $f(\Gamma)$ is not a point. From the least it follows, that $f(\Gamma)$ contains an arc, but it is impossible.

2. In that section we recall some definitions and facts which are necessary in the sequel.

Further on, by space we understand a compact metric space, which is contained in the m-dimensional Euclidean space R^m.

Let X be as above. By an n-dimensional chain lying in X over an abelian group $\mathfrak{A}$ we understand the linear form $\varkappa = a_1\sigma_1 + \ldots + a_k\sigma_k$, where the coefficients $a_1, \ldots, a_k$ belong to $\mathfrak{A}$, and $\sigma_1, \ldots, \sigma_k$ are n-dimensional oriented simplexes in X, i. e. systems of $n+1$ points (vertices) of X, given in a definite order modulo an even permutations. If the diameter of the vertices of the simplex σ is less than ε, then σ is said to be an ε-simplex. If each simplex of the chain $\varkappa$ is an ε-simplex, then $\varkappa$ is said to be an ε-chain. The boundary $\partial\varkappa$ of $\varkappa$ is defined as usual. Two ε-chains $\varkappa_1$ and $\varkappa_2$ in X are δ-homologous in X if there exists a δ-chain $\varkappa$ in X such that $\partial\varkappa = \varkappa_1 - \varkappa_2$ (notation: $\varkappa_1 \underset{\delta}{\sim} \varkappa_2$ in X). As usually if for an ε-chain $\varkappa$, we have $\partial\varkappa = 0$ (respectively, if the carrier of $\partial\varkappa$ is $\Phi \subset X$, i. e. all vertices of $\partial\varkappa$ are elements of Φ), then $\varkappa$ is called $\varepsilon =$ cycle in X (resp. ε-cycle in X rel Φ). By an n-dimensional true chain in X we understand the sequence $\varkappa = \{\varkappa_i\}$ of n-dimensional ε_i-chains $\varkappa_i$ such that $\lim_{i\to\infty} \varepsilon_i = 0$. If $\varkappa = \{\varkappa_i\}$ and $\varkappa' = \{\varkappa_i'\}$ are two ture chains, then we define $\varkappa \pm \varkappa' = \{\varkappa_i \pm \varkappa_i'\}$ and $\partial\varkappa = \{\partial\varkappa_i\}$. If, for some closed $\Phi \subset X$, Φ is a carrier of every $\varkappa_i$, then Φ is called a carrier of $\varkappa$. The true chain $\varkappa = \{\varkappa_i\}$ is said to be a true cycle relative to Φ, if Φ is the carrier of $\partial\varkappa$ (Φ may be empty). The true cycle $z = \{z_i\}$ is a convergent true cycle if the true cycle $\{z_i - z_{i+1}\}$ is homologous to zero in X.

Let $X \subset \mathbb{R}^m$. By an n-dimensional polyhedral chain x, lying in $U = \mathbb{R}^m \setminus X$ (or in $U = B^m \setminus X$, where B^m is an open cube in $\mathbb{R}^m$ for which $X \subset [B^m]$) we understand a chain over a group of all integers, the polyhedron of which lies in U and contains x.

In the sequel by a cycle lying in the compactum X (rel Φ) we understand a convergent true cycle over the group of the rational numbers. For the proof of the following results see, for example [1].

Let $X = X' \cup X''$ be a sum of two closed subsets X' and X'' and z^n be a convergent true cycle in X. Denote by z_1^n the part of z^n which lies in X' (z_1^n consists of all simplecses of z^n, everyone of which contains a vertice, belonging to X') and let $z^{n-1} = \partial z_1^n$.

Lemma 1 [1, p. 208]. *If* $z^n \sim 0$ *in* $X = X' \cup X''$ *then* $z^{n-1} \sim 0$ *in* $X_0 = X' \cap X''$.

Definition 1. *Let X be a metric space. The n-dimensional diameter* $\alpha^n X$ *of X is the infimum of all* $\delta > 0$, *such that there exists a continuous* δ*-map from X in an n-dimensional polyhedron.*

Definition 2. *The compact metric space X is called a* (V_n)*-continuum if* $\dim X = n$ *and for every two disjoint closed subsets F and G of X, such that* $\operatorname{Int} F \neq \emptyset$ *and* $\operatorname{Int} G \neq \emptyset$, *there is an* $\varepsilon > 0$, *such that for every partition C in X between F and G we have* $\alpha^{n-2} C \geq \varepsilon$.

A compactum $X \subset \mathbb{R}^n$ and a cycle z^k, lying in $\mathbb{R}^n \setminus X$ (or in $B^n \setminus X$) are said (as usually) to be linked if z^k is not homologous to zero in $\mathbb{R}^n \setminus X$ (in $B^n \setminus X$). If X is the minimal in the sense of Zorn compactum with this condition, then X and z^k are said to be irreducibly linked.

Lemma 2 [1, p. 241]. *Every k-dimensional compactum* $X \subset \mathbb{R}^n$, *which is irreducibly linked with certain cycle* z^p $(p = n - k - 1)$ *in certain (topological) ball, is a* (V_p)*-continuum.*

3. Denote with $I^n_{\pm i}$ the opposite faces $\{x \in I^n \mid x_i = \pm 1\}$; $i = 1, \dots, n$ of the n-dimensional cube I^n.

Take some number $a \in (0, 1)$ and let $i, j' \in \{1, \dots, n\}$; $i \neq j'$. Denote by $\eta^i_{\pm j}$ the map $\eta^i_{\pm j} : I^n \to I^n$ satisfied the following conditions:

(i) the restriction of $\eta^i_{\pm j}$ on the set $I^n \setminus I^n_{\pm j}$ is a homeomorphism and $\eta^i_{\pm j}(I^n_{\pm j}) = I^n_{\pm j}$.

(ii) for $H = \{x \in I^n_{\pm j} \mid |x_i| \leq a\}$, $\eta^i_{\pm j}(H) = \{x \in I^n_{\pm j} \mid x_i = 0\}$.

(iii) $\eta^i_{\pm j}(x) = x$ if $\varrho(x, I^n_{\pm j}) \geq a$.

It is easy to check that such a map exists; get for example

$$\eta^i_{\pm j}(x) = (x_1, \dots, x_{i-1}, \psi(\delta_{\pm i}(x), x_i), x_{i+1}, \dots, x_n),$$

where $\delta_{\pm j}(x) = \varrho(x, I^n_{\pm j})$; $\psi(\delta, t) : [0, 1] \times I \to I$ is a function defined by

$$\psi(\delta, t) = \begin{cases} \varphi(\delta, t) & \text{for } 0 \leq \delta \leq a \\ t & \text{for } a < \delta \leq 1 \end{cases}$$

and $\varphi : [0, 1] \times I \to I$ is obtained by the formula

$$\varphi(\delta, t) = \begin{cases} t\delta/a & \text{for } 0 \leq |t| \leq a \\ [t(1-\delta) + \operatorname{sign} t \cdot (\delta - a)]/(1-a) & \text{for } a < |t| \leq 1. \end{cases}$$

Consider the space $\mathbb{R}^k$ as a subset of $\mathbb{R}^n\,(n>k)$; $\mathbb{R}^k=\{x\in\mathbb{R}^n \mid x_{k+1}=\ldots=x_n=0\}$. Denote by z^{k-1} the $(k-1)$-dimensional cycle, the carrier of which is the $(k-1)$-dimensional sphere $S^{k-1}=\mathbb{R}^k\cap I^n$. Let for $i=1,2,\ldots,k$ C_i be a partition in I^n between I^n_{+i} and I^n_{-i} and $C=\bigcap_{i=1}^{k} C_i$.

Lemma 3. *z^{k-1} is not homologous to zero in $I^n\setminus C$.*

Proof. Let $\delta=\frac{1}{2}\min\left(\{\varrho(C_i, I^n_{+i}\cup I^n_{-i}) \mid i=1,\ldots,k\}\cup\left\{\varrho\left(C,\bigcup_{i=1}^{k}(I^n_{+i}\cup I^n_{-i})\right)\right\}\right)$ and $a=1-\delta$. For the number a we may choose the mappings $\eta^i_{\pm j}$, $i,j\in\{1,2,\ldots,k\}$ in such a way that the conditions (i) — (iii) hold. Denote $\eta_j^i=\eta^i_{-j}\circ\eta^i_{+j}$ and $h_j=\eta^i_1\circ\ldots\circ\eta^i_k$. It is easy to check that $C_i'=h_i(C_i)$ is a partition between I^n_{+i} and I^n_{-i}; $h_i(x)=x$ if $\varrho(x,T)\geq a$, where $T=\bigcup_{i=1}^{k}(I^n_{+i}\cup I^n_{-i})$ and the restriction of h_i the set $I^n\setminus T$ is a homeomorphism. Therefore $\bigcap_{i=1}^{k} C_i'=\bigcap_{i=1}^{k} C_i=C$. Suppose that, $z^{k-1}\sim 0$ in $I^n\setminus C$. Then there exists $\varepsilon>0$ such that $z^{k-1}\sim 0$ in $\Phi=I^n\setminus\{x\in I^n \mid \varrho(x,C)<\varepsilon\}$. We may assume that z^{k-1} is a convergent true cycle in Φ with carrier $S^{k-1}\subset\Phi$. Because C_k' is a partition in I^n, then C_k' is a partition in Φ and $\Phi=\Phi'\cup\Phi''$; $\Phi'\cap\Phi''=C_k'\cap\Phi=\Phi_1$. Let x^{k-1} be the part of z^{k-1} which lies in Φ'. Denote $\partial x^{k-1}=z^{k-2}$. It is easy to check that the carrier of z^{k-2} is the set $S^{k-1}\cap C_k'$. According to lemma 1. $z^{k-2}\sim 0$ in Φ_1. By induction we obtain for $1\leq i<k$, the cycle z^{k-i-1}, lying in $\Phi_{i-1}=\bigcap_{i=k-i+1}^{k} C_i'\cap\Phi$ and such that $z^{k-i-1}\sim 0$ in Φ_{i-1}. Therefore, the cycle z^0 is homologous to zero in $\Phi_{k-1}=\bigcap_{i=2}^{k} C_i'\cap\Phi$ and the carrier of z^0 is the set set $\Phi_{k-1}\cap S^{k-1}=\bigcap_{i=2}^{k} C_i'\cap S^{k-1}$ $\{(+1,0,\ldots,0),(-1,0,\ldots 0)\}$. It is easy to check that Φ_{k-1} is not connected between, $(1,0,\ldots,0)$ and $(-1,0,\ldots,0)$ (because $C=\bigcap_{i=2}^{k} C_i'\cap C_1'$ is partition between $(1,0\ldots,0)$ and $(-1,0,\ldots,0)$ in Φ_{k-1}), and therefore z^0 is not homologous to zero in Φ_{k-1}, which is a contradiction.

Theorem 2. *Let $M\subset I^n$ and $\dim M\leq k$, where $k\leq n-2$. For every two different points p_+ and p_- of $I^n\setminus M$ there exists a (V_{n-k-1})-continuum K, such that $\{p_+,p_-\}\subset K\subset I^n\setminus M$.*

Proof. Denote with p_{+i} and p_{-i} the coordinates of points p_+ and p_-, respectively. Since $p_+\neq p_-$ then there is such an integer i_0 for which $p_{+i_0}\neq p_{-i_0}$. Without loss of generality we may assume that $i_0=n$. Denote $\xi=(p_{+n}+p_{-n})/2$ and let P be an $(n-1)$-dimensional hyperplane $K=\{x\in I^n \mid x_n=\xi\}$ in I^n. Let Q be the sum of two connes over K with vertices p_+ and p_-, respectively. Consider the map $\varphi: I^n\to Q$, defined by the formula

$$\varphi(x)=|x_n|\,p_\varepsilon+(1-|x_n|)(x_1,\ldots,x_{n-1},\xi),$$

where $\varepsilon=\operatorname{sig} n x_n$. The restriction of φ on the set $I^n\setminus(I^n_{+n}\cup I^n_{-n})$ is a homeomorphism and $\varphi(I^n_{\pm n})=p_\pm$. Then the set $M_1=\varphi^{-1}(Q\cap M)$ is homeomorphic to $Q\cap M$ and, therefore, $\dim M_1\leq k$. It should be noted that we consider two different copies of the n-dimensional cube I^n, which are contained in the diagram $\varphi: I^n\to Q\subset I^n$.

Since dim $M_1 \leqq k$ there are partitions C_i between I^n_{+i} and I^n_{-i} for $i=1, 2, \ldots, k+1$ and such that $M_1 \cap C = \varphi$; $C = \bigcap_{i=1}^{k+1} C_i$. It follows from lemma 3 that the cycle z^k with carrier $S^k = R^{k+1} \bigcap Fr I^n$ is not homologous lemma to zero in $I^n \setminus C$. Since the restriction of φ on the set $I^n \setminus (I^n_{+n} \bigcup I^n_{-n})$ is a homeomorphism, the relation $\varphi(z^k) = z^k_1 + 0$ in $Q \setminus \varphi(C)$ holds. Let $K \subset \varphi(C)$ be a compactum which is irreducibly linked with z^k_1 in Q (by means of Zorn's lemma it is easy to check that such a compactum exists; see also [1, p. 210]). According to lemma 2 K is a (V_{n-k-1})-continuum.

Let us show that $\{p_+, p_-\} \subset K$. Suppose the contrary and let, for instance, $p_+ \notin K$. Take the conne L over $|z^k_1| = S^{k-1}$ with vertice p_+. It is clear that $L \cap K = \varnothing$. Since L is homeomorphic to I^{k+1}, we have $z^k_1 \sim 0$ in L and therefore $z^k_1 \sim 0$ in $Q \setminus K$, which is a contradiction,

Since every (V_n)-continuum is an n-dimensional Cantor manifold, the above theorem gives the following result, belonging to Hadjiivanov [6]:

Corollary 2. *Let* $M \subset I^n$ *and* dim $M \leqq k \leqq n-2$. *Then for every two points* p *and* q *of* I^n / M *there is an* $(n-k-1)$*-dimensional Cantor manifold* K, *such that* $\{p, q\} \subset K \subset I^n \setminus M$.

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