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ON THE BOUNDARY PROBLEMS IN A HALF-SPACE FOR A CLASS OF NONHYPOELLIPTIC EQUATIONS WITH CONSTANT STRENGTH

ILIJA D. ILIEV

In this paper a parametrix is constructed and an a priori inequality of the coercive type is proved for a class of boundary problems in a half-space. The class under consideration contains both hypoelliptic (e. g. the quasielliptic) equations, as well as some nonhypoelliptic equations.

1. Basic definitions and formulation of the results.

We consider a closed convex polyhedron P in R^n, satisfying the following conditions:

(A1) The vertices of P are vectors with components — non-negative integers.

(A2) P possesses a vertex at the origin and on each one of the coordinate axes.

(A3) All the vertices of P except one lie on the hyperplane $x_n=0$.

(A4) If $\alpha \in$ P and $0\leqq\beta<\alpha$, then $\beta\in$ P (as usual, given two vectors α, $\beta\in R^n$, we write $\alpha\geqq\beta$ if $\alpha_k\geqq\beta_k$, $k=1,\dots,n$, and $\alpha>\beta$, if $\alpha\geqq\beta$ and $\alpha\neq\beta$).

We denote by Λ the set of non-zero vertices of P and put $\mu(\xi)=\sum_{\alpha\in\Lambda}|\xi^\alpha|$, $\xi\in R^n$. From (A2) and (A3) it follows that

(1) $$\mu(\xi)\equiv\mu(\xi',\ \xi_n)=|\xi_n|^r+\mu(\xi',\ 0),$$

with some natural r.

An $n-1$-dimensional face of P is called the main one, if it does not lie on some of the coordinate planes. For convenience we shall assume throughout this paper, that the $(n-1)$-dimensional faces of P are closed sets.

We denote by S the set of the main points of P, i. e. the set of those points, which lie at least on one main face.

To each point $\alpha\in R^n$, $\alpha\geq 0$ corresponds the unique number $\langle\alpha\rangle$ with the property $\alpha\in\langle\alpha\rangle$ S $=\{\langle\alpha\rangle.\sigma:\sigma\in$ S$\}$. It follows immediately, that the function $\langle\alpha\rangle$ has the properties:

1) $\langle\alpha\rangle\geqq 0$; $\langle\alpha\rangle=0\Leftrightarrow\alpha=0$,

2) $\langle t\alpha\rangle=t\langle\alpha\rangle$, $t\geqq 0$,

3) $\langle\alpha+\beta\rangle\leqq\langle\alpha\rangle+\langle\beta\rangle$,

4) $\langle\alpha\rangle\leqq 1\Leftrightarrow\alpha\in$ P,

5) $\langle\alpha\rangle=1\Leftrightarrow\alpha\in$ S.

Given a fixed number $m>0$ and a polyhedron P, satisfying conditions (A1) — (A4), we deal with differential operators of the type

$$A(x,\ D)\,u=\sum_{\langle\alpha\rangle\leqq m}a_\alpha(x)\,D^\alpha u,\quad \text{where}\quad D=(D_1,\dots,D_n),$$

$$D_j=-i\frac{\partial}{\partial x_j},\ a_\alpha(x)\in C^\infty(R^n_+),\ R^n_+=\{x\in R^n:x_n\geqq 0\}.$$

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We denote by $\widetilde{A}(x, \xi)=\Sigma_{\langle\alpha\rangle=m} a_\alpha(x)\xi^\alpha$ $(x\in R^n_+,\ \xi\in R^n)$ the principal symbol of $A(x, D)$. As in [1], the operator $A(x, D)$ is called P-elliptic, if for some constant $c>0$ the inequality

$$(2)\qquad |\widetilde{A}(x, \xi)|\geq c(\mu(\xi))^m, \quad x\in R^n_+,\ \xi\in R^n,$$

holds.

It is well-known [1], that every P-elliptic operator is an operator with constant strength and moreover that the P-elliptic equations are hypoelliptic iff P satisfies the following condition:

(A4*) If $\alpha\in$ P and $0\leqq\beta<\alpha$, then $\beta\in$ P$\setminus$S.

When $n\geq 3$, $\xi'\neq 0$, it follows from inequality (2) that the equation with respect to λ

$$(3)\qquad \widetilde{A}(x, \xi', \lambda)=0$$

has an invariable number of roots $\lambda=\lambda(x, \xi')$ with a positive imaginary part, say d. From now on, superfluous stipulations to be avoided, we shall suppose that $n\geq 3$, $d\geq 1$.

Let $m_1, \dots, m_d$ be non-negative numbers. We consider the following boundary problem in R^n_+:

$$A(x, D)u=\sum_{\langle\alpha\rangle\leq m} a_\alpha(x)D^\alpha u=f(x), \quad x\in R^n_+,$$

$$(4)\qquad B_j(x, D)u|_{x_n=0}=\sum_{\langle\beta\rangle\leq m_j} b_{j\beta}(x)D^\beta u|_{x_n=0}=g_j(x'), \quad x'\in R^{n-1},\ 1\leq j\leq d.$$

Here the operator $A(x, D)$ satisfies (2), and $B_j(x, D)$ are connected with A by a condition of the Shapiro-Lopatinsky type. Namely, let $\lambda_j^+(x, \xi')$, $1\leq j\leq d$ be the roots of (3) with positive imaginary parts. Introduce the notations

$$\widetilde{A}^+(x, \xi', \lambda)=\prod_{j=1}^{d}(\lambda-\lambda_j^+(x, \xi')),$$

$\widetilde{B}_j(x, \xi)=\sum_{\langle\beta\rangle=m_j} b_{j\beta}(x)\xi^\beta$, $1\leqq j\leq d$, and suppose that

$$\widetilde{B}_j(x', 0, \xi', \lambda)\equiv\widetilde{B}_j^+(x', \xi', \lambda) \pmod{\widetilde{A}^+(x', 0, \xi', \lambda)}$$

as polynomials in λ. Let us write $\widetilde{B}_j^+(x', \xi', \lambda)$ in the form

$$\widetilde{B}_j^+(x', \xi', \lambda)=\sum_{k=1}^{d} b_{jk}^+(x', \xi')\lambda^{k-1}, \quad 1\leq j\leq d.$$

We require the following condition to be fulfilled: there exists a constant $c_1>0$, such that for every $x'\in R^{n-1}$, $\xi'\in R^{n-1}\setminus 0$

$$(5)\qquad |\det\{b_{jk}^+(x', \xi')\}|\geqq c_1(\mu'(\xi'))^\omega,$$

where $\omega=\sum_{k=1}^{d}(m_k-(k-1)/r)$, $\mu'(\xi')=\mu(\xi', 0)$.

When the conditions (2), (5) are fulfilled, we call the boundary problem (4) $\mathcal{P}$-elliptic in R^n_+ and also shall say that the system of operators B_j covers A.

Example 1. The system of Dirichlet operators $B_j \equiv D_n^{j-1}$, $1 \leq j \leq d$, covers every $\mathcal{P}$-elliptic operator A.

Example 2. Denote by $\mathcal{P}$ the convex polyhedron in R^3 whose vertices are (0, 0, 0), (2, 0, 0), (0, 2, 0), (2, 2, 0), (0, 0, 4). Then the problem

$$A(x, D)u \equiv a_{200}(x) D_1^2 u + a_{020}(x) D_2^2 u + a_{220}(x) D_1^2 D_2^2 u$$

$$+ a_{004}(x) D_3^4 u + a_{111}(x) D_1 D_2 D_3 u = f(x) \quad \text{in} \quad R^3_+,$$

$$B_1(x, D) u|_{x_3=0} \equiv D_3 u|_{x_3=0} = g_1(x'),$$

$$B_2(x, D) u|_{x_3=0} \equiv (D_3^3 u + b_{011}(x) D_2 D_3 u + b_{100}(x) D_1 u)|_{x_3=0} = g_2(x'),$$

where $x = (x_1, x_2, x_3) = (x', x_3)$, $Re\, a_{200} \geq a^0$, $Re\, a_{020} \geq a^0$, $Re\, a_{220} \geq a^0$, $a_{004} \geq a^0$ for $x \in R^3_+$ $(a^0 > 0)$, is $\mathcal{P}$-elliptic in R^3_+. In this case, as it is easy to be seen, we have $m = 1$, $m_1 = 1/4$, $m_2 = 3/4$, $d = 2$.

Given a fixed real s we denote by $H^{s,\mu}(R^n)$ the space of those temperate distributions $u \in \mathcal{S}'$, whose Fourier transformation $\widehat{u}$ is a function and

$$\| u \|^2_{s,\mu} = (2\pi)^{-n} \int_{R^n} (1 + \mu(\xi))^{2s} |\widehat{u}(\xi)|^2 \, d\xi < \infty,$$

$H^{s,\mu}(R^n)$ is Hilbert space with norm $\| \cdot \|_{s,\mu}$.

It can be checked that the condition (A4) is equivalent to the assertion: there exist constants c, N so that $1 + \mu(\xi + \eta) \leq c(1 + |\xi|)^N (1 + \mu(\eta))$ for all ξ, $\eta \in R^n$. Consequently $H^{s,\mu}(R^n)$ is a special case of the spaces, studied in [2].

For $s \geq 0$ we define the space $H^{s,\mu}(R^n_+)$, following the general scheme [2]:

$$H^{s,\mu}(R^n_+) = \{ u \in \mathcal{S}'(R^n_+) : \exists\, \widetilde{u} \in H^{s,\mu}(R^n),\ \widetilde{u}|_{R^n_+} = u \}.$$

The norm in $H^{s,\mu}(R^n_+)$ will be noted by $\| \cdot \|_{s,\mu}(R^n_+)$.

After all, let for $s \geq s_0 = \max(m, m_1 + 1/r, \ldots, m_d + 1/r)$ by $\mathcal{H}^{s,\mu}(R^n_+)$ be denoted the space

$$H^{s-m,\mu}(R^n_+) \times \prod_{j=1}^{d} H^{s-m_j-1/2r,\mu'}(R^{n-1}) \quad (\mu' = \mu'(\xi') = \mu(\xi', 0)).$$

$\mathcal{H}^{s,\mu}(R^n_+)$ is Hilbert space with norm

$$\langle\langle (f, g) \rangle\rangle^2_{s,\mu} = \| f \|^2_{s-m,\mu}(R^n_+) + \sum_{j=1}^{d} \| g_j \|^2_{s-m_j-1/2r,\mu'}(R^{n-1}),$$

where $g = (g_1, \ldots, g_d)$.

To the boundary problem (4) corresponds the operator $\mathcal{A}(x, D)$, defined by the equality

$$\mathcal{A}(x, D) u = \{ A(x, D) u,\ B_1(x, D) u|_{x_n=0}, \ldots, B_d(x, D) u|_{x_n=0} \}.$$

Theorem 1. *Suppose that the coefficients $a_\alpha(x)$, $b_{j\beta}(x)$ in (4) are infinitely smooth functions in R^n_+, bounded together with all their derivatives.*

Then for $s \geq s_0$, $\mathcal{A}(x, D)$ *is a continuous operator from* $H^{s,\mu}(R^n_+)$ *into* $\mathcal{H}^{s,\mu}(R^n_+)$: $\langle \mathcal{A} u \rangle_{s,\mu} \leq C \| u \|_{s,\mu}(R^n_+)$.

One of our main aims in this paper is to prove, that if some additional assumptions are made, then the contrary (in some special sence) inequality is valid.

Let $x=(x', x'')$, $x'=(x_1, \ldots, x_p)$, $x''=(x_{p+1}, \ldots, x_n)$ be a fixed partition of the variables. We formulate the following condition for P:

(A5) If $\nu=(\nu', \nu'')$ is an exterior normal to a main face of P then ν'' is a vector with positive components.

For the coefficients in (4) we make the following assumptions:

(C1) The coefficients in the principal parts a_α, $\langle \alpha \rangle = m$, $b_{j\beta}$, $\langle \beta \rangle = m_j$, $1 \leq j \leq d$, are infinitely smooth functions, independent of x', which become stable for $|x''|$ large enough.

(C2) The coefficientes $a_\alpha(x)$, $\langle \alpha \rangle < m$, $b_{j\beta}(x)$, $\langle \beta \rangle < m_j$ $1 \leq j \leq d$, are infinitely smooth functions, bounded in R^n_+ together with their derivatives.

Theorem 2. *Suppose that* P *satisfies in addition the condition* (A5) *and let* (4) *be a* P*-elliptic problem in* R^n_+, *whose coefficients satisfy* (C1) — C2). *Then for* $s \geq s_0$ *the following a priori inequality is valid*

$$\| u \|_{s,\mu}(R^n_+) \leq c(\langle \mathcal{A}(x, D) u \rangle_{s,\mu} + \| u \|_0(R^n_+)), \quad u \in H^{s,\mu}(R^n_+). \tag{6}$$

In the following theorem we construct a parametrix for $\mathcal{A}(x, D)$.

Theorem 3. *Under the assumptions of theorem 2, there exists a number* $\sigma > 0$, *so that for an arbitrary* $s_1 \geq s_0$ *there exist operators* $\mathcal{R}$ *and* T *with the properties*:

(i) $\mathcal{R}$ *is a continuous operator from* $\mathcal{H}^{s,\mu}(R^n_+)$ *into* $H^{s,\mu}(R^n_+)$, $s_1 \leq s \leq s_1+\sigma$.

(ii) T *is a continuous operator from* $H^{s-\sigma,\mu}(R^n_+)$ *into* $H^{s,\mu}(R^n_+)$, $s_1 \leq s \leq s_1+\sigma$.

(iii) $\mathcal{R}\mathcal{A}(x, D) = I + T$.

It is worth noting, that theorems 2, 3 remain valid also under small perturbations (inclusive with respect to the variable x') of the coefficients of the operators.

We shall formulate several corollaries.

Let K be a compact in R^n_+ and suppose that $K_0 = K \cap \{x_n = 0\}$ has nonzero measure in R^{n-1}. We denote for $s \geq 0$

$$H_0^{s,\mu}(K) = \{u \in H^{s,\mu}(R^n_+) : \operatorname{supp} u \subset K\}, \quad H_0^{s,\mu'}(K_0) = \{u \in H^{s,\mu'}(R^{n-1}) : \operatorname{supp} u \subset K_0\},$$

and for $s \geq s_0$

$$\mathcal{H}_0^{s,\mu}(K) = H_0^{s-m,\mu}(K) \times \prod_{j=1}^{d} H_0^{s-m_j-1/2r,\mu'}(K_0).$$

Corollary 1. *Under the assumptions of theorem 2 the operator* $\mathcal{A}(x, D)$, *regarded as a continuous operator from* $H_0^{s,\mu}(K)$ *into* $\mathcal{H}_0^{s,\mu}(K)$, *has a finite dimensional kernel and a closed range.*

Denote $H^\infty(R^n_+) = \bigcap_{s \geq 0} H^{s,\mu}(R^n_+)$, $\mathcal{H}^\infty(R^n_+) = \bigcap_{s \geq s_0} \mathcal{H}^{s,\mu}(R^n_+)$.

Corollary 2. *Suppose that the assumptions of the theorem 3 hold and* $s \geq s_0$. *Then if* $u \in H^{s_0,\mu}(R^n_+)$ *and* $\mathcal{A}(x, D) u \in \mathcal{H}^{s,\mu}(R^n_+)$, *then* $u \in H^{s,\mu}(R^n_+)$. *In particular, if* $\mathcal{A}(x, D) u \in \mathcal{H}^\infty(R^n_+)$, *then* $u \in H^\infty(R^n_+)$.

Corollary 3. *Under the assumptions of theorem* 3, *for* $\mathcal{A}(x, D)$ *regarded as an operator from* $H^{s,\mu}(R^n_+)$ *into* $\mathcal{H}^{s,\mu}(R^n_+)$, $s \geq s_0$, *it is fulfilled* $\operatorname{Ker} \mathcal{A} \subset H^\infty(R^n_+)$.

The results about smoothness of the solutions, formulated in corollaries 2, 3, are first of all interesting, because they are valid also for a wide class of nonhypoelliptic equations. We shall note, that in this special case, when the operator $A(x, D)$ is hypoelliptic, the condition (A5) is fulfilled with $p=0$ [1].

Theorems 1—3 are proved in point 4. In point 2 the Poisson kernels of the problem (4) are estimated, and in p. 3 some specific properties of the spaces $H^{s,\mu}(R^n)$, $H^{s,\mu}(R^n_+)$ are formulated.

Let us note, that boundary problems in R^n_+ for hypoelliptic equations with variable coefficients were studied in [3; 4], and for nonhypoelliptic equations with constant coefficients — in [5; 6].

2. Estimating the Poisson kernels of the problem (4). Let $x_0' \in R^{n-1}$ be a fixed point. In this paragraph we shall prove two estimates for the system of functions $\Omega_k(x_0', \xi', x_n)$, $1 \leq k \leq d$, being decreasing solutions of the systems of ordinary differential equations

$$\widetilde{A}\left(x_0', 0, \xi', -i\frac{\partial}{\partial x_n}\right)\Omega_k(x_0', \xi', x_n) = 0, \quad x_n \geq 0,$$

(7)

$$\widetilde{B}_j\left(x_0', 0, \xi', -i\frac{\partial}{\partial x_n}\right)\Omega_k(x_0', \xi', x_n)\big|_{x_n=0} = \delta_{jk}, \quad 1 \leq j \leq d$$

($\xi' \in R^{n-1}$, δ_{jk} — the Kronecker symbol).

The functions $\Omega_k(x_0', \xi', x_n)$ are actually the partial Fourier transformations with respect to x' of the Poisson kernels (or the Green functions) $G_k(x_0', x)$ of the problem (4). It follows from (5), that for $\xi' \neq 0$ and fixed k, the system (7) has only one decreasing solution. Following the elliptic boundary problems theory [7—9], we are going to look for an integral representation of Ω_k. Let us write $\widetilde{A}^+(x_0', 0, \xi', \lambda)$ in the form

$$\widetilde{A}^+(x_0', 0, \xi', \lambda) = \sum_{k=0}^{d} a_k^+(x_0', \xi')\lambda^{d-k}$$

and let $B = \{b^{jk}(x_0', \xi')\}_{j,k=1}^d$ be an inverse matrix of $\{b_{jk}^+(x_0', \xi')\}_{j,k=1}^d$. We introduce the notations

$$A_j^+(x_0', \xi', \lambda) = \sum_{k=0}^{j} a_k^+(x_0', \xi')\lambda^{j-k}, \quad N_k(x_0', \xi, \lambda) = \sum_{j=1}^{d} b^{jk}(x_0', \xi')A_{d-j}^+(x_0', \xi', \lambda),$$

$1 \leq k \leq d$. Then $\Omega_k(x_0', \xi', x_n)$ for $\xi' \neq 0$ can be represented by the formula

(8)
$$\Omega_k(x_0', \xi', x_n) = \frac{1}{2\pi i}\int_{\gamma^+}\frac{e^{i\lambda x_n}N_k(x_0', \xi', \lambda)}{\widetilde{A}^+(x_0', 0, \xi', \lambda)}\,d\lambda.$$

In (8) γ^+ is a Jordan contour in a half-plane $Im\,\lambda > 0$, which embraces lying in this half-plane roots of the equation $\widetilde{A}(x_0', 0, \xi', \lambda) = 0$. As in [7] it can be checked, that the function $\Omega_k(x_0', \xi', x_n)$, defined by (8), satisfies the system (7).

Theorem 4. *Let* (4) *be a* P*-elliptic problem with continuous and bounded in* R^n_+ *coefficients and let* Ω_k *be defined by* (8). *Then if* $l\geq 0$ *is an integer and* $0<\alpha<1$, *the following inequalities are valid*

$$\int_0^\infty |D_n^l\,\Omega_k(x_0',\xi',x_n)|^2\,dx_n\leq c\,(\mu'(\xi'))^{2(-m_k+(l-1/2)/r)},$$

$$\int_0^\infty\int_0^\infty \frac{|D_n^l\,\Omega_k(x_0',\xi',x_n)-D_n^l\,\Omega_k(x_0',\xi',y_n)|^2}{|x_n-y_n|^{1+2\alpha}}\,dx_n\,dy_n$$

$$\leq c\,(\mu'(\xi'))^{2(-m_k+(l+\alpha-1/2)/r)},\quad 1\leq k\leq d,$$

with constant c, *independent* of $x_0'\in R^{n-1}$, $\xi'\in R^{n-1}\setminus 0$.

The proof of theorem 4 follows from the listed below lemmas.

Lemma 1. *For all* $\alpha\in R^n$, $\alpha\geqq 0$ *and* $\xi\in R^n$ *it is fulfilled*

(9)
$$|\xi_1|^{\alpha_1}\ldots,|\xi_n|^{\alpha_n}\leq(\mu(\xi))^{\langle\alpha\rangle}.$$

Proof. It is sufficient to prove lemma 1 in the case $\alpha\in S$ because every $\alpha\in R^n$, $\alpha\geq 0$, can be written as $\alpha=\langle\alpha\rangle.\alpha'$, where $\alpha'\in S$. But in this special case (9) follows immediately from the Young inequality.

Lemma 2. *Suppose* $A(x, D)$ *satisfies the condition* (2) *for* P*-ellipticity and let* $\lambda=\lambda(x,\xi')$ *be a root of the equation* (3) *Then for* $x\in R^n_+$, $\xi'\in R^{n-1}$ *it is fulfilled*

(10)
$$|\lambda(x,\xi')|\leq c\,(\mu'(\xi'))^{1/r},\quad |\operatorname{Im}\lambda(x,\xi')|\geq c_1(\mu'(\xi'))^{1/r},\quad c_1>0,$$

c, c_1 *independent of* x.

Proof. For $\xi'\neq 0$ let us write the equation $\widetilde{A}(x,\xi',\lambda)=0$ in the form

$$\widetilde{A}(x,\xi',\lambda)=\sum_{k=0}^{mr}a_k(x,\xi')\,\lambda^k$$

$$=(\mu'(\xi'))^m\sum_{k=0}^{mr}[a_k(x,\xi')(\mu'(\xi'))^{k/r-m}]\left[\frac{\lambda}{(\mu'(\xi'))^{1/r}}\right]^k=0.$$

It easily follows from (A3) and (1), that if $\alpha=(\alpha',\alpha_n)\in R^n$, $\alpha\geq 0$, then the equality $\langle\alpha\rangle=\langle(\alpha',0)\rangle+\alpha_n/r$ holds. Consequently we have

$$a_k(x,\xi')=\sum_{\substack{\langle\alpha\rangle=m\\ \alpha_n=k}}a_\alpha(x)\,\xi'^{\alpha'}=\sum_{\substack{\langle(\alpha',0)\rangle=m-k/r\\ \alpha_n=k}}a_\alpha(x)\,\xi'^{\alpha'}.$$

From lemma 1 and the fact, that $a_\alpha(x)$ are bounded, it follows the estimate $|a_k(x,\xi')|\leq c\,(\mu'(\xi'))^{m-k/r}$, c independent of x, ξ'. The P-ellipticity condition guarantees the inequality $|a_{mr}(x,\xi')|\equiv|a_{0,mr}(x)|\geq c>0$, consequently $\lambda/(\mu'(\xi'))^{1/r}$ is bounded because it is a root of an equation with bounded coefficients.

In order to prove the second inequality in (10), we note that

$$|\widetilde{A}(x,\xi)|=|a_{0,mr}(x)|\prod_{j=1}^{mr}|\xi_n-\lambda_j(x,\xi')|\geq c\,(\mu(\xi))^m.$$

Since $|a_{0,mr}(x)|$ is bounded for $x\in R^n_+$ and for $2\leq j\leq mr$ the estimate $|\xi_n-\lambda_j(x,\xi')|\leq|\xi_n|+|\lambda_j(x,\xi')|\leq c\,(\mu(\xi))^{1/r}$ is valid, we obtain $|\xi_n-\lambda_1(x,\xi')|\geq c_1(\mu(\xi))^{1/r}\geq c_1.(\mu'(\xi'))^{1/r}$, $c_1>0$ independent of x, ξ. It remains to set $\xi_n=\operatorname{Re}\lambda_1(x,\xi')$.

Corollary 4. *Let $A(x, D)$ be a P-elliptic operator. Then there are numbers $D>\delta>0$, such that if λ does not belong to the set*

$$\{\lambda \in \mathbb{C} : |\lambda| < D,\ Im\,\lambda > \delta\}, \tag{11}$$

the following estimate holds $|\widetilde{\mathbf{A}}^{+}(x, \xi', \lambda(\mu'(\xi'))^{1/r})| \geq c(\mu'(\xi'))^{d/r}$, $c>0$.

Lemma 3. *If $x_0' \in R^{n-1}$, $\xi' \in R^{n-1}\setminus 0$, $|\lambda| \leq L < \infty$, then the inequality*

$$|N_k(x_0', \xi', \lambda(\mu'(\xi'))^{1/r})| \leq c(\mu'(\xi'))^{-m_k+(d-1)/r} \tag{12}$$

holds with constant c, independent of x_0', ξ', λ.

Proof. Using the definition of an inverse matrix, the inequality (3) and the estimate $|b_{jk}^{+}(x_0', \xi')| \leq c(\mu'(\xi'))^{m_j-(k-1)/r}$, we get easily the inequality

$$|b^{jk}(x_0', \xi')| \leq c(\mu'(\xi'))^{-m_k+(j-1)/r}\ , \tag{13}$$

where c does not depend of x_0', ξ'. It follows from the Viète formulae and lemma 2, that $|a_k^{+}(x_0', \xi')| \leq c(\mu'(\xi'))^{k/r}$, consequently we have

$$|A_j^{+}(x_0', \xi', \lambda(\mu'(\xi'))^{1/r})| \leq c(\mu'(\xi'))^{j/r} \tag{14}$$

for $\lambda \in \mathbb{C}$, $|\lambda| \leq L$. Now the inequality (12) follows from (13) and (14).

Proof of theorem 4. Substitute $\lambda = \lambda_1(\mu'(\xi'))^{1/r}$ in (8) and transform the contour γ_1^{+}, obtained after this change, to a contour γ, which is actually a boundary of the domain, defined by (11) (and which consequently does not depend on x_0', ξ'). Using lemma 3 and corollary 4, we obtain

$$|D_n^l\, \Omega_k(x_0', \xi', x_n)| \leq c(\mu'(\xi'))^{-m_k+l/r} . \exp(-\delta(\mu'(\xi'))^{1/r} . x_n)$$

($\delta>0$ is the number, which appears in corollary 4). By raising the second power and integrating with respect to x_n from 0 to ∞ we get the first of the inequalities in theorem 4. The second can be proved analogically.

3. Spaces $H^{s,\mu}(R^n)$, $H^{s,\mu}(R_{+}^{n})$. In this paragraph we shall enumerate several specific properties of the spaces $H^{s,\mu}(R^n)$, $H^{s,\mu}(R_{+}^{n})$, which remain outside the general theory in [2].

In $H^{s,\mu}(R_{+}^{n})$, $s\geq 0$ besides the norm

$$\|u\|'_{s,\mu}(R_{+}^{n}) = \inf\{\|\widetilde{u}\|_{s,\mu} : \widetilde{u} \in H^{s,\mu}(R^n),\ \widetilde{u}|_{R_{+}^{n}} = u\}, \tag{15}$$

we use the following norms

a) if $s.r$ is integer

$$\|u\|_{s,\mu}^{2}(R_{+}^{n}) = \frac{1}{(2\pi)^{n-1}} \int_0^\infty \int_{R^{n-1}} (1+\mu'(\xi'))^{2s} |F'u(\xi', x_n)|^2\, d\xi'\, dx_n$$
$$+ \frac{1}{(2\pi)^{n-1}} \int_0^\infty \int_{R^{n-1}} |F' D_n^{sr} u(\xi', x_n)|^2\, d\xi'\, dx_n, \tag{16}$$

b) if $s.r$ is not integer

$$\|u\|_{s,\mu}^{2}(R_{+}^{n}) = \frac{1}{(2\pi)^{n-1}} \int_0^\infty \int_{R^{n-1}} (1+\mu'(\xi'))^{2s} |F'u(\xi', x_n)|^2\, d\xi'\, dx_n$$
$$+ \frac{1}{(2\pi)^{n-1}} \int_0^\infty \int_0^\infty \int_{R^{n-1}} \frac{|F' D_n^{[sr]} u(\xi', x_n) - F' D_n^{[sr]} u(\xi', y_n)|^2}{|x_n - y_n|^{1+2(sr-[sr])}}\, d\xi'\, dx_n\, dy_n. \tag{17}$$

In (16), (17) F' is the partial Fourier transformation with respect to x' and $[sr]$ denotes the whole part of sr.

Let us denote by L the Hestens extension operator

$$\tilde{u}(x', x_n) = Lu(x', x_n) = \begin{cases} u(x', x_n), & x_n \geq 0, \\ \sum_{k=1}^{N} \lambda_k u(x', -x_n/k), & x_n < 0. \end{cases} \tag{18}$$

In (18) N is large enough, and the numbers λ_k satisfy the system $\sum_{k=1}^{N} (-1/k)^j \lambda_k = 1$, $j=0, \ldots, N-1$. Without difficulties, we obtain the inequality

$$\| Lu \|_{s,\mu} \leq c \| u \|_{s,\mu} (R^n_+), \quad s \geq 0, \quad u \in H^{s,\mu}(R^n_+). \tag{19}$$

With the help of (19), we deduce immediately, that the norm (15) is equivalent to (16) for sr integer and to (17) otherwise.

In the following lemmas by Ω will be denoted R^n or R^n_+ and $L^\infty(\Omega) = \{\varphi(x) \in C^\infty(\Omega) : \sup_\Omega |D^\alpha \varphi| < \infty \forall \alpha\}$.

Lemma 4. *The mapping* $u \to D^\alpha u$ *is continuous from* $H^{s,\mu}(\Omega)$ *into* $H^{s-\langle\alpha\rangle,\mu}(\Omega)$.

Lemma 5. *Let* $\varphi(x) \in L^\infty(\Omega)$. *Then the mapping* $u \to \varphi u$ *is continuous from* $H^{s,\mu}(\Omega)$ *into* $H^{s,\mu}(\Omega)$ *and*

$$\| \varphi u \|_{s,\mu}(\Omega) \leq c_s | \varphi |_s \| u \|_{s,\mu}(\Omega),$$

$u \in H^{s,\mu}(\Omega)$, *where* $|\varphi|_s = \sup_{\langle\alpha\rangle \leq [s]+1} |D^\alpha \varphi|$ *and* c_s *is an universal constant.*

Lemma 6. *If* $s_1 > s_2 > s_3$, *then for any* $\varepsilon > 0$ *there exists a constant* c_ε, *so that*

$$\| u \|_{s_2,\mu}(\Omega) \leq \varepsilon \| u \|_{s_1,\mu}(\Omega) + c_\varepsilon \| u \|_{s_3,\mu}(\Omega), \quad u \in H^{s_1,\mu}(\Omega).$$

In the next two lemmas we suppose that P satisfies in addition (A5). Let $x = (x', x'')$, $x' = (x_1, \ldots, x_p)$, $x'' = (x_{p+1}, \ldots, x_n)$ be the partition of the variables, connected with this condition.

Lemma 7. *Let* P *satisfy* (A5). *Then there exists a constant* $\varrho > 0$, *such that if* $\varphi(x'') \in L^\infty(\Omega)$, $u(x) \in H^{s,\mu}(\Omega)$, *the inequality*

$$\| \varphi(x'') D^\alpha u(x) - D^\alpha(\varphi(x'') u(x)) \|_{s,\mu}(\Omega) \leq c \| u \|_{s+\langle\alpha\rangle-\varrho,\mu}(\Omega)$$

is valid with c, *independent of* n.

Lemma 8. *Let* P *satisfy* (A5). *Then there exists a number* $\varrho > 0$, *such that if* $\varphi(x'') \in C_0^\infty(R^{n-p})$, $u(x) \in H^{s,\mu}(\Omega)$, *then the inequality*

$$\| \varphi(x'') u(x) \|_{s,\mu}(\Omega) \leq \sup_\Omega |\varphi(x'')| \, \| u \|_{s,\mu}(\Omega) + c \| u \|_{s-\varrho,\mu}(\Omega)$$

holds.

Lemma 9. *Let* $s > 1/2r$ *and* $u(x) \in H^{s,\mu}(\Omega)$. *Then the mapping* $x_n \to u(\cdot, x_n)$ *is a continuous mapping of* x_n *onto* $H^{s-1/2r,\mu'}(R^{n-1})$. *The inequality*

$$\| u(\cdot, x_n) \|_{s-1/2r,\mu'}(R^{n-1}) \leq c \| u \|_{s,\mu}(\Omega)$$

holds with constant c, *independent of* u, x_n.

We are not going to prove here lemmas 4 – 9.

4. **Proof of theorems 1, 2. 3.** Proof of Theorem 1. Theorem 1 follows immediately from lemmas 4, 5 and 9.

Proof of theorem 2. We consider first the case, where the operators A, B_j are of the form

$$A(D)\equiv \sum_{\langle\alpha\rangle=m} a_\alpha D^\alpha,\quad B_j(D)\equiv \sum_{\langle\beta\rangle=mj} b_{j\beta} D^\beta,\quad 1\leq j\leq d, \tag{20}$$

a_α, $b_{j\beta}=$const. Let $s\geq s_0$ be fixed. We can assume without loss of generality that $u\in C^\infty_{(0)}(R^n_+)$. Here $C^\infty_{(0)}(R^n_+)$ is the space of functions $u\in C^\infty(R^n_+)$, equal to zero for $|x|$ large enough. We abbreviate $f=A(D)u$, $g_j=B_j(D)u|_{x_n=0}$, $1\leq j\leq d$. Let f_0 be an extension of f in the whole space R^n, defined with the help of the Hestens formula (20), with N large enough. Denote by F and F^{-1} (F', and F'^{-1}, respectively) the Fourier transformation and its inverse in R^n (in R^{n-1}, respectively). Set $u_0(x)=F^{-1}(Ff_0(\xi)/A(\xi))$, $g_{j0}(x')=B_j(D)u_0(x)|_{x_n=0}$, $1\leq j\leq d$, and for $\xi'\neq 0$ $v(\xi', x_n)=F'u(\xi', x_n)-F'u_0(\xi', x_n)$. We have for $\xi'\neq 0$

$$\begin{gathered} A\left(\xi', -i\frac{\partial}{\partial x_n}\right)v(\xi', x_n)=0,\quad x_n\geq 0,\\ B_j\left(\xi', -i\frac{\partial}{\partial x_n}\right)v(\xi', x_n)|_{x_n=0}=F'g_j(\xi')-F'g_{j0}(\xi'),\quad 1\leq j\leq d. \end{gathered} \tag{21}$$

Since $v(\xi', x_n)$ belongs to the space of decreasing solutions of the system (21), then for $\xi'\neq 0$, $x_n\geq 0$ the representation

$$v(\xi', x_n)=\sum_{j=1}^{d}(F'g_j(\xi')-F'g_{j0}(\xi'))\,\Omega_j(\xi', x_n)$$

is valid (see p. 2). In this way we obtain the equality

$$F'u(\xi', x_n)=F'u_0(\xi', x_n)+\sum_{j=1}^{d}(F'g_j(\xi')-F'g_{j0}(\xi'))\,\Omega_j(\xi', x_n).$$

Using this last equality, the norms (16), (17) and the estimates for Ω_j, proved in theorem 4, we obtain the inequality (6) just as in [9], proposition 14. 1.

We pass to the general case. Denote by R^{n-p}_+ the space of points $x''=(x_{p+1},\dots,x_n)$, for which $x_n\geq 0$. We fix K so large, that outside $\mathfrak{U}=\{x''\in R^{n-p}_+: |x''|\leq K\}$ the coefficients in the principal parts of the operators in (4) are constants. Let $\delta>0$. Choose points $x''_k\in\mathfrak{U}\cap\{x_n=0\}$, $1\leq k\leq N_0$, $x''_k\in\mathfrak{U}\cap\{x_n\geq\frac{3}{2}\delta\}$, $N_0+1\leq k\leq N$, so that the domains $\mathfrak{U}_k=\{x''\in R^{n-p}_+: |x''-x''_k|<\delta\}$, $1\leq k\leq N$, cover $\mathfrak{U}:\mathfrak{U}\subset\bigcup_{k=1}^{N}\mathfrak{U}_k$. Let $\{\varphi_k(x'')\}$ be the partition of unity for $\mathfrak{U}$, connected with the covering $\{\mathfrak{U}_k\}$. We denote $\mathfrak{U}_0=R^{n-p}_+\setminus\mathfrak{U}$, $\varphi_0(x'')=1-\sum_{k=1}^{N}\varphi_k(x'')$. Fix a point $x''_0\in\mathfrak{U}_0\cap\{x_n=0\}$. We construct functions $\psi_k(x'')\in C^\infty(R^{n-p}_+)$ so that $\operatorname{supp}\psi_k\subset\mathfrak{U}_k$, $\psi_k=1$ on $\operatorname{supp}\varphi_k$ $(0\leq k\leq N)$. Let us introduce the operators

$$\widetilde{\mathcal{A}}(x'', D)u=\{\widetilde{A}(x'', D)u, \widetilde{B}_1(x'', D)u|_{x_n=0},\dots,\widetilde{B}_d(x'', D)u|_{x_n=0}\},$$

$$\widetilde{\mathcal{A}}_k=\widetilde{\mathcal{A}}(x''_k, D),\quad 0\leq k\leq N.$$

For $1\leq k\leq N_0$ we have $\mathcal{A}\varphi_k=\widetilde{\mathcal{A}}_k\varphi_k+(\mathcal{A}-\widetilde{\mathcal{A}})\varphi_k+(\widetilde{\mathcal{A}}-\widetilde{\mathcal{A}}_k)_{\varphi_k}$. It follows from the proved already part of the theorem, that

$$\|\varphi_k u\|_{s,\mu}(R^n_+)\leq c_0(\langle\widetilde{\mathcal{A}}_k\varphi_k u\rangle_{s,\mu}+\|\varphi_k u\|_0(R^n_+)).$$

Using the results of p. 3, we obtain

$$\langle(\mathcal{A}-\widetilde{\mathcal{A}})\varphi_k u\rangle_{s,\mu}\leq c\|\varphi_k u\|_{s-\varrho,\mu}(R^n_+)\leq\varepsilon\|\varphi_k u\|_{s,\mu}(R^n_+)+c\|\varphi_k u\|_0(R^n_+),\ \varrho>0;$$

$$\langle(\widetilde{\mathcal{A}}-\widetilde{\mathcal{A}}_k)\varphi_k u\rangle_{s,\mu}=\langle\psi_k(\widetilde{\mathcal{A}}-\widetilde{\mathcal{A}}_k)\varphi_k u\rangle_{s,\mu}\leq\varepsilon\|\varphi_k u\|_{s,\mu}(R^n_+)+c\|\varphi_k u\|_0(R^n_+)$$

for δ small enough.

Putting $\varepsilon=1/4c_0$ and combining the above inequalities, we get (for $1\leq k\leq N_0$)

$$\|\varphi_k u\|_{s,\mu}(R^n_+)\leq 2c_0\langle\mathcal{A}\varphi_k u\rangle_{s,\mu}+c'\|\varphi_k u\|_0(R^n_+). \tag{22}$$

In a similar way with the help of the identity $\mathcal{A}\varphi_0=\widetilde{\mathcal{A}}_0\varphi_0+(\mathcal{A}-\widetilde{\mathcal{A}})\varphi_0$, we obtain an inequality of the type (22) with $k=0$.

For $k\geq N_0+1$ we can use theorem 3 in [1]. This theorem guarantees the inequality

$$\|\varphi_k u\|_{s,\mu}(R^n)\leq 2c_0\|A\varphi_k u\|_{s-m,\mu}(R^n)+c'\|\varphi_k u\|_0(R^n).$$

Obviously, the last inequality is equivalent to (22) for $N_0+1\leq k\leq N$.

Finally, we obtain

$$\|u\|_{s,\mu}(R^n_+)\leq\sum_{k=0}^{N}\|\varphi_k u\|_{s,\mu}(R^n_+)\leq\sum_{k=0}^{N}(2c_0\langle\mathcal{A}\varphi_k u\rangle_{s,\mu}+c'\|\varphi_k u\|_0(R^n_+))$$

$$\leq 2c_0\sum_{k=0}^{N}\langle\varphi_k\mathcal{A}u\rangle_{s,\mu}+2c_0\sum_{k=0}^{N}\langle(\mathcal{A}\varphi_k-\varphi_k\mathcal{A})u\rangle_{s,\mu}+c\|u\|_0(R^n_+)$$

$$\leq c_1\langle\mathcal{A}u\rangle_{s,\mu}+c_2\|u\|_{s-\varrho,\mu}(R^n_+)+c\|u\|_0(R^n_+),\ \varrho>0.$$

It remains to use the interpolation inequality in lemma 6. Theorem 2 is proved.

P o o f of t h e o r e m 3. We again consider first the special case, when the operators A, B_j have the form (20).

Let $s'>s_0$ be a real number. We shall prove, that there are operators $\mathcal{R}$, $\mathcal{S}$ and T, satisfying for $s_0\leq s\leq s'$ the conditions

(a) $\mathcal{R}$ is a continuous operator from $\mathcal{H}^{s,\mu}(R^n_+)$ into $H^{s,\mu}(R^n_+)$,

(b) $\mathcal{S}$ is a continuous operator from $\mathcal{H}^{s,\mu}(R^n_+)$ into $\mathcal{H}^{s',\mu}(R^n_+)$,

(c) T is a continuous operator from $H^{s,\mu}(R^n_+)$ into $H^{s',\mu}(R^n_+)$,

(d) $\mathcal{A}\mathcal{R}=\mathfrak{J}+\mathcal{S}$, $\mathcal{R}\mathcal{A}=I+T$.

Here $\mathfrak{J}$ and I are the unit operators in $\mathcal{H}^{s,\mu}(R^n_+)$ and $H^{s,\mu}(R^n_+)$, respectively.

Assume the operators $\mathcal{R}$ and $\mathcal{S}$ are already constructed, so that the conditions (a), (b), (d) be satisfied. Denote $T=\mathcal{R}\mathcal{A}-I$. We shall establish (c). Obviously T is a bounded operator in $H^{s,\mu}(R^n_+)$, $s_0\leq s\leq s'$. It follows from $\mathcal{A}\mathcal{R}\mathcal{A}$

$=(\mathfrak{J}+\mathfrak{S})\mathcal{A}=\mathcal{A}(I+T)$ that $\mathfrak{S}\mathcal{A}=\mathcal{A}T$. Then according to (b) and theorem 2 we have for $u\in H^{s,\mu}(R^n_+)$, $s_0\leqq s\leqq s'$

$$\|Tu\|_{s',\mu}(R^n_+)\leq c(\langle \mathcal{A}Tu\rangle_{s',\mu}+\|Tu\|_{s,\mu}(R^n_+))=c(\langle \mathfrak{S}\mathcal{A}u\rangle_{s',\mu}+\|Tu\|_{s,\mu}(R^n_+))$$

$$\leq c(\langle \mathcal{A}u\rangle_{s,\mu}+\|u\|_{s,\mu}(R^n_+))\leqq c\|u\|_{s,\mu}(R^n_+).$$

Thus (c) is proved.

Let us construct $\mathfrak{R}$ and $\mathfrak{S}$ now. Let L be the continuous extension operator for $H^{s,\mu}(R^n_+)$ into $H^{s,\mu}(R^n)$, $0\leqq s\leq s'-m$, defined by the Hestens formula. Choose a function $\varphi(\xi)\in C_0^\infty(R^n)$,

$$0\leqq\varphi\leq 1,\qquad \varphi(\xi)=\begin{cases}1 & \text{for } |\xi|\leq 1,\\ 0 & \text{for } |\xi|\geq 2,\end{cases}$$

and fix it. Let γ be the restriction operator from R^n onto R^n_+ : $\gamma n=n|_{R^n_+}$. We define the operators

$$R_0=\gamma F^{-1}\frac{(\mu(\xi))^m}{(\varphi(\xi)+(\mu(\xi))^m)A(\xi)}FL,$$

$$R_j=\gamma F'^{-1}\frac{(\mu'(\xi'))^{m_j+1/2}}{\varphi(\xi',0)+(\mu'(\xi'))^{m_j+1/2}}\Omega_j(\xi',x_n)F',$$

$$1\leq j\leq d.$$

It is not difficult to check, that the operators $\mathfrak{R}$, $\mathfrak{S}$,

$$\mathfrak{R}(f,g_1,\ldots,g_d)=R_0f+\sum_{j=1}^{d}R_j(g_j-B_jR_0f|_{x_n=0}),$$

$$\mathfrak{S}=\mathcal{A}\mathfrak{R}-\mathfrak{J}$$

satisfy the conditions (a), (b). Thus theorem 3 is proved in this special case when the operators A, B_j have the form (20).

Let us consider now that general cese. We shall use the notations, introduced in proving theorem 2. For $0\leq k\leq N$ set

$$\mathcal{A}_k(x,D)=\widetilde{\mathcal{A}}_k(D)+\psi_k(x'')(\mathcal{A}(x,D)-\widetilde{\mathcal{A}}_k(D)).$$

We shall prove, that for a suitable $\delta>0$, there exist operators $\mathfrak{R}_k$, T_k, $0\leq k\leq N$, with the properties

(a1) $\mathfrak{R}_k$ is a continuous operator from $\mathcal{H}^{s,\mu}(R^n_+)$ into $H^{s,\mu}(R^n_+)$ for $0\leq k\leq N_0$ and form $H^{s-m,\mu}(R^n)$ into $H^{s,\mu}(R^n)$ for $N_0+1\leq k\leq N$,

(b1) T_k is a continuous operator from $H^{s-\sigma,\mu}(R^n_+)$ into $H^{s,\mu}(R^n_+)$ for $0\leq k\leqq N_0$ and from $H^{s-\sigma,\mu}(R^n)$ into $H^{s,\mu}(R^n)$ for $N_0+1\leq k\leq N$,

(c1) $\mathfrak{R}_k\mathcal{A}_k(x,D)=I+T_k$.

In (a1), (b1) $\sigma>0$ and $s_1\leq s\leqq s_1+\sigma$.

Assume that the operators $\mathfrak{R}_k$, T_k, $0\leq k\leq N$, with the above properties are already constructed. We set

$$\mathfrak{R}=\sum_{k=0}^{N}\mathfrak{R}_k\varphi_k(x''),\quad T=\mathfrak{R}\mathcal{A}(x,D)-I.$$

It follows from (al) that $\mathfrak{R}$ satisfies (i) in theorem 3. It remains to show, that T satisfies (ii). We have

$$T=\sum_{k=0}^{N}\mathfrak{R}_k\varphi_k(x'')\mathcal{A}(x,\ D)-I=\sum_{k=0}^{N}\mathfrak{R}_k\varphi_k(x'')\mathcal{A}_k(x,\ D)-I$$

$$=\sum_{k=0}^{N}(\mathfrak{R}_k\mathcal{A}_k(x,\ D)\varphi_k(x'')+P_k)-I=\sum_{k=0}^{N}(T_k\varphi_k(x'')+P_k).$$

For the operator $P_k=\mathfrak{R}_k(\varphi_k(x'')\mathcal{A}_k(x,\ D)-\mathcal{A}_k(x,D)\varphi_k(x''))$ we have according to (al) and lemmas 4, 5, 7

$$\| P_k u\|_{s,\mu}=\| P_k\psi_k(x'')u\|_{s,\mu}\leqq c\|\psi_k(x'')u\|_{s-\varrho,\mu}\leqq c\| u\|_{s-\varrho,\mu}(R^n_+)$$

($\varrho>0$; where it is not mentioned, the norms are taken over R^n_+ $(0\leqq k\leqq N_0)$ and over $R^n(N_0+1\leqq k\leqq N)$). Similarly

$$\| T_k\varphi_k(x'')u\|_{s,\mu}\leqq c\|\varphi_k(x'')u\|_{s-\sigma,\mu}\leqq c\| u\|_{s-\sigma,\mu}(R^n_+).$$

Consquently T satisfies (ii) with $\sigma'=\min(\varrho,\ \sigma)>0$.

It remains to prove (al), (bl), (cl). Let first $0\leqq k\leqq N_0$. It follows from the first part of proving, that there are operators $\widetilde{\mathfrak{R}}_k$, $\widetilde{T}_k$, such that for $s_1\leqq s\leqq s_1+\sigma$ $(\sigma>0)$ $\widetilde{\mathfrak{R}}_k$ is a continuous operator from $\mathcal{H}^{s,\mu}(R^n_+)$ into $H^{s,\mu}(R^n_+)$, $\widetilde{T}_k$ is a continuous operator from $H^{s-\sigma,\mu}(R^n_+)$ into $H^{s,\mu}(R^n_+)$ and $\widetilde{\mathfrak{R}}_k\widetilde{\mathcal{A}}_k=I+\widetilde{T}_k$. We have $\widetilde{\mathfrak{R}}_k\mathcal{A}_k(x,\ D)=I+\widetilde{T}_k+P_k$, where $P_k=\widetilde{\mathfrak{R}}_k\psi_k(x'')(\mathcal{A}(x,D)-\widetilde{\mathcal{A}}_k(D))$.
Using the results of p. 3, it is easy to see, that if $\delta>0$ is small enough, we can write $\widetilde{T}_k+P_k$ in the form $\widetilde{T}_k+P_k=P_k'+P''_k$, where

$$\| P_k'u\|_{s,\mu}(R^n_+)\leqq\frac{1}{2}\| u\|_{s,\mu}(R^n_+)\ \ (s_1\leqq s\leqq s_1+\sigma),$$

$$\| P''_k u\|_{s,\mu}(R^n_+)\leqq c\| u\|_{s-\sigma,\mu}(R^n_+).$$

Consequently, we can take

$$\mathfrak{R}_k=(I+P_k')^{-1}\widetilde{\mathfrak{R}}_k,\ \ T_k=(I+P_k')^{-1}P''_k.$$

Thus for $0\leqq k\leqq N_0$ operators $\mathfrak{R}_k$, T_k, satisfying (al) (bl) (cl), are constructed. In the case $N_0+1\leqq k\leqq N$ the existence of $\mathfrak{R}_k$, T_k follows from theorem 4 in [1].

Theorem 3 is completely proved.

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Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

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