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SELECTION AND FACTORIZATION THEOREMS FOR SET-VALUED MAPPINGS

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A unified method for proving selection and factorization theorems for set-valued mappings is presented. Some elements of this method were announced in the papers of Nedev and Čoban (1974), Nedev (1975) and Čoban and Valov (1975).

Applications are made in general dimension theory and some extension theorems of Hanner-Dowker type are obtained.

A number of selection theorems appeared in the literature beginning with the E. Michael's paper [12]. These theorems were obtained by different authors with various methods. Moreover, as it was pointed out by the author in [18], that the proofs of some of these theorems were erroneous and this has not been recognized for a long time (see [16]).

Here we give a unified approach to this topic. Some elements of such an approach were announced by S. Nedev and M. Čoban in [19], where the great closeness between the problems of finding a selection and finding a factorization for given set-valued carrier was pointed out as well. In this paper selection and factorization theorems are obtained by the same method. The key role is played by the notion of carrier having the selection-factorization property (abbr.-s. f. p. carriers). Selection and factorization theorems are proved for s. f. p. carriers and then in the yet known cases it is shown by simple verification that the corresponding carrier is s. f. p. one.

In order to make the exposition of our method self-contained and to give the reader the possibility to compare easily this method with other ones, we present here the proofs or outlines of proofs of a number of known results. As a rule these proofs are completely or partially modified, simplified or generalized, so that a form is given to them, which is, according to our opinion, more convenient for our purposes.

1. Some definitions and preliminary results. If A is a set then $|A|$ is the cardinality of A. If X is a topological space and $A\subset X$, then $[A]$ and $\langle A\rangle$ stand for the closure and the interior of A, respectively. If φ is a collection of subsets of X, then the order $\operatorname{Ord}(\varphi, A)$ is the smallest cardinal number τ such that for every point $x\in A$ we have $\tau>|\{M\in\varphi : x\in M\}|$. In the case when $\operatorname{Ord}(\varphi, A)\leqq\aleph_0$, φ is called point-finite on A. The collection φ is called locally-finite [1] if for every $x\in X$ there exists a neighbourhood Ox of x which intersects no more than finitely many elements of φ; φ is called discrete if for every $x\in X$ there is a neighbourhood of x intersecting at most one element of φ.

The dimension $\dim X$ of the space X is the smallest integer $n\geqq -1$ such that every finite open covering of X has an open refinement whose order on X is not greater than $n+2$. The space is called τ-pointwise-τ'-paracompact if every open covering ω of X with $|\omega| < \tau$ and $\operatorname{Ord}(\omega, X) \leq \tau'$ has an open locally finite refinement. The τ^+-pointwise-τ^+-paracompact spaces are said to be τ-paracompact. (τ^+ means the smallest cardinal number, which is greater than τ.) The spaces which are τ-paracompact for every τ are called paracompact [5]. We shall use also the notation ∞-pointwise-∞-paracompact space instead of paracompact space. In a similar sense the symbol ∞ will appear in some other notations.

The space X is called τ-collectionwise normal if X is a T_1-space and every discrete collection φ of closed subsets of X with $|\varphi| \leq \tau$ can be separated by an open discrete collection $\gamma = \{V_F \mid F \in \varphi\}$ (i. e. $F \subset V_F$ for every $F \in \varphi$). The 2-collectionwise normal spaces are called normal. It is easily seen that every normal space is $\aleph_0$-collectionwise normal one. ∞-collectionwise normal spaces are called collectionwise normal [3].

Lemma 1 (S. Lefshetz, [10]). *Let X be a normal space and $\omega = \{U_\alpha \mid \alpha \in A\}$ be a point-finite open covering of X. Then there is an open covering $\gamma = \{V_\alpha \mid \alpha \in A\}$ of X, which is an index-closure refinement of ω, i. e. $[V_\alpha] \subset U_\alpha$ for every $\alpha \in A$.*

Outline of the proof. Let T be the topology of X. Denote by $\mathfrak{M}$ the family of all couples (A', f') where $A' \subset A$ and $f' : A' \to T$ is a mapping with the properties:

1) $[f'(\alpha)] \subset U_\alpha$ for every $\alpha \in A'$;

2) $X = (\cup \{f'(\alpha) \mid \alpha \in A\}) \cup (\cup \{U_\alpha \mid \alpha \in A \setminus A'\})$.

We partially order $\mathfrak{M}$ in the obvious manner. Every simple ordered subset of $\mathfrak{M}$ has an obvious upper bound and hence, by Kuratowski — Zorn's lemma, $\mathfrak{M}$ has a maximal element (A_0, f_0). It is not difficult to check that $A_0 = A$, which completes the proof.

Recall now the definition of partition of unity. Let X be a topological space. A collection φ of continuous functions from X to the non-negative reals is called partition of unity on X if $\Sigma\{f(x) \mid f \in \varphi\} = 1$ for every $x \in X$. If ω is a covering of X and φ is a partition of unity on X, then, by definition, φ is subordinated to ω if every f in φ vanished outside some U in ω.

Lemma 2 ([11]). *Every open locally finite covering ω of a normal space X has a partition of unity subordinated to it.*

Proof. Let $\omega = \{U_\alpha \mid \alpha \in A\}$ and let $\gamma = \{V_\alpha \mid \alpha \in A\}$ be an open index-closure refinement of ω (see lemma 1). Using the Urysohn's lemma, for every $\alpha \in A$ we construct a continuous function $g_\alpha : X \to [0, 1]$ such that $g_\alpha([V_\alpha]) = 1$ and $g_\alpha(X \setminus U_\alpha) = 0$. Next define $g(x) = \Sigma\{g_\alpha(x) \mid \alpha \in A\}$ and $f_\alpha(x) = g_\alpha(x)/g(x)$ for every $x \in X$ and $\alpha \in A$. The collection $\varphi = \{f_\alpha \mid \alpha \in A\}$ is the required partition of unity.

Lemma 3 (M. R. Mather, [7]; see [17]). *If the partition of unity φ on a topological space X is subordinated to the covering ω of X, then ω has a locally finite open refinement.*

Proof. For every $i = 1, 2, \ldots$ and every $f \in \varphi$ define $V_f^i = \{x \in X \mid f(x) > i^{-1}\}$. Let us show that the family $\{V_f^i \mid f \in \varphi\}$ is locally finite for every $i = 1, 2, \ldots$. Take an arbitrary $x_0 \in X$ and let $\{f_1, \ldots, f_n\} \subset \varphi$ be such that $\Sigma_{j=1}^n f_j(x_0) > 1 - i^{-1}$. Define $Ox_0 = \{x \in X \mid \Sigma_{j=1}^n f_j(x) > 1 - i^{-1}\}$. It is obvious now that $f \notin \{f_1, \ldots, f_n\}$ implies $Ox_0 \cap V_f^i = \varnothing$. Thus $\{V_f^i \mid f \in \varphi\}$ is a locally finite collection. Next we put $P_f^i = \{x \in X \mid f(x) \geq (i+1)i^{-2}\}$, $G_f^i = \{x \in X \mid f(x) > (i+1)i^{-2}\}$, $P_i = \cup \{P_f^i \mid f \in \varphi\}$ and $G_i = \cup \{G_f^i \mid f \in \varphi\}$ for every $f \in \varphi$ and $i = 1, 2, \ldots$. Then V_f^i and G_f^i are open,

$G^i_f \subset P^i_f \subset V^i_f$ for every $f \in \varphi$ and $i=1, 2, \ldots$ and $X = \cup\{G_i \mid i=1, 2, \ldots\}$. Define now $U^1_f = V^1_f$ and $U^i_f = V^i_f \setminus (P_1 \cup \ldots \cup P_{i-1})$ for every $f \in \varphi$ and $i=2, 3, \ldots$. The collection $\gamma = \{U^i_f \mid f \in \varphi,\ i=1, 2, \ldots\}$ is a locally finite open covering of X, which refines ω. In fact, if $x \in X$ then $x \in U^{i_x}_{f_{i_x}}$, where $i_x = \min\{i \mid x \in V^i_{f_i}$ for some $f_i \in \varphi\}$. Moreover, if $x \in G_n$ and O_i is a neighbourhood of x intersecting at most initely many members of $\{V^i_f \mid f \in \varphi\}$ then $G_n \cap O_1 \cap \ldots \cap O_n$ is a neighbourhood of x, which intersects only finitely many elements of γ. This completes the proof.

The collection φ of subsets of a topological space X is called closure-preserving if $[\cup\{F \mid F \in \varphi'\}] = \cup\{[F] \mid F \in \varphi'\}$ for every subcollection φ' of φ; the collection φ is called hereditarily-closure preserving if every collection ψ of the form $\psi = \{H_F \mid F \in \varphi\}$, where $H_F \subset F$ for every $F \in \varphi$, is closure preserving. Note that every locally finite collection of sets is hereditarily closure preserving. Yet, every closed, point-finite and closure preserving collection is locally finite.

Lemma 4. *Let the covering ω of a topological space X has a refinement γ with some of the following properties*:

1. *γ is a (closed or open) locally finite collection;*
2. *γ is a (closed or open) (hereditarily) closure-preserving collection;*
3. $\mathrm{Ord}(\gamma, X) \leq n$.

Then ω has an index-refinement with the same property.

Proof. For every $\Gamma \in \gamma$ fix a $U_\Gamma \in \omega$ such that $\Gamma \subset U_\Gamma$ and define $H_U = \cup\{\Gamma \in \gamma \mid U_\Gamma = U\}$ for every U in ω. Then the covering $\{H_U \mid U \in \omega\}$ is the required one.

Lemma 5. *Let $\tau > \aleph_0$ and $\tau' \geq \aleph_0$ be cardinal numbers and X be a T_1-space. If every open covering ω of X with $|\omega| < \tau$ and $\mathrm{Ord}(\omega, X) \leq \tau'$ has a closed closure preserving refinement, then X is normal and τ-pointwise-τ'-paracompact.*

Proof (the idea is taken from E. Michael's proof in [13]). First of all let us show that X is normal. Let F_1 and F_2 be disjoint closed subsets of X. By lemma 4 there is a closed covering $\{P_1, P_2\}$ which is an index-refinement of the covering $\{X \setminus F_1, X \setminus F_2\}$. Define $U_1 = X \setminus P_1$ and $U_2 = X \setminus P_2$. Then $U_1 \cap U_2 = \emptyset$, $F_1 \subset U_1$, $F_2 \subset U_2$ and both U_1 and U_2 are open sets. So X is normal.

Let now $\omega = \{U_\alpha \mid \alpha \in A\}$ be an open covering of X such that $|\omega| < \tau$ and $\mathrm{Ord}(\omega, X) \leq \tau'$. We assume that A is a well-ordered set and define by induction a sequence $\{\mu_i = \{P^i_\alpha \mid \alpha \in A\},\ i=1, 2, \ldots\}$ of closed closure-preserving coverings μ_i of X such that:

1. $P^i_\alpha \subset U_\alpha$ for every $\alpha \in A$ and $i=1, 2 \ldots$
2. $P^i_\alpha \cap P^j_\beta = \emptyset$ if $\beta > \alpha$ and $j > i$.

We begin with $\mu_1 = \{P^1_\alpha \mid \alpha \in A\}$, which is a closed closure-preserving index-refinement of ω existing by the assumption and by lemma 4. Assume now that $\mu_1, \mu_2, \ldots, \mu_n$ have already been constructed. Define $U^{n+1}_\alpha = U_\alpha \setminus \cup\{P^i_\beta \mid \beta < \alpha,\ i=1, 2, \ldots, n\}$ for every $\alpha \in A$. Let us see that $\{U^{n+1}_\alpha \mid \alpha \in A\}$ is an open covering of X. Suppose $x \in X$ and let $\alpha_x = \min\{\alpha \in A \mid x \in U_\alpha\}$. Obviously, $x \in U^{n+1}_{\alpha_x}$. Then for μ_{n+1} we take a closure preserving index-refinement of $\{U^{n+1}_\alpha \mid \alpha \in A\}$. So the sequence $\{\mu_i \mid i=1, 2, \ldots\}$ is well constructed.

Now for every i and $\alpha \in A$ we put $V_\alpha^i = P_\alpha^i \setminus ((\cup \{P_\beta^i \mid \beta \neq \alpha\}) \cup (\cup \{P_\alpha^j \mid j \leq i-2\})$ and define $\gamma' = \{V_\alpha^i \mid \alpha \in A,\ i = 1, 2, \dots\}$. We are going to show that γ' is an open covering of X with $|\gamma'| < \tau$ and $\operatorname{Ord}(\gamma', X) \leq \aleph_0$. Fix $x \in X$ and let $\alpha_i = \min\{\alpha \in A \mid x \in P_\alpha^i\}$ for every i. Define $\alpha_0 = \min\{\alpha_i \mid i = 1, 2, \dots\}$ and $k = \min\{i \mid \alpha_i = \alpha_0\}$. Then $x \in V_{\alpha_k}^{k+1}$. In fact, $x \in P_{\alpha_k}^k$, whence, by 2., $x \notin P_\beta^{k+1}$ for $\beta > \alpha_k$. Also $x \notin P_\beta^{k+1}$ for $\beta < \alpha_k$ by the definition of α_k. Finally $x \notin P_{\alpha_k}^j$ for $j \leq k-1$ again by the definition of α_k. So, $x \in V_{\alpha_k}^{k+1}$ and γ' is a covering. The openness of γ' and the inequality $|\gamma'| < \tau$ are obvious. To show that $\operatorname{Ord}(\gamma', X) \leq \aleph_0$ it is sufficient to remark that the collections $\{V_\alpha^i \mid \alpha \in A\}$ are disjoint ones, because if $x \in V_{\alpha_k}^i$ then $x \notin V_\beta^j$ for $j \geq k+2$ and $\beta \in A$ (here k is the same as was defined above). Therefore, by the assumption of the lemma, there is a closed closure-preserving index-refinement $\{Q_\alpha^i \mid \alpha \in A,\ i = 1, 2, \dots\}$ of γ'. Since X is a normal space there are open sets Γ_i and G_i in X such that $Q_i \subset \Gamma_i \subset [\Gamma_i] \subset G_i \subset [G_i] \subset V_i$, where $Q_i = \cup\{Q_\alpha^i \mid \alpha \in A\}$ and $V_i = \cup\{V_\alpha^i \mid \alpha \in A\}$ for every $i = 1, 2, \dots$. Now we define $W_\alpha^1 = V_\alpha^1 \cap G_1$ and $W_\alpha^i = V_\alpha^i \cap G_i \setminus \cup\{[\Gamma_j] \mid j \leq i-1\}$ for every $\alpha \in A$ and $i = 2, 3, \dots$. In order to finish the proof it is sufficient to show that $\gamma = \{W_\alpha^i \mid \alpha \in A,\ i = 1, 2, \dots\}$ is an open locally-finite covering of X. For, let $x \in X$ and let $i_0 = \min\{i \mid x \in G_i\}$. Then $x \in V_\alpha^{i_0}$ for some $\alpha \in A$ and $x \notin [\Gamma_j]$ for all $j < i_0$. So $x \in W_\alpha^{i_0}$. The local-finiteness of γ may be seen as follows. First we remark that the collection $\gamma_i = \{W_\alpha^i \mid \alpha \in A\}$ is discrete for every i. In fact, if $x \notin [G_i]$, then $O_i = X \setminus [G_i]$ is a neighbourhood of x which intersects no element of γ_i; in the case $x \in [G_i]$ we have $x \in V_\alpha^i$ for some $\alpha \in A$ and then $O_i = V_\alpha^i$ is a neighbourhood of x, which intersects no element of γ_i except W_α^i. Finally for $x \in X$ let i_x be such that $x \in \Gamma_{i_x}$. Then $\Gamma_{i_x} \cap O_1 \cap O_2 \cap \dots \cap O_{i_x}$ is a neighbourhood of x intersecting at most i_x elements of γ.

Lemma 6 (S. Nedev [17]). *Let X' be a closed subset of a τ-collectionwise normal space X and let ω be a collection of open subsets of X, which covers X' and is such that $|\omega| \leq \tau$ and $\operatorname{Ord}(\omega, X) \leq \aleph_0$. Then there exists a locally finite (in X) collection of open subsets of X which covers X' too and is a closure-refinement of ω.*

Proof. Put $X_n' = \{x \in X \mid \operatorname{Ord}(\omega, \{x\}) \leq n+1\}$. It is obvious that $X' = \cup_{n=1}^\infty X_n'$ and that X_n' is closed for every $n = 1, 2, \dots$. Now put $F_U^1 = X_1' \cap U$ for every $U \in \omega$. Then $\{F_U^1 \mid U \in \omega\}$ is a discrete collection of closed subsets of X. Moreover, $|\{F_U^1 \mid U \in \omega\}| \leq |\omega| \leq \tau$. By the τ-collectionwise normality there is a discrete collection $\{V_U^1 \mid U \in \omega\}$ of open subsets of X such that $F_U^1 \subset V_U^1 \subset [V_U^1] \subset U$ for every $U \in \omega$. Define $W_1 = \cup\{V_U^1 \mid U \in \omega\}$. We have $W_1 \supset \cup\{F_U^1 \mid U \in \omega\} = X_1'$, whence there is an open set $G_1 \subset X$ such that $X_1' \subset G_1 \subset [G_1] \subset W_1$. Now for every n we put $A_n = \{(U_1, \dots, U_n) \in \omega^n \mid i \neq j \text{ implies } U_i \neq U_j\}$. Define $F_{U_1U_2}^2 = U_1 \cap U_2 \cap (X_2' \setminus W_1)$ for every $(U_1, U_2) \in A_2$. It is not difficult to check that $\{F_{U_1U_2}^2 \mid (U_1, U_2) \in A_2\}$ is a discrete collection of closed subsets of X. Consequently there is a discrete collection $\{V_{U_1U_2}^2 \mid (U_1, U_2) \in A_2\}$ of open subsets of X such that $F_{U_1U_2}^2 \subset V_{U_1U_2}^2 \subset [V_{U_1U_2}^2] \subset U_1 \cap U_2$ for each $(U_1, U_2) \in A_2$. Define $W_2 = \cup\{V_{U_1U_2}^2 \mid (U_1, U_2) \in A_2\}$. Then $X_2' \subset W_1 \cup W_2$, whence there is an open subset

G_2 of X such that $X'_2 \cup [G_1] \subset G_2 \subset [G_2] \subset W_1 \cup W_2$. Next we put $F^3_{U_1U_2U_3} = U_1 \cap U_2 \cap U_3 \cap (X'_3 \setminus (W_1 \cup W_2))$ for every $(U_1, U_2, U_3) \in A_3$ and remark that $\{F^3_{U_1U_2U_3} \mid (U_1, U_2, U_3) \in A_3\}$ is a closed discrete collection and so on.

Let now $A = \cup_{n=1}^{\infty} A_n$ and let G be an open subset of X such that $X' \subset G \subset [G] \subset \cup_{n=1}^{\infty} G_n$. For every $\alpha \in A$, $\alpha = (U_1, U_2, \ldots, U_n)$ we define $V_\alpha = G \cap V^n_{U_1U_2\ldots U_n}$. Then $\gamma = \{V_\alpha \mid \alpha \in A\}$ is the required collection. In fact, it is obvious that γ covers X' and refines ω so that we must only show that γ is locally finite in X. Let $x \in X$. In the case $x \in X \setminus [G]$ the last set is a neighbourhood of x intersecting no element of γ. If $x \in [G]$, then $x \in G_n$ for some n and G_n is a neighbourhood of x intersecting no elements of the collections $\{V_\alpha \mid \alpha \in A_i\}$ for $i > n$. But the collections $\{V_\alpha \mid \alpha \in A_i\}$ are discrete ones and consequently for every i there is a neighbourhood O_i of x which intersects at most one element of $\{V_\alpha \mid \alpha \in A_i\}$. Thus, the neighbourhood $G_n \cap O_1 \cap O_2 \cap \ldots \cap O_n$ of x intersects at most n elements of γ.

We will see in the following that the property of the τ-collectionwise normal spaces which is formulated in lemma 6 characterizes the τ-collectionwise normality. By this reason we introduce the concept of τ-pointwise-τ'-collectionwise normality in the following manner: a T_1-space X will be called τ-pointwise-τ'-collectionwise normal iff for every closed set $F \subset X$ and every collection ω of open subsets of X, which covers F and is such that $|\omega| < \tau$ and $\mathrm{Ord}(\omega, F) \leq \tau'$, there is a locally-finite (in X) collection of open subsets of X which covers F too and is a closure-refinement of ω. Moreover, it is clear what we mean when we say that X is τ-pointwise-τ'-collectionwise normal at a fixed closed subset F.

L e m m a 7. *Let F be a closed subset of a normal space X and let for every open collection ω, which covers F and is such that* $\mathrm{Ord}(\omega, F) \leq \tau'$ *and* $|\omega| < \tau$ *there is a closed closure-preserving covering $\{P_u \mid U \in \omega\}$ of X for which $F \cap P_U \subset U$ whatever $U \in \omega$ is. Then X is τ-pointwise-τ'-collectionwise normal at F.*

The proof is almost a repetition of the proof of lemma 5. Namely, we define by induction a sequence $\{\mu_i = \{P^i_U \mid U \in \omega\} \mid i = 1, 2, \ldots\}$ of closed closure-preserving coverings μ_i of X such that

1. $F \cap P^i_U \subset U$ for every $U \in \omega$ and $i = 1, 2, \ldots$

2. $(F \cap P^i_U) \cap (F \cap P^j_V) = \emptyset$ for $j > i$ and $V > U$ (we assume that ω is well-ordered). Next we define $\gamma' = \{V^i_U \mid U \in \omega, i = 1, 2, \ldots\}$ as in lemma 5 and see in the same way that γ' covers F and is such that $|\gamma'| < \tau$ and $\mathrm{Ord}(\gamma', F) \leq \aleph_0$. Take a closed closure-preserving covering $\{Q^i_U \mid U \in \omega, i = 1, 2, \ldots\}$ of X corresponding to γ' (according to the assumption of the lemma) and put $Q_i = \cup\{F \cap Q^i_U \mid U \in \omega\}$ and $V_i = \cup\{U \cap V^i_U \mid U \in \omega\}$. By the normality of X there are open sets Γ_i and G_i such that $Q_i \subset \Gamma_i \subset [\Gamma_i] \subset G_i \subset [G_i] \subset V_i$ for every $i = 1, 2, \ldots$. Moreover, there is an open set G such that $F \subset G \subset [G] \subset \cup\{\Gamma_i \mid i = 1, 2, \ldots\}$. Finally we define $W^1_U = U \cap G \cap V^1_U \cap G_1$ and $W^i_U = U \cap G \cap V^i_U \cap G_i \setminus \cup\{[\Gamma_j] \mid j \leq i-1\}$ for every $U \in \omega$, $i = 2, 3, \ldots$ Put $\gamma = \{W^i_U \mid U \in \omega, i = 1, 2, \ldots\}$. Obviously γ covers F and refines ω. The local-finiteness of γ at the points in $X \setminus [G]$ is obvious and at the points of $[G]$ can be seen as in lemma 5.

Lemma 8. *Let X be a normal space and $\varphi=\{F_1, F_2, \ldots, F_n\}$ and $\omega=\{U_1, U_2, \ldots, U_n\}$ be finite collections of subsets of X such that U_i is open, F_i is closed and $F_i\subset U_i$ for every $i=1, 2, \ldots, n$. Then there is a collection $\gamma=\{V_1, V_2, \ldots, V_n\}$ of open subsets of X such that $F_i\subset V_i\subset U_i$ for every $i=1, 2, \ldots, n$ and* $\mathrm{Ord}(\gamma, X)=\mathrm{Ord}(\varphi, X)$. *(Moreover: $\cap_{j=1}^{s}[V_{i_j}]\neq\emptyset$ implies $\cap_{j=1}^{s}F_{i_j}\neq\emptyset$ whatever $i_1, i_2, \ldots, i_s$.)*

This lemma is largely known, easy and useful.

Proof. Define $\psi_1=\{H\subset X \mid H$ is a common part of some number of elements of $\varphi\}$. Put $W_1=X\setminus\cup\{H\in\psi_1 \mid H\cap F_1=\emptyset\}$ and take an open set V_1 such that $F_1\subset V_1\subset[V_1]\subset W_1\cap U_1$. Next put $F'_1=[V_1]$, $F'_i=F_i$ for $i=2, 3, \ldots, n$ and $\varphi'=\{F'_1, F'_2, \ldots, F'_n\}$. Obviously $\cap_{j=1}^{s}F'_{i_j}\neq\emptyset$ implies $\cap_{j=1}^{s}F_{i_j}\neq\emptyset$ whatever $i_1, i_2, \ldots, i_s$. Define $\psi_2=\{H\subset X \mid H$ is a common part of some number of elements of $\varphi'\}$. Put $W_2=X\setminus\cup\{H\in\psi_2 \mid H\cap F'_2=\emptyset\}$, take an open set V_2 such that $F'_2\subset V_2\subset[V_2]\subset W_2\cap U_2$ and go on by induction.

Corollary 1. *If every finite open covering ω of a normal space X has a closed finite refinement φ with* $\mathrm{Ord}(\varphi, X)\leqq n+2$ *then* $\dim X\leqq n$.

Lemma 9 (M. Katetov [9]). *Let ω be an open covering of a space X. If X has a hereditarily-closure-preserving closed covering μ whose every element has* $\dim\leqq n$ *and intersects only a finite number of elements of ω then ω has an open refinement γ with* $\mathrm{Ord}(\gamma, X)\leqq n+2$.

Proof (taken from [7]). Let us arrange the elements of μ in a transfinite sequence $F_0, F_1, \ldots, F_\alpha, \ldots$, $\alpha\leqq\xi$, of type $\xi+1$ and let $\omega=\{U_s \mid s\in S\}$. For every $\alpha\leqq\xi$ we define by means of transfinite induction a family $\mu_\alpha=\{F_{\alpha,s} \mid s\in S\}$ consisting of closed subsets of X and satisfying the conditions

1. $F_{\alpha,s}\subset F_\alpha$ for every $s\in S$;
2. $\mathrm{Ord}(\{(U_s\setminus\cup\{F_{\beta,s} \mid \beta<\alpha\}) \mid s\in S\}, F_\lambda)\leqq n+2$ for every $\lambda<\alpha$;
3. The family $\{(U_s\setminus\cup\{F_{\beta,s} \mid \beta\leqq\alpha\}) \mid s\in S\}$ is an open covering of X.

Let us suppose either that the families μ_α satisfying 1—3 have already been defined for $\alpha<\alpha_0$ or that $\alpha_0=0$. To begin with we shall show that

4. $\cup\{(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha<\alpha_0\}) \mid s\in S\}=X$.

This formula is evidently valid if $\alpha_0=0$. Let us suppose, therefore, that $\alpha_0>0$ and let us suppose that there is a point

5. $x\in X\setminus\cup\{(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha<\alpha_0\}) \mid s\in S\}$.

Let $U_{s_1}, U_{s_2}, \ldots, U_{s_k}$ be all elements of ω containing x. By virtue of 5 there exists an $\alpha(i)<\alpha_0$ for every $i\leqq k$ such that $x\in F_{\alpha(i), s_i}$. It follows, however, that for $\alpha=\max\{\alpha(1), \ldots, \alpha(k)\}$ we have $x\notin\cup\{(U_s\setminus\cup\{F_{\beta,s} \mid \beta\leqq\alpha\}) \mid s\in S\}$ contrary to 3. Thus, formula 4 is proved. Since μ is hereditarily-closure-preserving we infer that the covering $\{(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha<\alpha_0\}) \mid s\in S\}$ is open (if $\alpha_0=0$ this is obvious). Since $\dim F_{\alpha_0}\leqq n$, there exists an open covering $\nu=\{V_s \mid s\in S\}$ of the space F_{α_0} such that $\mathrm{Ord}(\nu, F_{\alpha_0})\leqq n+2$ and $V_s\subset F_{\alpha_0}\cap(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha<\alpha_0\})$. Let $\mu_{\alpha_0}=\{F_{\alpha_0,s} \mid s\in S\}$, where $F_{\alpha_0,s}=(F_{\alpha_0}\cap[U_s\setminus\cup\{F_{\alpha,s} \mid \alpha<\alpha_0\}])\setminus V_s$. Since $F_{\alpha_0}\cap(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha\leqq\alpha_0\})=F_{\alpha_0}\cap(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha<\alpha_0\})\setminus F_{\alpha_0,s}=V_s$ and $F_\beta\cap(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha\leqq\alpha_0\})\subset F_\beta\cap(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha\leqq\beta\})$ holds for $\beta<\alpha_0$ we infer that 1. and 2. are valid for $\alpha=\alpha_0$. We have also $F_{\alpha_0}\cap(\cup\{(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha\leqq\alpha_0\}) \mid s\in S\})=\cup\{F_{\alpha_0}\cap(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha\leqq\alpha_0\}) \mid s\in S\}=\cup\{V_s \mid s\in S\}=F_{\alpha_0}$ so that $F_{\alpha_0}\subset\cup\{(U_s\setminus\cup\{F_{\alpha,s} \mid \alpha\leqq\alpha_0\}) \mid s\in S\}$, which, together with 4 and 1 for $\alpha=\alpha_0$, prove that 3 is valid for $\alpha=\alpha_0$ because the set $\cup\{F_{\alpha,s} \mid \alpha\leqq\alpha_0\}$ is closed by virtue of 1 and by the fact that μ is hereditarily closure-preserving. We may assume, therefore, that the family μ_α has already been defined for $\alpha \leq \xi$. Conditions 2 and 3 imply, in view of the equality $\cup\{F_\alpha \mid \alpha \leq \xi\} = X$, that the family $\{(U_s \setminus \cup\{F_{\alpha,s} \mid \alpha \leq \xi\}) \mid s \in S\}$ is the required refinement.

Corollary 2. *If the space X has a closed hereditarily-closure-preserving (or locally finite) covering whose every element has $\dim \leq n$, then $\dim X \leq n$.*

Theorem 1 (C. H. Dowker [23]). *If ω is a locally finite open covering of a space X with $\dim X \leq n$, then ω has an open refinement γ with $\mathrm{Ord}(\gamma, X) \leq n+2$.*

Proof (R. Engelking). Let $\omega = \{U_s \mid s \in S\}$. Denote by A the family of all non-empty finite subsets of S and for any $\alpha \in A$ let $F_\alpha = (\cap\{[U_s] \mid s \in \alpha\}) \cap (\cap\{(X \setminus U_s) \mid s \notin \alpha\})$. Then the families $\mu = \{F_\alpha \mid \alpha \in A\}$ and ω satisfy the assumptions of lemma 9.

Lemma 10. *Let X be a normal space and $F_1, F_2, \ldots$ be a sequence of closed subsets of X with $\dim F_i \leq n_i$ for every $i = 1, 2, \ldots$ If ω is a locally finite open covering of X, then there is a sequence $\omega_0, \omega_1, \ldots$ of open locally finite coverings of X such that*

1. $\omega_0 = \omega$;
2. *ω_{i+1} is an index-closure-refinement of ω_i for every $i = 0, 1, 2, \ldots$;*
3. *$\mathrm{Ord}(\omega_k, F_i) \leq n_i + 2$ for $i \leq k$.*

Proof. The collection $\{U \cap F_1 \mid U \in \omega\}$ is a locally finite open covering of F_1 so that, by theorem 1 and lemma 4, there is a collection $\gamma_1 = \{\Gamma_U \mid U \in \omega\}$ of open subsets of F_1 which covers F_1, is an index-refinement of $\{U \cap F_1 \mid U \in \omega\}$ and is such that $\mathrm{Ord}(\gamma_1, F_1) \leq n_1 + 2$. Let $\Gamma_U = F_1 \cap G_U$, where G_U is an open subset of X for every $U \in \omega$. Now put $V_U = (G_U \cap U) \cup (U \setminus F_1)$ for every $U \in \omega$ and $\omega_1' = \{V_U \mid U \in \omega\}$. Obviously ω_1' is an open index-refinement of ω and $\mathrm{Ord}(\omega_1', F_1) \leq n_1 + 2$. We take ω_1 to be an open index-closure-refinement of ω_1'. Next we finish the proof by induction.

Example 1 (T. Przimusinski [21]). *For every cardinal number $\tau \geq \aleph_0$ there exists a zero-dimensional, perfectly normal, pointwise-paracompact, τ-collectionwise normal (hence τ-paracompact) space which is not τ^+-collectionwise normal.*

(Recall that the space X is called pointwise-paracompact iff every open covering of X has an open point-finite refinement; a space X is called perfectly normal if it is a normal space whose every closed subset is a G_δ-set in it (i. e. every closed set is a common part of countably many open sets).)

Proof. For every cardinal number $\theta \leq \tau^+$ let Y_θ be the discrete space with $|Y_\theta| = \theta$ and denote by $\mathscr{K}$ the family of all coverings k of Y_{τ^+} consisting of disjoint elements and such that $|k| \leq \tau$. For every $k \in \mathscr{K}$ fix a function $f_k : k \xrightarrow{\text{onto}} Y_{|k|}$ and let $g_k : Y_{\tau^+} \to Y_{|k|}$ be defined by $g_k(x) = f_k(F)$ iff $x \in F \in k$. The diagonal mapping $g = \{g_k \mid k \in \mathscr{K}\}$ is a homeomorphic embedding of Y_{τ^+} into $Y = \Pi\{Y_{|k|} \mid k \in \mathscr{K}\}$ so that we can identify Y_{τ^+} with $g(Y_{\tau^+}) \subset Y$. Let X be the topological space defined as follows: the points of X are just the points of Y; the set $U \subset Y$ is open in X iff $U = V \cap W$, where V is open in Y and $W \subset Y \setminus Y_{\tau^+}$.

In order to prove the τ-collectionwise normality of X let $\{F_\alpha \mid \alpha \in A\}$ be a discrete family of closed sets in X with $|A| \leq \tau$. The covering $k_0 = \{F_\alpha \cap Y_{\tau^+},$

$(Y_{\tau^+}\setminus\cup\{F_\alpha\mid\alpha\in A\})\ \alpha\in A\}$ belongs to $\mathcal{H}$ and the sets $U_\alpha=\pi_{k_0}^{-1}(f_{k_0}(F_\alpha\cap Y_{\tau^+}))$, where $\alpha\in A$ and $\pi_{k_0}: Y\to Y_{|k_0|}$ is the projection, are open and disjoint in X and $F_\alpha\cap Y_{\tau^+}\subset U_\alpha$. The sets $V_\alpha=(U_\alpha\setminus\cup\{F_\beta\mid\beta\in A,\ \beta\neq\alpha\})\cup F_\alpha$ are also open and disjoint in X and $F_\alpha\subset V_\alpha$.

In order to show that X is not τ^+-collectionwise normal we remark that the collection of points of Y_{τ^+} is discrete in X and has cardinality τ^+. If there is an open disjoint collection in X, which separates the points of Y_{τ^+}, then there is an open disjoint collection γ in Y with $|\gamma|=\tau^+$. But this is impossible because Y is a product whose all factors are spaces of density $\leqq\tau$. (Let us recall that the set $Z'\subset Z$ is said to be dense in the space Z if $[Z']=Z$; density $d(Z)$ of the space Z is, by definition, the smallest cardinal number λ such that there is a dense subset Z' in Z with $|Z'|=\lambda$.) It is known that a product whose all factors are spaces of density τ contains no disjoint open collections of cardinality $\geqq\tau^+$. This fact is an obvious consequence from the following theorem of Hewitt-Marczewski-Pondiczery:

If $|A|\leqq 2^\tau$ and for every $\alpha\in A$ the space X_α has density $\leqq\tau$, then $d(\Pi\{X_\alpha\mid\alpha\in A\})\leqq\tau$.

To finish the proof we assume that $Y_\theta=\{\alpha\mid\alpha$ is a cardinal number, $\alpha<\theta\}$ and that $f_k(y)=0$ whenever $k=\{\{y\},\ Y_{\tau^+}\setminus\{y\}\}$ for some $y\in Y_{\tau^+}$. Next consider $\widehat{X}=\{x=\{x_k\mid k\in\mathcal{H}\}\in Y\mid x_k\neq 0$ for all but finitely many $k\}$ and define $X'=Y_{\tau^+}\times\{0\}\cup((\widehat{X}\setminus Y_{\tau^+})\times(\{1,\ 2^{-1},\dots,\ n^{-1},\dots\})$ with the topology induced by $X\times(\{0,\ 1,\ 2^{-1},\dots,\ n^{-1},\dots\})$. Now X' satisfies all our requirements.

Remark 1. Since every paracompact space is collectionwise normal, the case $\tau=\aleph_0$ of example 1 shows that the words "locally finite" in theorem 1 cannot be replaced by the words "point finite".

Remark 2. In the case $2^{\tau^+}=2^\tau$ one can modify (by the aid of Hewitt-Marczewski-Pondiczery's theorem mentioned above) the construction of the space X and obtain a τ-collectionwise normal space with $d(Z)=\tau$, which is not τ^+-collectionwise normal (see [24]). The converse is also true: if $2^{\tau^+}>2^\tau$, then every τ-collectionwise normal space of density τ is collectionwise normal.

2. Special spaces and collections of subsets. We recall that Y_τ stands for the discrete space of cardinality τ. The product $Y_\tau^{\aleph_0}$ of countably many copies of Y_τ will be denoted by $B(\tau)$ and named the Baire space of weight $\tau\aleph_0$. The space $B(\tau)$ will be considered in the metric p, determined by the condition that $p(y^1,\ y^2)=2^{-k}$ if $y_j^1=y_j^2$ for every $j=1,\ 2,\dots,\ k-1$ and $y_k^1\neq y_k^2$, where $y^i=\{y_1^i,\ y_2^i,\dots\}$, $i=1,\ 2$. It is easily seen that p induces the initial product topology on $B(\tau)$ and that p is non-archimedean, i. e. that $p(x,y)\leqq\max\{p(x,z),\ p(z,y)\}$ for all $x,\ y,\ z\in B(\tau)$. The following known lemma shows that every subspace of $B(\tau)$ has $\dim\leqq 0$.

Lemma 1. *If the topology of a space X is induced by a nonarchimedean metric ϱ then* $\dim X\leqq 0$.

Proof. At first we remark that if $\varepsilon\geqq\eta>0$ then $O_\varepsilon(x)\cap O_\eta(y)\neq\emptyset$ implies $O_\varepsilon(x)\supset O_\eta(y)$ whatever $x,\ y\in X$ are. Now let ω be an open covering of X. We are going to show that ω has an open disjoint refinement. To begin with, define the collection γ_1 as follows: $V\in\gamma_1$ iff there are $x\in X$ and $U\in\omega$ such that $V = O_1(x) \subset U$. Next define the collection γ_2 as follows: $V \in \gamma_2$ iff 1) V intersects no element of γ_1 and 2) there are $x \in X$ and $U \in \omega$ such that $V = O_{2^{-1}}(x) \subset U$. And so on: suppose $\gamma_1, \ldots, \gamma_n$ have already been defined and then define γ_{n+1} as follows: $V \in \gamma_{n+1}$ iff 1) V intersects no element of γ_i for $i \leq n$ and 2) there are $x \in X$ and $U \in \omega$ such that $V = O_{2^{-n}}(x) \subset U$. Put $\gamma = \cup_{n=1}^{\infty} \gamma_n$. Obviously, γ is a disjoint collection of open subsets of X and γ refines ω. To finish the proof it is sufficient to show that γ covers X. So let $x \in X$ and let $U \in \omega$ be such that $x \in U$. There is a positive integer k such that $O_{2^{-k}}(x) \subset U$. There are two possibilities:

a) $O_{2^{-k}}(x)$ intersects no element of γ_i for $i \leq k$. Then $x \in O_{2^{-k}}(x) \in \gamma_{k+1}$;

b) There is a $V \in \gamma_i$ for some $i = 1, 2, \ldots, k$ such that $V \cap O_{2^{-k}}(x) \neq \varnothing$. Then $x \in O_{2^{-k}}(x) \subset V$ and this completes the proof.

We now introduce the following notations: R stands for the reals; $l_1(\tau)$ is a Banach space of weight $\tau \aleph_0$, namely $l_1(\tau) = \{y : T \to R \mid \Sigma\{|y(t)| \mid t \in T\} < \infty\}$ with the norm $\|y\| = \Sigma\{|y(t)| \mid t \in T\}$, where T is some set of cardinality τ; $H(\tau)$ is the generalized Hilbert space of weight $\tau \aleph_0$, namely $H(\tau) = \{y : T \to R \mid \Sigma\{y^2(t) \mid t \in T\} < \infty$ with the norm $\|y\|^2 = \Sigma\{y^2(t) \mid t \in T\}$.

Let now (Y, ϱ) be a metric space. We shall use the notations: $\mathcal{F}(Y) = \{F \subset Y \mid F$ is a non-void complete subspace of $Y\}$, $\mathcal{F}_\tau(Y) = \{F \in \mathcal{F}(Y) \mid$ every open covering of F has an open refinement of cardinality less than $\tau\}$: $\mathcal{F}'_\tau(Y)$ is defined as follows: $\mathcal{F}'_\tau(Y) = \mathcal{F}_\tau(Y)$ in the case when the metric space Y is not complete and $\mathcal{F}'_\tau(Y) = \mathcal{F}_\tau(Y) \cup \{Y\}$ otherwise. For $\mathcal{F}_{\aleph_0}(Y)$ and $\mathcal{F}'_{\aleph_0}(Y)$ we shall use the following special notations: $\mathcal{F}_{\aleph_0}(Y) = \mathcal{C}(Y)$ and $\mathcal{F}'_{\aleph_0}(Y) = \mathcal{C}'(Y)$. In the cases when Y is in addition a linear topological space and is considered as such a space we shall assume that every element of $\mathcal{F}(Y) (\mathcal{F}_\tau(Y), \mathcal{F}'_\tau(Y))$ is convex.

Finally, if $A \subset Y$, then $O_\varepsilon A = \{x \in X$ there is an $y \in A$ such that $\varrho(y, x) < \varepsilon\}$ (we have already used this notation in the previous sections).

3. S. f. p. carriers. If X and Y are sets, then every mapping $\Phi : X \to 2^Y$ (2^Y being the set of all subsets of Y) is called set-valued carrier (or mapping) of X into Y. For every subset M of Y we denote:

$$\Phi^{-1}(M) = \{x \in X \mid \Phi(x) \cap M \neq \varnothing\}.$$

Let now X and Y be topological spaces. The carrier $\Phi : X \to 2^Y$ is called lower semi-continuous -abbr. l. s. c. (or upper semi-continuous—abbr. u. s. c.) if $\Phi^{-1}(M)$ is an open (closed) subset of X for every open (closed) subset M of Y. Also, Φ is l. s. c. (u. s. c.) at the point $x \in X$ iff for every open set U in Y such that $\Phi(x) \cap U \neq \varnothing$ $(\Phi(x) \subset U)$ there is a neighbourhood O_x of x in X such that $\Phi(y) \cap U \neq \varnothing$ $(\Phi(y) \subset U)$ for every $y \in O_x$. It is clear that Φ is l. s. c. (u. s. c.) iff it is l. s. c. (u. s. c.) at every point $x \in X$.

Let X be a normal space and (Y, ϱ) be a metric space. We shall say that the carrier $\Phi : X \to \mathcal{F}(Y)$ has the selection-factorization property, or briefly, is s. f. p., iff for every closed subset F of X and every locally-finite collection γ of open subsets of Y such that $\Phi^{-1}(\gamma) = \{\Phi^{-1}(U) \mid U \in \gamma\}$ covers F, there is an open locally finite (in F) covering of F which refines $\Phi^{-1}(\gamma)$ (see [19]).

First of all we prove the following important assertion.

Lemma 1. (M. Čoban, V. Vâlov [4]). *Let X be a normal space, (Y, ϱ) be a metric space of weight τ and $\Phi: X \to \mathcal{F}(Y)$ be s. f. p. Then there exist:*

a) *a sequence* $\{\gamma_n = \{V^n_\alpha \mid \alpha \in A_n\} \mid n = 1, 2, \dots\}$ *of open locally finite coverings* γ_n *of* Y;

b) *a sequence* $\{\omega_n = \{U^n_\alpha \mid \alpha \in A_n\} \mid n = 1, 2, \dots\}$ *of open locally finite coverings* ω_n *of* X;

c) *a sequence* $\{\varphi_n = \{f_{\alpha n} \mid \alpha \in A_n\} \mid n = 1, 2, \dots\}$ *of partitions of unity on* X;

d) *a sequence* $\{\pi_n \mid n = 1, 2, \dots\}$ *of mappings* $\pi_n: A_{n+1} \to A_n$ *such that:*

1. $U^n_\alpha = f^{-1}_{\alpha n}((0, 1])$, $[U^n_\alpha] \subset \Phi^{-1}(V^n_\alpha)$ *for every n and* $\alpha \in A_n$,

2. $U^n_\alpha = \cup\{U^{n+1}_\beta \mid \beta \in \pi^{-1}_n(\alpha)\}$ *for every n and* $\alpha \in A_n$,

3. $V^n_\alpha = \cup\{V^{n+1}_\beta \mid \beta \in \pi^{-1}_n(\alpha)\}$ *for every n and* $\alpha \in A_n$,

4. $f_{\alpha n}(x) = \Sigma\{f_{\beta, n+1}(x) \mid \beta \in \pi^{-1}_n(\alpha)\}$ *for every n and* $\alpha \in A_n$, $x \in X$,

5. $\operatorname{diam}(V^n_\alpha) < 2^{-n}$ *for every n and* $\alpha \in A_n$,

6. $|A_n| \leq \tau$ *for every n.*

Proof. We fix a sequence $\{\gamma'_n = \{W^n_\beta \mid \beta \in B_n\} \mid n = 1, 2, \dots\}$ of open locally finite coverings of Y such that $\operatorname{diam}(W^n_\beta) < 2^{-n}$ for every n and $\beta \in B_n$. Since Φ is s. f. p., there exists, according to lemmas 1.1 and 1.4, a locally finite open covering $\{G^1_\beta \mid \beta \in B_1\}$ of X which is an index-closure refinement of the covering $\{\Phi^{-1}(W^1_\beta) \mid \beta \in B_1\}$. According to lemma 1.2 there is a partition of unity $\{g_{\beta 1} \mid \beta \in B_1\}$, index-subordinated to $\{G^1_\beta \mid \beta \in B_1\}$. Now we put $A_1 = B_1$, $V^1_\alpha = W^1_\alpha$ and $f_{\alpha 1} = g_{\alpha 1}$ for every $\alpha \in A_1$ and define $U^1_\alpha = f^{-1}_{\alpha 1}((0, 1]) \subset G^1_\alpha \subset [G^1_\alpha] \subset \Phi^{-1}(V^1_\alpha)$. Next for every $\alpha \in A_1$ we consider the collection $\gamma_\alpha = \{V^1_\alpha \cap W^2_\beta \mid \beta \in B_2\}$. Since Φ is s. f. p. there is a locally finite (in $[U^1_\alpha]$) open covering $\{G^2_\beta \mid \beta \in B_2\}$ of $[U^1_\alpha]$ which is an index-closure-refinement of $\Phi^{-1}(\gamma_\alpha)$. Therefore there is a partition of unity $\{g_{\alpha\beta 2} \mid \beta \in B_2\}$ on $[U^1_\alpha]$, which is index-subordinated to $\{G^2_{\alpha\beta} \mid \beta \in B_2\}$. Now put $A_2 = A_1 \times B_2$ and define $\pi_1: A_2 \to A_1$ to be the projection onto the first factor. For every $\lambda = (\alpha, \beta) \in A_2$ define $f_{\lambda 2}(x) = f_{\alpha 1}(x) \cdot g_{\alpha\beta 2}(x)$ if $x \in G^2_{\alpha\beta}$ and $f_{\lambda 2}(x) = 0$ otherwise. Finally define $U^2_\alpha = f^{-1}_{\alpha 2}((0, 1])$ for every $\alpha \in A_2$ and this finishes the second step of our indication. Since the next steps are now obvious, the lemma is proved.

Lemma 2 ([19]). *In the conditions of lemma 1 let, in addition, $F_1, F_2, \dots, F_k, \dots$ be a sequence of closed subsets of X such that $\dim F_k \leq n_k$ for every $k = 1, 2, \dots$. Then there is a sequence $\{\mu_k = \{H^k_\alpha \mid \alpha \in A_k\} \mid k = 1, 2, \dots\}$ of closed locally finite coverings μ_k of X such that:*

7) $H^k_\alpha \subset U^k_\alpha$ *for every k and* $\alpha \in A_k$,

8) $H^k_\alpha \supset \cup\{H^{k+1}_\beta \mid \beta \in \pi^{-1}_k(\alpha)\}$ *for every k and* $\alpha \in A_k$,

9) $\operatorname{Ord}(\mu_k, F_i) \leq n_i + 2$ *for every* $i \leq k$.

Proof. Let $\nu_1 = \{G^1_\alpha \mid \alpha \in A_1\}$ be an open index-closure-refinement of ω_1 such that $\operatorname{Ord}(\nu_1, F_1) \leq n_1 + 2$ (see lemma 1.10). Let $\{\Gamma^1_\alpha \mid \alpha \in A_1\}$ be an index-closure open refinement of ν_1. Define $\mu_1 = \{[\Gamma^1_\alpha] \mid \alpha \in A_1\}$ so that $H^1_\alpha = [\Gamma^1_\alpha]$. Now consider the covering $\nu'_2 = \{U^2_\alpha \cap \Gamma^1_{\pi_1(\alpha)} \mid \alpha \in A_2\}$ and let $\nu_2 = \{G^2_\alpha \mid \alpha \in A_2\}$ be an open index-refinement of ν'_2 with $\operatorname{Ord}(\nu_2, F_1) \leq n_1 + 2$ and $\operatorname{Ord}(\nu_2, F_2) \leq n_2 + 2$ (again lemma 1.10). Let $\{\Gamma^2_\alpha \mid \alpha \in A_2\}$ be an open index-closure-refinement of ν_2 and define $\mu_2=\{[\Gamma^2_\alpha]\mid\alpha\in A_2\}$, i. e. $H^2_\alpha=[\Gamma^2_\alpha]$ for $\alpha\in A_2$. The next step is to consider the covering $\{U^3_\alpha\cap\Gamma^2_{\pi_2(\alpha)}\ \alpha\in A_3\}$ and so on, by induction.

Lemma 3 (K. Fort [8]). *Let $\Phi_i:X\to\mathcal{C}(Y)$ be l. s. c. where (Y,ϱ) s a complete metric space and let $\Phi_i(x)\subset O_{2^{-i}}(\Phi_{i+1}(x))$ and $\Phi_{i+1}(x)\subset O_{2^{-i}}(\Phi_i(x))$ for every $x\in X$ and $i=1,2,\dots$. Let us define the carrier $\Phi:X\to 2^Y$ by the formula*

$$(*)\qquad \Phi(x)=\{y\in Y\mid y=\lim_{i\to\infty}y_i \text{ where } y_i\in\Phi_i(x) \text{ and } \varrho(y_i,y_{i+1})\leqq 2^{-i} \text{ for } i=1,2,\dots\}.$$

Then: 1) $\Phi:X\to\mathcal{C}(Y)$ and 2) Φ is l. s. c.

Proof. In order to prove that $\Phi(x)$ is a non-empty compact subset of Y for every $x\in X$, we consider the product $Z=\Pi\{\Phi_i(x)\mid i=1,2,\dots\}$, which is compact according to the famous Tychonoff theorem. Let $Z_i\subset Z$ be such that $r=\{y_1,y_2,\dots,y_k,\dots\}\in Z_i$ iff $\varrho(y_k,y_{k+1})\leqq 2^{-k}$ for $k=1,2,\dots,i$. It is clear that Z_i is a non-empty closed subset of Z for every i. We put $Z_0=\bigcap_{i=1}^{\infty}Z_i$ and remark that Z_0 is non-empty compact too. Now define a mapping $f:Z_0\to Y$ by the formula: $f(z)=\lim_{i\to\infty}y_i$ where $z=\{y_1,y_2,\dots\}$. It is not difficult to check that f is a continuous mapping and that $\Phi(x)=f(Z_0)$. Thus $\Phi(x)$ is a non-void compact.

In order to prove that Φ is l. s. c. we take a point $x_0\in X$ and an open subset U of Y such that $\Phi(x_0)\cap U\neq\emptyset$. Take $y_0\in\Phi(x_0)\cap U$ and denote $\varepsilon=\varrho(y_0,Y\setminus U)$. According to the definition, $y_0=\lim_{i\to\infty}y_i^0$, where $y_i^0\in\Phi_i(x_0)$ and $\varrho(y^0,y^0_{i+1})\leqq 2^{-i}$ for every i. Therefore there is an i_0 such that $\varrho(y_0,y^0_{i_0})<\varepsilon 2^{-1}$ and $\Sigma_{i=i_0}^{\infty}2^{-i}<\varepsilon 2^{-1}$. Since Φ_{i_0} is l. s. c., there is a neighbourhood O_{x_0} of x_0 such that $x\in O_{x_0}$ implies $\Phi_{i_0}(x)\cap O_{\varepsilon 2^{-1}}y_0\neq\emptyset$. For such an x take $y_{i_0}\in\Phi_{i_0}(x)\cap O_{\varepsilon 2^{-1}}y_0$. Obviously there is a sequence $\{y_i\mid i=1,2,\dots\}$ such that $y_i\in\Phi_i(x)$ and $\varrho(y_i,y_{i+1})\leqq 2^{-i}$ for $i=1,2,\dots$. Let $y=\lim_{i\to\infty}y_i$. Then, by the definition, $y\in\Phi(x)$ and moreover $\varrho(y_0,y)\leqq\varrho(y_0,y_{i_0})+\varrho(y_{i_0},y)<\varepsilon 2^{-1}+\Sigma_{i=i_0}^{\infty}2^{-i}<\varepsilon$ so that $\Phi(x)\cap U\neq\emptyset$.

Lemma 4. (S. Nedev [18]). *Let $\Phi_i:X\to\mathcal{C}(Y)$ be u. s. c., where (Y,ϱ) is a complete metric space and let $\Phi_{i+1}(x)\subset O_{2^{-i}}(\Phi_i(x))$ for every $x\in X$ and $i=1,2,\dots$. Let us define $\Phi:X\to 2^Y$ by the formula $(*)$ (see lemma 3). Then 1) $\Phi:X\to\mathcal{C}(Y)$ and 2) Φ is u. s. c.*

Moreover, if $x\in X$ is such that $|\Phi_i(x)|\leqq n$ holds for infinitely many values of the index i, then $|\Phi(x)|\leqq n$ holds too (this is obvious).

Proof. Having done the proof of lemma 3 we have only to prove that Φ is u. s. c. First of all we shall prove the following.

Let $x\in X$ and let U be an open subset of Y such that $\Phi(x)\subset U$. Then there is an i_0 such that $\Phi_i(x)\subset U$ for every $i\geqq i_0$.

Suppose that this is not the case and denote $3\varepsilon=\varrho(\Phi(x),X\setminus U)$. Remark that $\varepsilon>0$ because of the compactness of $\Phi(x)$. Our assumption gives us that for every i there is a $j_i>i$ such that $\Phi_{j_i}(x)\setminus O_{3\varepsilon}(\Phi(x))\neq\emptyset$. Take i_0 so large that $\Sigma_{i=i_0}^{\infty}2^{-i}<\varepsilon$ and such that $\Phi_{i_0}(x)\setminus O_{3\varepsilon}(\Phi(x))\neq\emptyset$. Then for every $i>i_0$ we have $\Phi_i(x)\setminus O_{2\varepsilon}(\Phi(x))\neq\emptyset$. In fact, let $i>i_0$ and take $y_{j_i}\in\Phi_{j_i}(x)\setminus O_{3\varepsilon}(\Phi(x))$. Then, by induction beginning with y_{j_i} and going down we select $y_k\in\Phi_k(x)$ for $i\leqq k\leqq j_i$ such that $\varrho(y_k,y_{k+1})\leqq 2^{-k}$. After this we have $\varrho(\Phi(x),y_i)\geqq\varrho(\Phi(x),y_{j_i})$

$-\varrho(y_{j_i}, y_i)\geq 3\varepsilon-2^{-j_i+1}-2^{-j_i}-\ldots-2^{-i-1}>3\varepsilon-\varepsilon=2\varepsilon$. Now define the sets Z_{i_0k}, $k=1, 2, \ldots$, in Z (see the proof of lemma 3) as follows: $z=\{y_1, y_2, \ldots, y_i, \ldots\}\in Z_{i_0k}$ iff $\varrho(y_i, y_{i+1})\leq 2^{-i}$ for $i=1, 2, \ldots, i_0+k$ and $y_{i_0+k}\in\Phi_{i_0+k}(x)\setminus O_{\eta_k}(\Phi(x))$ where $\eta_k=\varepsilon+\Sigma_{i=i_0}^{i_0+k}2^{-i}$. It is obvious again that Z_{i_0k} is a non-void closed subset of Z and that $Z_{i_0k}\supset Z_{i_0,k+1}$ for every $k=1, 2, \ldots$. Consequently there is $z=\{y_1, y_2, \ldots\}\in\cap_{k=1}^{\infty}Z_{i_0k}$. But if $y=\lim\limits_{i\to\infty} y_i$ then $y\notin O_\varepsilon(\Phi(x))$ (because that $y_i\notin O_\varepsilon(\Phi(x))$ for $i>i_0$) and this contradicts to $y\in\Phi(x)$.

Now we are able to prove that Φ is u. s. c. Let x_0 be a point in X and U be an open subset of Y such that $\Phi(x_0)\subset U$. We take i so large that $\varepsilon 2^{-1}>2^{-i}+2^{-i-1}+\ldots$ and that $\Phi_i(x_0)\subset O_{\varepsilon 2^{-1}}(\Phi(x_0))$ (see the previous paragraph). Since Φ_i is u. s. c. there is a neighbourhood Ox_0 of x_0 such that $x\in Ox_0$ implies $\Phi_i(x)\subset O_{\varepsilon 2^{-1}}(\Phi(x_0))$. For such an x let $y\in\Phi(x)$. According to the definition $y=\lim_{j\to\infty} y_j$, where $y_j\in\Phi_j(x)$ and $\varrho(y_j, y_{j+1})\leqq 2^{-j}$ for every j. Therefore, $\varrho(\Phi(x_0), y)\leq\varrho(\Phi(x_0), y_i)+\varrho(y_i, y)<\varepsilon 2^{-1}+\Sigma_{j=i}^{\infty}2^{-j}<\varepsilon$. Thus, $\Phi(x)\subset O_\varepsilon(\Phi(x_0))\subset U$ for every $x\in Ox_0$, i. e. Φ is u. s. c. at x_0.

Lemma 5. *Let X be τ-pointwise-τ'-paracompact space where $\tau>\aleph_0$ and $\tau'\geq\aleph_0$ and Y be a metric space of weight $<\tau$. Then every l. s. c. $\Phi: X\to\mathcal{F}_{\tau'}(Y)$ is s. f. p.*

Proof. Let F be a closed subset of X and γ be a locally finite collection of open subsets of Y such that $\Phi^{-1}(\gamma)=\{\Phi^{-1}(U)\mid U\in\gamma\}$ covers F. Consider the covering $\omega=\Phi^{-1}(\gamma)\cup\{X\setminus F\}$. We have $|\omega|<\tau$ and $\mathrm{Ord}(\omega, X)\leq\tau'$. In fact, $|\omega|<\tau$ since $|\gamma|<\tau$. The inequality $\mathrm{Ord}(\omega, X)\leq\tau'$ follows from the fact that $\Phi(x)\in\mathcal{F}_{\tau'}(Y)$ for every $x\in X$. Namely, since γ is locally finite, every $y\in\Phi(x)$ has a neighbourhood Oy intersecting only a finite number of elements of γ. Since $\Phi(x)\in\mathcal{F}_{\tau'}(Y)$ the covering $\{O_y\mid y\in\Phi(x)\}$ has a refinement of cardinality $<\tau'$ so that $\Phi(x)$ intersects less than τ' elements of γ. Thus, $\mathrm{Ord}(\omega, X)\leq\tau'$ and consequently ω has an open locally finite refinement ω' because of the τ-pointwise-τ'-paracompactness of X. Hence $\{V\cap F\mid V\in\omega'\}$ is a locally finite open covering of F which refines $\Phi^{-1}(\gamma)$.

Lemma 6. *Let X be a τ-pointwise-τ'-collectionwise normal space where $\tau>\aleph_0$ and $\tau'\geq\aleph_0$ and let Y be a metric space of weight $<\tau$. Then every l. s. c. $\Phi: X\to\mathcal{F}'_{\tau'}(Y)$ is s. f. p.*

Proof. Let F be a closed subset of X and γ be a locally finite collection of open subsets of Y such that $\Phi^{-1}(\gamma)=\{\Phi^{-1}(U)\mid U\in\gamma\}$ covers F. Fix an $U_0\in\gamma$ such that $U_0\neq\emptyset$ and denote $F_1=F\setminus\Phi^{-1}(U_0)$. It is clear that if $x\in F_1$ then $\Phi(x)\neq Y$ so that $\Phi(x)\in\mathcal{F}_{\tau'}(Y)$ for every $x\in F_1$. Hence $\mathrm{Ord}(\Phi^{-1}(\gamma), F_1)\leq\tau'$ (see the proof of lemma 5). According to the definition of τ-pointwise-τ'-collectionwise normality there is a locally finite collection ω of open subsets of X, which refines $\Phi^{-1}(\gamma)$ and covers F_1. Define $\omega'=\omega\cup\{\Phi^{-1}(U_0)\}$. Then $\{V\cap F\mid V\in\omega'\}$ is a locally finite open covering of refining $\Phi^{-1}(\gamma)$.

4. Selection theorems. The carrier $\psi: X\to 2^Y$ is called a selection for the carrier $\Phi: X\to 2^Y$ iff $\psi(x)\subset\Phi(x)$ for every $x\in X$.

Proposition 1. *Let $\Phi: X\to\mathcal{F}(Y)$ be s. f. p. carrier, where X is a normal space and Y is a metric space. Then there exist an u. s. c. carrier $\psi: X\to\mathcal{C}(Y)$ and a l. s. c. carrier $\varphi: X\to\mathcal{C}(Y)$ such that $\varphi(x)\subset\psi(x)\subset\Phi(x)$ for every $x\in X$.*

Proof. Let $\{\omega_n=\{U^n_\alpha \mid \alpha\in A_n\} \mid n=1, 2, \dots\}$ and $\{\gamma_n=\{V^n_\alpha \mid \alpha\in A_n\} \mid n=1,2,\dots\}$ be as in lemma 3.1. Fix a point $y^n_\alpha\in V^n_\alpha$ for every n and $\alpha\in A_n$. Define $\psi_i: X \to \mathcal{C}(\widetilde{Y})$ and $\varphi_i: X\to \mathcal{C}(\widetilde{Y})$, where $\widetilde{Y}$ is the completion of Y, by the formulas: $\psi_i(x)=\{y^i_\alpha \mid x\in[U^i_\alpha]\}$ and $\varphi_i(x)=\{y^i_\alpha \mid x\in U^i_\alpha\}$ for every $i=1, 2,\dots$. Then the following holds:

1) $\varphi_i(x)\subset\psi_i(x)\subset O_{2^{-i}}(\Phi(x))$ in $\widetilde{Y}$ for every $x\in X$ and $i=1, 2,\dots$ In fact, let $y^i_\alpha\in\psi_i(x)$. Then $x\in[U^i_\alpha]\subset\Phi^{-1}(V^i_\alpha)$ so that $\Phi(x)\cap V^i_\alpha\neq\emptyset$. Take $y\in\Phi(x)\cap V^i_\alpha$. Then $\varrho(y^i_\alpha, \Phi(x))\leq\varrho(y^i_\alpha, y)\leq\operatorname{diam}(V^i_\alpha)<2^{-i}$.

2) ψ_i is u. s. c. since $O^{\psi_i}_x=X\setminus\bigcup\{[U^i_\alpha] \mid x\notin[U^i_\alpha]\}$ is a neighbourhood of x with the property: $\psi_i(z)\subset\psi_i(x)$ for every $z\in O^{\psi_i}_x$.

3) φ_i is l. s. c. since $O^{\varphi_i}_x=\bigcap\{U^i_\alpha \mid x\in U^i_\alpha\}$ is a neighbourhood of x such that $\varphi_i(x)\subset\varphi_i(z)$ for every $z\in O^{\varphi_i}_x$.

4) $\psi_{i+1}(x)\subset O_{2^{-i}}(\psi_i(x))$. In fact, let $y^{i+1}_\alpha\in\psi_{i+1}(x)$. This implies $x\in[U^{i+1}_\alpha]\subset[U^i_{\pi_i(\alpha)}]$, whence $y^i_{\pi_i(\alpha)}\in\psi_i(x)$. Thus $\varrho(y^{i+1}_\alpha, \psi_i(x))\leq\varrho(y^{i+1}_\alpha, y^i_{\pi_i(\alpha)})\leq\operatorname{diam}(V^i_{\pi_i(\alpha)})<2^{-i}$.

5) $\varphi_{i+1}(x)\subset O_{2^{-i}}(\varphi_i(x))$ and $\varphi_i(x)\subset O_{2^{-i}}(\varphi_{i+1}(x))$. The first of these inclusions is proved in 4). For the second let $y^i_\alpha\in\varphi_i(x)$. Then $x\in U^i_\alpha=\bigcup\{U^{i+1}_\beta \mid \beta\in\pi_i^{-1}(\alpha)\}$ so that $x\in U^{i+1}_\beta$ for some $\beta\in\pi_i^{-1}(\alpha)\subset A_{i+1}$. Therefore $y^{i+1}_\beta\in\varphi_{i+1}(x)$ and hence $\varrho(y_\alpha, \varphi_{i+1}(x))\leq\varrho(y^i_\alpha, y^{i+1}_\beta)\leq\operatorname{diam}(V^i_\alpha)<2^{-i}$ (recall that $y^{i+1}_\beta\in V^{i+1}_\beta\subset\bigcup\{V^{i+1}_\gamma \mid \gamma\in\pi^{-1}(\alpha)\}=V^i_\alpha$).

Define now $\psi: X\to 2^{\widetilde{Y}}$ and $\varphi: X\to 2^{\widetilde{Y}}$ by the formulas:

$$\psi(x)=\{y\in\widetilde{Y} \mid y=\lim_{i\to\infty} y_i,\ \varrho(y_i, y_{i+1})\leq 2^{-i},\ y_i\in\psi_i(x) \quad \text{for} \quad i=1, 2\dots\} \text{ and}$$

$$\varphi(x)=\{y\in\widetilde{Y} \mid y=\lim_{i\to\infty} y_i,\ \varrho(y_i, y_{i+1})\leq 2^{-i},\ y_i\in\varphi_i(x) \quad \text{for} \quad i=1, 2,\dots\}.$$

Making use of the lemmas 3.3 and 3.4 we obtain that φ and ψ have all required properties.

Proposition 2 (see [19, prop. 2]). *Let X be a normal space, Y be a metric space and $F_1, F_2,\dots$ be a sequence of closed subsets of X such that $\dim F_k\leq n_k$ for $k=1, 2,\dots$. Then every s. f. p. carrier $\Phi: X\to\mathcal{F}(Y)$ has an u. s. c. selection $\psi: X\to\mathcal{C}(Y)$ such that $|\psi(x)|\leq n_k+1$ for every $x\in F_k$, $k=1,2,\dots$*

Proof. We return now to lemma 3.2 where a sequence $\{\mu_k=\{H^k_\alpha \mid \alpha\in A_k\} \mid k=1, 2,\dots\}$ is constructed. Let y^i_α be as in the proof of proposition 1 and define $\psi_i: X\to 2^{\widetilde{Y}}$ by the formula $\psi_i(x)=\{y^i_\alpha \mid x\in H^i_\alpha\}$ for every $x\in X$ and $i=1, 2,\dots$. Next put $\psi(x)=\{y\in\widetilde{Y} \mid y=\lim_{i\to\infty} y_i,\ y_i\in\psi_i(x),\ \varrho(y_i, y_{i+1})\leq 2^{-i}$ for $i=1, 2,\dots\}$ for every $x\in X$. Lemma 3.4 shows that $\psi: X\to\mathcal{C}(Y)$ is the required selection.

Proposition 3. *Let X be a normal space, Y be a closed convex subset of a Banach space and $\Phi: X\to\mathcal{F}(Y)$ be s. f. p. Then Φ has a continuous single-valued selection.*

Proof. Let $\{f_{\alpha i} \mid \alpha\in A_i,\ i=1, 2,\dots\}$ and $\{\gamma_i=\{V^i_\alpha \mid \alpha\in A_i\} \mid i=1, 2,\dots\}$ be as in lemma 3.1. Pick an $y^i_\alpha\in V^i_\alpha$ for every $\alpha\in A_i$ and $i=1, 2,\dots$, and put $f_i(x)=\Sigma\{f_{\alpha i}(x)y^i_\alpha \mid \alpha\in A_i\}$ for every $x\in X$ and $i=1,2,\dots$. Then $f_i: X\to Y$ is a continuous mapping because of the local-finiteness of the system $\{f_{\alpha i}^{-1}((0, 1]) \mid \alpha \in A_i\}$ for every $i=1, 2, \dots$. Moreover,

1) $f_i(x) \in O_{2^{-i}}(\Phi(x))$ for every $x \in X$. In fact, if $f_{\alpha i}(x) \neq 0$ then $\Phi(x) \cap V_\alpha^i \neq \emptyset$ so that $\varrho(y_\alpha^i, \Phi(x)) \leq \operatorname{diam}(V_\alpha^i) < 2^{-i}$. Hence 1) holds by the convexity of $O_{2^{-i}}(\Phi(x))$ (remember that now $\Phi(x)$ is convex),

2) $f_{i+1}(x) \in O_{2^{-i}}(f_i(x))$ for every $x \in X$ and every $i=1, 2, \dots$. In fact $\|f_{i+1}(x) - f_i(x)\| = \|\Sigma\{f_{\alpha,i+1}(x) y_\alpha^{i+1} \mid \alpha \in A_{i+1}\} - \Sigma\{f_{\alpha i}(x) y_\alpha^i \mid \alpha \in A_i\}\| = \|\Sigma\{f_{\beta,i+1}(x) y_\beta^{i+1} \mid \beta \in A_{i+1}\} - \Sigma\{(\Sigma\{f_{\beta,i+1}(x) \mid \beta \in \pi_i^{-1}(\alpha)\}) y_\alpha^i \mid \alpha \in A_i\}\| \leqq \Sigma\{(\Sigma\{f_{\beta,i+1}(x) \| y_\beta^{i+1} - y_\alpha^i\| \mid \beta \in \pi_i^{-1}(\alpha)\} \mid \alpha \in A_i\} < 2^{-i} \Sigma\{f_{\beta,i+1}(x) \mid \beta \in A_{i+1}\} = 2^{-i}$. Thus $f = \lim_{i \to \infty} f_i$ exists, is a continuous mapping and $f(x) \in \Phi(x)$ for every $x \in X$.

Theorem 1 (see [18]). *The following properties of a T_1-space X are equivalent:*

a) *X is normal and τ-pointwise- τ'-paracompact where $\tau > \aleph_0$ and $\tau' \geqq \aleph_0$;*

b) *every l. s. c. carrier $\Phi\colon X \to \mathfrak{F}_{\tau'}(Y)$ where Y is a closed convex subset of a Banach space of weight $w(Y) < \tau$ admits a selection;*

c) *every l. s. c. carrier $\Phi\colon X \to \mathfrak{F}_{\tau'}(l_1(\tau''))$, where $\tau'' < \tau$ admits a selection (in b) and c) "selection" means continuous single-valued selection).*

Remark 1. In the case $\tau = \aleph_1$ the implication c) $\to$ a) remains true if we replace $l_1(\tau'')$ by R.

Proof. a) $\to$ b) follows immediately by proposition 3 and lemma 3.5; b) $\to$ c) is obvious;

c) $\to$ a). We repeat the Michael's proof [12, 3.2'' (b) $\to$ (a)].

To show that X is τ-pointwise- τ'-paracompact it is sufficient, by lemma 1.3, to show that every open covering ω of X, such that $|\omega| < \tau$ and $\operatorname{Ord}(\omega, X) \leqq \tau'$, has a partition of unity subordinated to it. Let $Y = l_1(\tau'') = \{y\colon \omega \to R \mid \Sigma\{|y(u)| \mid u \in \omega\} < \infty\}$ and let $C = \{y \in Y \mid y(U) \geqq 0$ for all $U \in \omega$ and $\Sigma\{y(U) \mid U \in \omega\} = 1\}$. Clearly C is a closed convex subset of Y. Now, for $x \in X$, let $\Phi(x) = C \cap \{y \in Y \mid y(U) = 0$ for all $U \in \omega$ such that $x \notin U\}$. Clearly $\Phi(x)$ may be considered as a subspace of $l_1(\tau''')$, where $\tau''' = |\{U \mid x \in U \in \omega\}|$ whence $\Phi(x) \in \mathfrak{F}_{\tau'}(Y)$. We will now show that Φ is l. s. c. and we will then apply condition c) to prove our result.

Let us first of all show that for every $y \in C$ and $\varepsilon > 0$ there exists an $y' \in C$ such that $\|y - y'\| < \varepsilon$ and $y'(U) > 0$ for only finitely many $U \in \omega$. To find such a y' we need only pick $U_1, U_2, \dots, U_n \in \omega$ such that $y(U_i) > 0$ for all i and $y(U_1) + y(U_2) + \dots + y(U_n) = \delta > 1 - \varepsilon/2$ and then define $y' \in C$ by $y'(U) = 0$ for $U \notin \{U_1, U_2, \dots, U_n\}$, $y'(U_i) = y(U_i)$ for $i = 2, 3, \dots, n$ and $y'(U_1) = y(U_1) + (1 - \delta)$. Clearly $\|y - y'\| \leqq 2(1 - \delta) < \varepsilon$ and, therefore, y' satisfies all our requirements.

Next we will show that Φ is l. s. c. Let $x_0 \in X$ and the open set G of Y is such that $\Gamma(x_0) \cap G \neq \emptyset$. Take $y \in \Phi(x_0) \cap G$ and let $O_\varepsilon y \subset G$. Define y' and $U_1, U_2, \dots, U_n$ as in the previous paragraph. Since $y(U_i) > 0$ for $i = 1, 2, \dots, n$, it follows from the definition of Φ that $x_0 \in U_i$ for $i = 1, 2, \dots, n$, and hence $Ox_0 = U_1 \cap U_2 \cap \dots \cap U_n$ is a neighbourhood of x_0. It also follows from the definition of Φ that $y' \in \Phi(x)$ for every $x \in Ox_0$ so that Φ is l. s. c. at x_0.

By assumption c) there now exists a selection f for Φ. For each $U \in \omega$ define $f_U\colon X \to R$ by $f_U(x) = f(x)(U)$. It now follows immediately from the definitions that $\{f_U \mid U \in \omega\}$ is a partition of unity on X and this partition is subordinated to ω since f_U vanishes outside of U for every $U \in \omega$. Thus, X is

τ-pointwise-τ'-paracompact. It remains to show that X is normal. Let F_0 and F_1 be disjoint closed subsets of X. Define $\Phi: X \to \mathcal{C}(R)$ by setting $\Phi(x)=\{i\}$ if $x \in F_i$, $i=0,1$, and $\Phi(x)=[0,1]$ otherwise. By assumption c) there exists a selection f for Φ. Define $U_0=f^{-1}(-\infty, 1/2)$ and $U_1=f^{-1}(1/2, +\infty)$. Obviously U_0 and U_1 are disjoint open subset of X such that $F_0 \subset U_0$ and $F_1 \subset U_1$. This completes the proof.

Theorem 2. *The following properties of a T_1-space X are equivalent:*

a) *X is τ-pointwise-τ'-collectionwise normal* ($\tau>\aleph_0$, $\tau'\geq\aleph_0$);

b) *every l.s.c. carrier $\Phi: X \to \mathcal{F}'_{\tau'}(Y)$, where Y is a closed convex subset of a Banach space of weight $\omega(Y)<\tau$, admits a selection;*

c) *every l. s. c. carrier $\Phi: X \to \mathcal{F}'_{\tau'}(l_1(\tau''))$ where $\tau''<\tau$ admits a selection ("selection" means single-valued continuous selection).*

Proof. a) $\to$ b). This follows immediately by proposition 3 and lemma 3.6; b) $\to$ c) is obvious;

c) $\to$ a). Let F be a closed subset of X and ω be an open in X covering of F such that $|\omega|=\tau''<\tau$ and $\operatorname{Ord}(\omega, F)\leq\tau'$. Let $Y=l_1(\tau'')=\{y: \omega \to R \mid \Sigma\{\|y(U)\| \mid U \in \omega\}<\infty\}$ and define C as in the proof that c) $\to$ a) in theorem 1. Define $\Phi: X \to \mathcal{F}_{\tau'}'(Y)$ as follows: $\Phi(x)=Y$ if $x \in X\setminus F$ and $\Phi(x)=\{y \in C \mid y(U)=0$ for all $U \in \omega$ such that $x \notin U\}$ for $x \in F$. As in the previous proof we see that $\Phi(x) \in \mathcal{F}_{\tau'}(Y)$ for every $x \in F$. Next we will show that Φ is l. s. c. Let $x_0 \in X$ and the open set G in Y be such that $\Phi(x_0)\cap G\neq\emptyset$. In the case $x_0 \in X\setminus F$ we define $Ox_0=X\setminus F$ and in the case $x_0 \in F$ we define Ox_0 as in the previous proof. Thus Φ is l. s. c. By assumption c) there exists a selection f for Φ. For every $U \in \omega$ and $x \in X$ we put $f_U(x)=|f(x)(U)|$. Then we have $\Sigma\{f_U(x) \mid U \in \omega\}=\|f(x)\|$ for every $x \in X$ and hence $\Sigma\{f_U(x) \mid U \in \omega\}=1$ for every $x \in F$. Consequently (by the continuity of the norm $\|\cdot\|$) there is a neighbourhood OF of F in X such that $[OF]\subset \cup\{U \mid U \in \omega\}$ and $\|f(x)\|\geq 1/2$ for every $x \in OF$. For $x \in OF$ define $g_U(x)=f_U(x)/\|f(x)\|$. Obviously, the family $\{g_U \mid U \in \omega\}$ is partition of unity on OF. Define $W_U=\{x \in OF \mid g_U(x)>0\}$ for every $U \in \omega$. It follows that the partition of unity $\{g_U \mid U \in \omega\}$ is subordinated to the covering $\{W_U \mid U \in \omega\}$ of OF. By lemmas 1.3 and 1.4 there is a locally finite open covering $\{G_U \mid U \in \omega\}$ of OF, which is an index-refinement of $\{W_U \mid U \in \omega\}$. By the normality of X (see the previous proof) there is an open set G such that $F\subset G\subset[G]\subset OF$. Finally we define $V_U=U\cap G\cap G_U$ for every $U \in \omega$. The collection $\gamma=\{V_U \mid U \in \omega\}$ is an open locally finite (in X) covering of F which refines ω.

Theorem 3. *The following properties of a T_1-space X are equivalent:*

a) *X is normal and τ-pointwise-τ'-paracompact where $\tau>\aleph_0$ and $\tau'\geq\aleph_0$;*

b) *for every l. s. c. carrier $\Phi: X \to \mathcal{F}_{\tau'}(Y)$, where Y is a metric space of weight $w(Y)<\tau$ there are an u. s. c. $\psi: X \to \mathcal{C}(Y)$ and a l. s. c. $\varphi: X \to \mathcal{C}(Y)$ such that $\varphi(x)\subset\psi(x)\subset\Phi(x)$ for every $x \in X$;*

c) *every l. s. c. carrier $\Phi: X \to \mathcal{F}_{\tau'}(Y_{\tau''})$, where $\tau''<\tau$ admits an u. s. c. selection $\psi: X \to \mathcal{F}(Y_{\tau''})$.*

Proof. a) $\to$ b) follows immediately by proposition 1 and lemma 3.5; b) $\to$ c) is obvious;

c) $\to$ a). Let ω be an open covering of X with $|\omega|=\tau''<\tau$ and $\operatorname{Ord}(\omega, X)\leq\tau'$. Our purpose is to show that ω has a closed closure-preserving refinement and then use lemma 1.5. Identify $Y_{\tau''}$ with ω and define $\Phi: X \to \mathcal{F}_{\tau'}(\omega)$ by setting $\Phi(x)=\{U \mid x \in U \in \omega\}$ for every $x \in X$. Φ is l. s. c. since ω is open and

$\Phi(x)\in\mathcal{F}_{\tau'}(\omega)$ since $\mathrm{Ord}(\omega, X)\leq\tau'$. By assumption c) there is an u. s. c. selection $\psi: X\to\mathcal{F}(\omega)$ for Φ. Denote $F_U=\psi^{-1}(U)$ for every $U\in\omega$. Obviously F_U is a closed subset of X and $F_U\subset U$ for every $U\in\omega$. Moreover, the collection $\{F_U\mid U\in\omega\}$ is a covering for X (also obvious) and is a closure-preserving one because every subset of $Y_{\tau''}$ is closed.

Theorem 4. *The following properties of a T_1-space are equivalent*:

a) *X is τ-pointwise-τ'-collectionwise normal where $\tau>\aleph_0$, $\tau'\geq\aleph_0$;*

b) *for every l. s. c. $\Phi: X\to\mathcal{F}'_{\tau'}(Y)$, where Y is a metric space of weight $<\tau$, there exist an u. s. c. $\psi: X\to\mathcal{C}(Y)$ and an l. s. c. $\varphi: X\to\mathcal{C}(Y)$ such that $\varphi(x)\subset\psi(x)\subset\Phi(x)$ for every $x\in X$;*

c) *every l. s. c. $\Phi: X\to\mathcal{F}'_{\tau'}(Y_{\tau''})$, where $\tau''<\tau$, has an u. s. c. selection $\psi: X\to\mathcal{F}(Y_{\tau''})$.*

Proof. That a) $\to$ b) follows by proposition 1 and lemma 3.6; that b) $\to$ c) is obvious.

c) $\to$ a). Let F be a closed subset of X and ω be a collection of open subsets of X covering F and such that $|\omega|=\tau''<\tau$ and $\mathrm{Ord}(\omega, X)\leq\tau'$. We will show that there is a closed closure-preserving covering μ of X such that the collection $\{H\cap F\mid H\in\mu\}$ refines ω and then we shall use lemma 1.7. In order to do this we identify $Y_{\tau''}$ with ω and define $\Phi: X\to\mathcal{F}'_{\tau'}(\omega)$ by setting $\Phi(x)=\{U\mid x\in U\in\omega\}$ for every $x\in F$ and $\Phi(x)=\omega$ for every $x\in X\setminus F$. It is obvious now that Φ is l. s. c. and hence, by c), it has an u. s. c. selection ψ. Therefore, the covering $\mu=\{\psi^{-1}(U)\mid U\in\omega\}$ satisfies all our requirements (see the proof of the previous theorem).

Theorem 5. *Let X be a T_1-space and $F_1, F_2, \ldots, F_k, \ldots$ be a sequence of closed subsets of X. Then the following a), b), c) are equivalent*:

a) *X is normal and τ-pointwise-τ'-paracompact, where $\tau>\aleph_0$, $\tau'\geq\aleph_0$ and $\dim F_k\leq n_k$ for every $k=1, 2, \ldots$;*

b) *every l. s. c. $\Phi: X\to\mathcal{F}_{\tau'}(Y)$, where Y is a metric space of weight $<\tau$, has an u. s. c. selection $\psi: X\to\mathcal{C}(Y)$ such that $|\psi(x)|\leq n_k+1$ for every $x\in F_k$, $k=1, 2, \ldots$;*

c) *every l. s. c. $\Phi: X\to\mathcal{F}_{\tau'}(Y_{\tau''})$, where $\tau''<\tau$, has an u. s. c. selection $\psi: X\to\mathcal{F}(Y_{\tau''})$ such that $|\psi(x)|\leq n_k+1$ for every $x\in F_k$, $k=1, 2, \ldots$;*

Proof. That a) $\to$ b) follows from proposition 2 and lemma 3.5; that b $\to$ c) is obvious.

c) $\to$ a). Having the theorem 3 proven we have only to show that $\dim F_k\leq n_k$ for every $k=1, 2, \ldots$. Fix k and let $\{V_1, V_2, \ldots, V_n\}$ be a finite open covering of F_k. For every i, $i=1, 2, \ldots, n$, there is an open subset U_i of X such that $V_i=F_k\cap U_i$. Therefore $\omega=\{X\setminus F_k, U_1, U_2, \ldots, U_n\}$ is an open covering of X. We identify Y_{n+1} with ω and define $\Phi: X\to\mathcal{C}(\omega)$ by setting $\Phi(x)=\{U\mid x\in U\in\omega\}$ for every $x\in X$. It is clear that Φ is l. s. c. and by c) there is an u. s. c. $\psi: X\to\mathcal{C}(\omega)$ which is a selection for Φ and is such that $|\psi(x)|\leq n_k+1$ for every $x\in F_k$. Put $P_i=\psi^{-1}(U_i)\cap F_k$ for every $i=1, 2, \ldots, n$. It follows that $\mu=\{P_1, P_2, \ldots, P_n\}$ is a closed index-refinement of $\{V_1, V_2, \ldots, V_n\}$ with $\mathrm{Ord}(\mu, F_k)\leq n_k+2$. Thus, by corollary 1.1, $\dim F_k\leq n_k$.

In the same way we obtain the following theorem.

Theorem 6. *Let X be a T_1-space and $F_1, F_2, \ldots, F_k, \ldots$ be a sequence of closed subsets of X. Then the following a), b), c) are equivalent*:

a) *X is τ-pointwise-τ'-collectionwise normal where $\tau>\aleph_0$, $\tau'\geq\aleph_0$ and $\dim F_k\leq n_k$ for every $k=1, 2, \ldots$;*

b) *every l. s. c.* $\Phi\colon X\to\mathcal{F}'_{\tau'}(Y)$, *where* Y *is a metric space of weight* $w(Y)<\tau$, *has an u. s. c. selection* $\psi\colon X\to\mathcal{C}(Y)$ *such that* $|\psi(x)|\leq n_k+1$ *for every* $x\in F_k$, $k=1,2,\ldots$;

c) *every l. s. c.* $\Phi\colon X\to\mathcal{F}'_{\tau'}(Y_{\tau''})$, *where* $\tau''<\tau$, *has an u. s. c. selection* $\psi\colon X\to\mathcal{F}(Y_{\tau''})$ *such that* $|\psi(x)|\leq n_k+1$ *for every* $x\in F_k$, $k=1,2,\ldots$

Corollary 1. *Let* τ *be an infinite cardinal number. Then a* T_1*-space* X *is* τ^+*-pointwise-*$\aleph_0$*-collectionwise normal if and only if* X *is* τ*-collectionwise normal.*

Proof. The "if" part is proved in lemma 1.6, so we have to prove the "only if" part. Let X be τ^+-pointwise-$\aleph_0$-collectionwise normal and let φ be a discrete collection of closed subsets of X, such that $|\varphi|=\tau'\leq\tau$. We consider $H(\tau')=\{y\colon \varphi\to R\,|\,\Sigma\{y^2(F)\,|\,F\in\varphi\}<\infty\}$. For every $F\in\varphi$ let $y_F\in H(\tau')$ be such that $y_F(F)=1$ and $y_F(F')=0$ for every $F'\neq F$, $F'\in\varphi$. Now we define $\Phi\colon X\to\mathcal{C}'(H(\tau'))$ by setting $\Phi(x)=\{y_F\}$ for every $x\in F\in\varphi$ and $\Phi(x)=H(\tau')$ for every $x\in X\setminus\cup\{F\,|\,F\in\varphi\}$. It is easily seen that Φ is l. s. c. Therefore, by b) of theorem 2, Φ admits a continuous selection, say f. Since $H(\tau')$ is a metric space and since $\|y_F-y_{F'}\|=\sqrt{2}$ for every $F'\neq F$, $F,F'\in\varphi$, there is a discrete collection $\{O_F\,|\,F\in\varphi\}$ of open subsets of $H(\tau')$ such that $y_F\in O_F$ for every $F\in\varphi$. It is now obvious that $\{f^{-1}(O_F)\,|\,F\in\varphi\}$ is a discrete collection of open subsets of X such that $F\subset f^{-1}(O_F)$ for every $F\in\varphi$.

Theorem 7. *The following properties of a* T_1*-space* X *are equivalent*:

a) X *is* τ*-collectionwise normal where* $\tau\geqq\aleph_0$;

b) *every continuous mapping* $f\colon F\to Y$, *where* F *is a closed subset of* X *and* Y *is a closed convex subset of a Banach space with* $\omega(Y)\leqq\tau$ *has a continuous extension on* X;

c) *every continuous mapping* $f\colon F\to H(\tau)$ *where* F *is a closed subset of* X, *has a continuous extension on* X (*i. e. there is a continuous mapping* $\tilde f\colon X\to H(\tau)$ *such that* $\tilde f(x)=f(x)$ *for every* $x\in F$).

Proof. a)$\to$b). We define l. s. c. $\Phi\colon X\to\mathcal{C}'(Y)$ by setting $\Phi(x)=\{f(x)\}$ for every $x\in F$ and $\Phi(x)=Y$ for every $x\in X\setminus F$. By corollary 1 and theorem 4, Φ has a selection $\tilde f$ which is the required extension; b)$\to$c) is obvious and for c)$\to$a) one can see the proof of corollary 1.

Teorem 8 ([18]). *The following properties of* T_1*-space* X *are equivalent*:

a) X *is* τ*-collectionwise normal for some* $\tau\geqq\aleph_0$;

b) *every continuous mapping* $f\colon F\to Y$, *where* F *is a closed subset of* X *and* Y *is a complete metric space of weight* $\leqq\tau$, *has a set-valued extensions* $\tilde f_1\colon X\to\mathcal{C}(Y)$ *and* $\tilde f_2\colon X\to\mathcal{C}(Y)$ *such that* $\tilde f_1$ *is u. s. c.* $\tilde f_2$ *is l. s. c. and* $\tilde f_2(x)\subset\tilde f_1(x)$ *for every* $x\in X$;

c) *every continuous mapping* $f\colon F\to Y_\tau$, *where* F *is a closed subset of* X, *has an u. s. c. extension* $\tilde f\colon X\to\mathcal{F}(Y_\tau)$.

Theorem 9 ([18]). *Let* X *be a* τ*-collectionwise normal space and* $F_1,F_2,\ldots,F_k,\ldots$ *be a sequence of closed subsets of* X *such that* $\dim F_k\leq n_k$ *for every* $k=1,2,\ldots$ *Then every continuous mapping* $f\colon F\to Y$, *where* F *is a closed subset of* X *and* Y *is a complete metric space of weight* $\leqq\tau$, *has an u. s. c. extension* $\tilde f\colon X\to\mathcal{C}(Y)$ *such that* $|\tilde f(x)|\leq n_k+1$ *for every* $x\in F_k$ *and every* $k=1,2,\ldots$.

The proofs of the above theorems are left to the reader.

5. Factorization theorems.

Theorem 1 (S. Nedev, M. Čoban [19]). *Let $\Phi: X \to \mathcal{F}(Y)$ be s. f. p., where X is a normal space and Y is a metric space of weight τ. Then there are:*

a) *a metric space Z of weight $w(Z) \leq \tau\aleph_0$;*

b) *a continuous mapping $f: X \to Z$ and*

c) *an l. s. c. carrier $\varphi: Z \to \mathcal{C}(Y)$, such that $\varphi(f(x)) \subset \Phi(x)$ for every $x \in X$; moreover, if F is a closed G_δ-set in X, then one may assume that $f(F)$ is a closed subset of Z.*

Proof. We remember the lemma 1.1. For every $n = 1, 2, \dots$, define a pseudo-metric $\varrho_n(x, y)$ on X by setting $\varrho_n(x, y) = \Sigma\{|f_{\alpha n}(x) - f_{\alpha n}(y)| \mid \alpha \in A_n\}$ for every $x, y \in X$. Next define $\varrho(x, y) = \Sigma\{\varrho_n(x, y)2^{-n} \mid n = 1, 2, \dots\} + |g(x) - g(y)|$, where g is a continuous function $g: X \to [0,1]$ such that $g^{-1}(0) = F$. It is clear that ϱ is a continuous pseudo-metric on X. Now for Z we take the quotient set X/ϱ with the topology induced by the metric ϱ; for f we take the natural projection $f: X \to X/\varrho = Z$. To define the carrier $\varphi: Z \to \mathcal{C}(Y)$ we pick an $y^n_\alpha \in V^n_\alpha$ and put $\varphi_n(z) = \{y^n_\alpha \mid z \in f(U^n_\alpha)\}$ for every $z \in Z$ and every $n = 1, 2, \dots$. Next we define $\varphi(z) = \{y \in \widetilde{Y} \mid y = \lim_{i\to\infty} y_i,\ \varrho(y_i, y_{i+1}) \leq 2^{-i},\ y_i \in \varphi_i(z),\ i = 1, 2, \dots\}$ for every $z \in Z$. Our purpose is to make use of lemma 3.3 so we have to check that the conditions of that lemma are fulfilled. But this is not difficult to be done if we take in consideration that $f(U^n_\alpha)$ is an open subset of Z, that $f^{-1}(f(U^n_\alpha)) = U^n_\alpha$ for every $\alpha \in A_n$ and $n = 1, 2, \dots$ and the proof of proposition 4.1. Therefore, φ is a l. s. c. carrier such that $\varphi(f(x)) \subset \Phi(x)$ for every $x \in X$. Finnally we have to show that $w(Z) \leq \tau\aleph_0$. Let $A = \cup\{A_n \mid n = 1, 2, \dots\}$. Obviously, $|A| \leq \tau\aleph_0$. We define a mapping $p: Z \to l_1(|A|)$ by setting $p(z)(\alpha) = f_{\alpha n}(x)$, where x is an arbitrary point in $f^{-1}(z)$ for every $z \in Z$, $\alpha \in A_n \subset A$ and $n = 1, 2, \dots$. It is almost obvious now that p is an isometrical embedding and this completes the proof.

In the situation described in theorem 1 we shall say (see [19]) that the triple (Z, f, φ) constitutes an l. s. c. weak-factorization for Φ, conserving the word "factorization" for the case where $\varphi(f(x)) = \Phi(x)$ for every $x \in X$. After the proof of theorem 1 it is almost obvious that the following conditions assure the existence of an l. s. c. factorization for Φ (see [19]):

1) $\Phi^{-1}(\gamma)$ is a locally finite collection in X for every open locally finite collection γ in Y.

2) $\Phi^{-1}(U)$ is an open F_σ-set in X for every open subset U of Y.

In order to exploit the single-valuedness of f in theorem 1, we consider the following theorem.

Theorem 2 (B. A. Passynkov, A. Zareloua, see [2]). *Let $f: X \to Y$ be a continuous mapping of a normal space X into a metric space Y of weight $w(Y) = \tau$ and let $F_1, F_2, \dots, F_k, \dots$ be a sequence of closed subsets of X such that $\dim F_k \leq n_k$ for every $k = 1, 2, \dots$. Then there are:*

a) *a metric space Z of weight $w(Z) \leq \tau$;*

b) *continuous mappings $g: X \to Z$ and $h: Z \to Y$;*

c) *a sequence $P_1, P_2, \dots, P_k, \dots$ of closed subsets of Z such that*

1) *$f(x) = h(g(x))$ for every $x \in X$;*

2) *$g(F_k) \subset P_k$ and $\dim P_k \leq n_k$ for every $k = 1, 2, \dots$*

Outline of the proof. In fact we will consider an outline of the Arkhangelskii method [2], which is the best method of proving of factorization theorems. We need the following lemma.

Lemma 1. *Let ω be a locally finite open covering of a normal space X. Then there are locally finite open coverings γ and δ of X such that:*

1) $|\gamma| \leqq |\omega|\aleph_0$, $|\delta| \leqq |\omega|\aleph_0$;

2) *γ is a star-refinement of ω;*

3) *every element of δ intersects only finitely many elements of γ.*

Let us give the definition of the concept of star-refinement. If γ is a collection of subsets of a set X, then, by definition, $\mathrm{St}(x, \gamma) = \cup\{U \mid x \in U \in \gamma\}$ (the star of a point x with respect to γ) for every $x \in X$. The collection γ is said to be a star-refinement of a collection ω if for every $x \in X$ there is an $U \in \omega$ such that $\mathrm{St}(x, \gamma) \subset U$.

Proof of the lemma 1. We identify $Y_{|\omega|}$ with ω and define an l. s. c. carrier $\Phi: X \to \mathcal{C}(\omega)$ by setting $\Phi(x) = \{U \mid x \in U \in \omega\}$ for every $x \in X$. From the local-finiteness of ω we infer that Φ is s. f. p., whence, by theorem 1, Φ has an l. s. c. weak-factorization (Z, f, φ). We consider the open covering $\omega' = \{\varphi^{-1}(U) \mid U \in \omega\}$ of Z. Since Z is a metric space of weight $\leq |\omega|\aleph_0$, there are open locally finite coverings γ' and δ' of Z such that γ' is a star-refinement of ω' and every element of δ' intersects finitely many elements of γ'. Now we define the required γ and δ by setting $\gamma = f^{-1}(\gamma')$ and $\delta = f^{-1}(\delta')$.

Proof of theorem 2. The previous lemma 1 and lemma 1.10 permit us to construct by induction a sequence $\{\omega_i \mid i = 1, 2, \dots\}$ of open locally finite coverings of X such that

1) ω_i is a closure-refinement of $f^{-1}(\gamma_i)$, where γ_i is an open locally finite covering of Y with $\mathrm{diam}(V) < 2^{-i}$ for every $V \in \gamma_i$;

2) ω_{i+1} is a star-refinement of ω_i such that every element of ω_{i+1} intersects only finitely many elements of ω_i;

3) $\mathrm{Ord}(\omega_i, F_k) \leq n_k + 2$ for $i \geqq k$.

We begin with ω_1' and δ_1, where ω_1' and δ_1 are locally finite open coverings of X such that ω_1' is a star-refinement of $f^{-1}(\gamma_1)$ and every element of δ_1 meets only finitely many elements of ω_1' and $|\omega_1'| \leq \tau\aleph_0$, $|\delta_1| \leq \tau\aleph_0$. Applying lemma 1.10 we take ω_1 to be an index-closure-refinement of ω_1' such that $\mathrm{Ord}(\omega_1, F_1) \leq n_1 + 2$; next we consider the covering $f^{-1}(\gamma_2) \wedge \omega_1 \wedge \delta_1 = \{U \cap V \cap W \mid U \in f^{-1}(\gamma_2), V \in \omega_1, W \in \delta_1\}$ and let ω_2' and δ_2 be constructed according to lemma 1 with respect to $f^{-1}(\gamma_2) \wedge \omega_1 \wedge \delta_1$. Again applying lemma 1.10 we take ω_2 to be an index-closure-refinement of ω_2' such that $\mathrm{Ord}(\omega_2, F_k) \leqq n_k + 2$ for $k = 1, 2$ and so on. Thus, the sequence $\{\omega_i \mid i = 1, 2, \dots\}$ is well constructed. Next we define a semi-metric d on X in the following manner:

$$d(x, y) = \begin{cases} 1 & \text{if } y \notin \mathrm{St}(x, \omega_1), \\ 2^{-i} & \text{if } y \in \mathrm{St}(x, \omega_i) \setminus \mathrm{St}(x, \omega_{i+1}), \\ 0 & \text{if } y \in \cap\{\mathrm{St}(x, \omega_i) \mid i = 1, 2, \dots\}, \end{cases}$$

for every $x, y \in X$. By 2) we have that if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$ then $d(x, z) < 2\varepsilon$ for every $\varepsilon > 0$ so that, according to the known metrization results, there is a pseudo-metric ϱ on X such that $d(x, y) \leqq \varrho(x, y) \leq 4d(x, y)$ for every $x, y \in X$. Now define Z to be the quotient set $Z = X/\varrho$ equipped with the metric-topology induced by ϱ and let $g: X \to Z$ be the natural projection. Put $V_U = \langle\{z \in Z \mid g^{-1}(z)\sigma \subset U\}\rangle$ for every $U \in \omega_i$ and every $i = 1, 2, \dots$. Then the collection $B = \{\lambda_i = \{V_U \mid U \in \omega_i\} \mid i = 1, 2, \dots\}$ is a σ-locally finite base for Z

with $|B|\leq\tau\aleph_0$ so that $w(Z)\leq\tau\aleph_0=\tau$ (if τ is a finite number, then the theorem is obvious). It is easily seen that $f(g^{-1}(z))$ is a single point in Y for every $z\in Z$ so that the mapping $h: Z\to Y$ for which $f=h\circ g$, is well determined. Let us check the continuity of h. Fix an arbitrary point $z_0\in Z$ and denote $y_0=h(z_0)$. If ε is a given positive number then there is an n such that $\mathrm{St}(y_0, \gamma_n)\subset O_\varepsilon(y_0)$. Take $x_0\in g^{-1}(z_0)$ and choose $U_0\in\omega_n$ such that $\mathrm{St}(x_0, \omega_{n+1})\subset U_0$. Now we will show that $h(O_{2^{-n-2}}(z_0))\subset O_\varepsilon(y_0)$ (and $O_{2^{-n-2}}(z_0)\subset V_{U_0}$). To do this it is sufficient to show that $g^{-1}(z)\subset U_0$ for every $z\in O_{2^{-n-2}}(z_0)$ and that $f(U_0)\subset O_\varepsilon(y_0)$. The latter is almost obvious, namely, there is a $W\in\gamma_n$ such that $U_0\subset f^{-1}(W)$. Hence $y_0\in f(U_0)\subset W\subset \mathrm{St}(y_0, \gamma_n)\subset O_\varepsilon(y_0)$. Let now $z\in O_{2^{-n-2}}(z_0)$, For every $x\in g^{-1}(z)$ we have $d(x, x_0)\leq\varrho(x, x_0)<2^{-n-2}$ so that $x\in \mathrm{St}(x_0\ \omega_{n+1})\subset U_0$.

Now, to finish the proof, we have only to construct the closed sets P_k, $k=1, 2, \ldots$. For this purpose we define $P_k^j=\{z\in Z\mid \mathrm{Ord}(\lambda_j, z)\leq n_k+2\}$ for every $k, j=1, 2, \ldots$. Obviously, every such P_k^j is a closed subset of Z and $g(F_k)\subset P_k^j$ for every $j=k, k+1, \ldots$ It remains to show that $\dim P_k\leq \dim F_k\leq n_k$ for every k, where $P_k=\cap_{j=k}^{\infty}P_k^j$. Considering the metric space P_k for a fixed k we define $\lambda_i'=\{V\cap P_k \mid V\in\lambda_i\}$ for $i=k, k+1, \ldots$. Then the sequence $\{\lambda_i'\mid i=k, k+1, \ldots\}$ of open locally finite coverings of P_k has the properties

I) λ_{i+1}' is a (star) refinement of λ_i' for every $i=k, k+1, \ldots$;

II) $\mathrm{Ord}(P_k, \lambda_i')\leq n_k+2$ for every $i=k, k+1, \ldots$;

III) $\mathrm{diam}(W)\leq 2^{-i+4}$ for every $W\in\lambda_i'$ and $i=k, k+1, \ldots$

Thus, $\dim P_k\leq n_k$ accorping to known results of dimension theory (see, for instance, P. Vopenka [22]).

Theorem 3. *Let X be a T_1-space and $F_0, F_1, F_2, \ldots$ be a sequence of closed subsets of X. Then following a), b), c) are equivalent:*

a) *X is τ-collectionwise normal with $\tau\geq\aleph_0$, F_0 is a G_δ-set in X and $\dim F_k\leq n_k$ for every $k=1, 2, \ldots$;*

b) *every l. s. c. carrier $\Phi: X\to \mathcal{C}'(Y)$, where Y is a metric space of weight $\leq\tau$ has an u. s. c. weak-factorization (Z, f, ψ) such that $f(F_0)$ is a closed subset of Z with $F_0=f^{-1}(f(F_0))$ and $|\psi(f(x))|\leq n_k+1$ for every $x\in F_k$, $k=1, 2, \ldots$*

Proof. That b)$\to$a) follows immediately by theorem 4.6 because $\psi_0 f$ is an u. s. c. selection for Φ. So let us prove that a)$\to$b). By theorem 1 we have an l. s. c. weak-factorization (Z_1, g, φ) for Φ such that $g(F_0)$ is a closed subset of Z_1 with $F_0=g^{-1}(g(F_0))$. Therefore, by theorem 2 there are a) a metric space Z with $w(Z)\leq w(Z_1)$, b) continuous mapping $f: X\to Z$ and $h: Z\to Z_1$ and c) a sequence $P_1, P_2, \ldots$ of closed subsets of Z, such that $g(x)=h(f(x))$ for every $x\in X$, $f(F_k)\subset P_k$ and $\dim P_k\leq n_k$ for every $k=1, 2, \ldots$. Moreover, $f(F_0)=h^{-1}(g(F_0))$ is a closed subset of Z with $F_0=f^{-1}(f(F_0))$. Now, obviously, the carrier $\varphi_0 h: Z\to \mathcal{C}(Y)$ is l. s. c. and, consequently, according to theorem 4.5 (or 4.6) it has an u. s. c. selection $\psi: Z\to\mathcal{C}(Y)$ such that $|\psi(z)|\leq n_k+1$ for every $z\in P_k$, $k=1, 2, \ldots$, and hence (Z, f, ψ) is the required u. s. c. weak-factorization.

It is clear that, in the same way, one can obtain some other factorization theorems analogous to theorem 3. Theorem 3 was especially formulated because we shall need it in 7.

6. Some properties of the dimension dim. In this section we list some known results from the general dimension theory. A part of them is needed in the last section and another part permits us to show by a corresponding example that the expression u. s. c., appearing in condition c) of theorems 4.5 and 4.6 cannot be replaced, in general, by l. s. c. The results of the dimension theory are listed here, as a rule, without proofs, put in some cases, which seem to be more convenient, we exploit the occasion to demonstrate that the selection and factorization methods are applicable also in the dimension theory and we give some proofs, based on the theory developed above.

Theorem 1 (known as countable-sum theorem). *If X is a normal space and $X=\cup_{i=1}^{\infty}F_i$, where F_i is a closed subset of X with* $\dim F_i\leq n$ *for every* $i=1,2,\ldots$, *then* $\dim X\leq n$.

Proof. To prove this theorem it is sufficient to consider the following two particular cases of theorem 4.5 (or 4.6).

(A) The following properties of a T_1-space X are equivalent:

a) X is normal and $\dim X\leq n$;

b) every l. s. c. $\Phi\colon X\to\mathcal{C}'(Y_{\aleph_0})$ has an u. s. c. selection $\psi\colon X\to\mathcal{F}(Y_{\aleph_0})$, such that $|\psi(x)|\leq n+1$ for every $x\in X$.

(B) Let X be a T_1-space and $F_1, F_2,\ldots$ be a sequence of closed subsets of X. Then the following c) and d) are equivalent:

c) X is normal and $\dim F_i\leq n$ for every $i=1,2,\ldots$;

d) every l. s. c. $\Phi\colon X\to\mathcal{C}'(Y_{\aleph_0})$ has an u. s. c. selection $\psi\colon X\to\mathcal{F}(Y_{\aleph_0})$ such that $|\psi(x)|\leq n+1$ for every $x\in\cup_{i=1}^{\infty}F_i$.

Now, by the assumption of theorem 1, c) holds. Consequently, c) $\to$ d) $=$(since $X=\cup_{i=1}^{\infty}F_i$) $=$ b) $\to$ a) and this is just the conclusion of theorem 1.

This proof of theorem 1 is somewhat attractive, but in fact one may prove the theorem in a simpler manner. Namely, let $\omega=\{U_1,U_2,\ldots,U_k\}$ be a finite open covering of X. Using the lemma 1.10 we construct a sequence $\{\gamma_i=\{V_j^i\,|\,j=1,2,\ldots,k\}\,|\,i=1,2,\ldots\}$ of open coverings of X with the following properties:

1) $\gamma_1=\omega$ and γ_{i+1} is an idnex-closure-refinement of γ_i for $i=2,3,\ldots$;

2) $\mathrm{Ord}\,(\delta_j,F_i)\leq n+2$ whenever $i\leq j$.

Finally we put $H_j=\cap\{[V_j^i]\,|\,i=1,2,\ldots\}$ for every $j=1,2,\ldots,k$ and remark that $\mu=\{H_1,H_2,\ldots,H_k\}$ covers X and is a closed index-refinement of ω such that $\mathrm{Ord}\,(\mu,X)\leq n+2$. Thus, corollary 1.2 finishes the proof (due to J. Chaber).

Another known result we are able to obtain here (and we shall need in 7) is the following one:

Theorem 2 (C. H. Dowker [23]). *Let F be a closed subset of a normal space X such that* $\dim F\leq n$ *and* $\dim H\leq n$ *for every closed subset* $H\subset X\setminus F$. *Then* $\dim X\leq n$.

Proof. Let $\omega=\{U_1,U_2,\ldots,U_k\}$ be a finite open covering of X. There is a collection $\mu=\{F_1,F_2,\ldots,F_k\}$ of closed subsets of F such that μ covers F, μ is an index-refinement of ω and $\mathrm{Ord}\,(\mu,F)\leq n+2$. By Lemma 1.8 there is a collection $\gamma=\{V_1,V_2,\ldots,V_k\}$ of open subsets of X such that $F_i\subset V_i\subset U_i$ for every $i=1,2,\ldots,k$ and $\mathrm{Ord}\,(\gamma,X)\leq n+2$. Let $V=\cup\{V_i\,|\,i=1,2,\ldots,k\}$. By the normality of X there is a closed G_δ-set H in X such that $F\subset H\subset V$. Define $U_i'=U_i\setminus H$ for every $i=1,2,\ldots,k$, and consider the collection

$\omega' = \{V_1, V_2, \ldots, V_k, U'_1, U'_2, \ldots, U'_k\}$, which is an open covering of X. Remark that $X \setminus H = \cup \{H_i \mid i = 1, 2, \ldots\}$, where H_i is a closed subset of X, $H_i \subset X \setminus F$ for every $i = 1, 2, \ldots$. So, by the assumption, $\dim H_i \leq n$. Now we identify Y_{2k} with ω' and define l. s. c. $\Phi: X \to \mathcal{C}(Y_{2k}) = \mathcal{C}(\omega')$ by setting $\Phi(x) = \{W \mid x \in W \in \omega'\}$ for every $x \in X$. Note that $|\Phi(x)| \leq n+1$ for every $x \in H$. By theorem 4.6 (or 4.5) Φ has an u. s. c. selection $\psi: X \to \mathcal{C}(\omega')$ such that $|\psi(x)| \leq n+1$ for every $x \in H_i$, $i = 1, 2, \ldots$. Thus, $|\psi(x)| \leq n+1$ for every $x \in X$, whence the collection $\nu = \{\psi^{-1}(W) \mid W \in \omega'\}$ is a closed index-refinement of ω', which covers X and is such that $\operatorname{Ord}(\nu, X) \leq n+2$.

Theorem 3. *Let X be a metric space of weight τ and let $F_1, F_2, \ldots$ be a sequence of closed subsets of X such that* $\dim F_k \leq n_k$ *for* $k = 1, 2, \ldots$ *Then there are: a subset M of $B(\tau)$ and a perfect mapping* $f: M \xrightarrow{\text{onto}} X$ *such that* $|f^{-1}(x)| \leq n_k + 1$ *for every $x \in F_k$ and $k = 1, 2, \ldots$*

Here "f is a perfect mapping" means, as usually, that f is continuous, $f^{-1}(x)$ is a compact subset of M for every $x \in X$ and the image of any closed subset of M under f is a closed subset of X.

In order to prove this theorem we need the following known result.

Theorem 4. *Let X be a (complete) metric space of weight τ. Then there are: a (closed) subset M_1 of $B(\tau)$ and a continuous compact open mapping* $g: M_1 \xrightarrow{\text{onto}} X$.

Here "open" means that the image of any open subset of M_1 under f is an open subset of X and "compact" means that $f^{-1}(x)$ is a compact subset of M_1 for every $x \in X$.

Proof. Consider a sequence $\{\omega_i \mid i = 1, 2, \ldots\}$ of open locally finite coverings of X such that $\operatorname{diam}(U) < 2^{-i}$ for every $U \in \omega_i$, $i = 1, 2, \ldots$. Next we identify ω_i with a suitable subset of Y_τ (note that $|\omega_i| \leq \tau$ for every $i = 1, 2, \ldots$) so that $Y = \Pi\{\omega_i \mid i = 1, 2, \ldots\}$ is a subspace of $B(\tau) = Y_\tau^{\aleph_0}$. Define

$$M_1 = \left\{ y = (U_1, U_2, \ldots, U_n, \ldots) \in Y \mid \text{there is a point } x \in X \text{ such that } x \in \bigcap_{n=1}^{\infty} U_n \right\}$$

and let $g: M_1 \to X$ be defined by the formula $g(y) = \cap_{i=1}^{\infty} U_i$ for every $y = (U_1, U_2, \ldots, U_n, \ldots) \in M_1$. The uniform continuity of g is obvious. In fact, if $y', y'' \in M_1$ and $p(y', y'') < 2^{-n} \Leftrightarrow U'_i = U''_i$ for every $i = 1, 2, \ldots n$, hence $g(y'), g(y'') \in U'_n = U''_n \in \omega_n$ so that $\varrho(g(y'), g(y'')) \leq \operatorname{diam}(U'_n) < 2^{-n}$.

Let us show that $g^{-1}(x)$ is a closed subset of Y whatever $x \in X$ is. Let $y^{(i)} \in g^{-1}(x)$ for every $i = 1, 2, \ldots$ and let $\lim_{i\to\infty} y^{(i)} = y = (U_1, U_2, \ldots, U_n, \ldots) \in Y$. For an arbitrary n there is an i such that $p(y, y^i) < 2^{-n}$, whence $U_n = U_n^{(i)} \ni x$

Now we are able to show that $g^{-1}(x)$ is a compact subset of M_1 for every $x \in X$. Denote $Y_n(x) = \{U \mid x \in U \in \omega_n\}$ for every $n = 1, 2, \ldots$ and let $Y(x) = \Pi\{Y_n(x) \mid n = 1, 2, \ldots\}$. Obviously $Y(x)$ is a compact subset of Y such that $Y(x) \supset g^{-1}(x)$. Thus, $g^{-1}(x)$ is compact. Finally we have to see that g is an open mapping, but this is almost obvious. In fact, if $y = (U_1, U_2, \ldots, U_n, \ldots) \in M_1$ and $x = g(y)$, then $g(O_{2^{-n}}(y)) \supset \cap_{i=1}^{n} U_i \ni x$ and this completes the proof (remark that if X is complete then one can extend g, by the uniform continuity, over $[M_1]$).

Let us now prove theorem 3. We take M_1 and g as in theorem 4 and consider the carrier $\Phi: X \to \mathcal{C}(M_1)$ defined by the formula: $\Phi(x) = g^{-1}(x)$ for every $x \in X$. The openness of g implies that Φ is l. s. c. Thus, by theorem 4.5 (or 4.6) there is an u. s. c. selection $\psi: X \to \mathcal{C}(M_1)$ for Φ snch that $|\psi(x)| \leq n_k+1$ for every $x \in F_k$, $k=1, 2, \dots$ We define $M=\cup\{\psi(x) \mid x \in X\}$ and $f=g/M$. The circumstance that ψ is u. s. c. implies that f is a closed mapping and this completes the proof.

Now let us recall the concept of another dimension-type invariant, namely the concept of the large inductive dimension $\operatorname{Ind} X$, defined in the usual manner:

1) $\operatorname{Ind} X=-1$ iff $X=\varnothing$;

2) $\operatorname{Ind} X \leq n$ iff for every couple (F, U) such that F is a closed subset of X and U is an open neighbourhood of F in X there is an open subset V of X with the properties: a) $F \subset V \subset [V] \subset U$ and b) $\operatorname{Ind}(\operatorname{Fr}(V)) \leq n-1$, where, by definition, $\operatorname{Fr}(V)=[V] \setminus \langle V \rangle$.

In the sequel we shall need the following known results:

Lemma 2. *For a normal space X,* $\dim X=0$ *iff* $\operatorname{Ind} X=0$.

Lemma 3. $\dim X \leq \operatorname{Ind} X$ *for every normal space X.*

Theorem 5 (K. Morita, K. Nagami, see [15]). *Let X and Y be normal spaces with* $\operatorname{Ind} X=0$ *and let* $f: X \xrightarrow{\text{onto}} Y$ *be a perfect mappiug such that* $|f^{-1}(y)| \leq n+1$ *for every* $y \in Y$. *Then* $\operatorname{Ind} Y \leq n$.

Theorem 6 (M. Katetov [9]). $\dim X=\operatorname{Ind} X$ *for every metric space X.*

Proof. That $\dim X \leq \operatorname{Ind} X$ follows from lemma 3. That $\operatorname{Ind} X \leq \dim X$ follows from theorem 5, the particular case when $F_k=X$ for every $k=1, 2, \dots$ of theorem 3 and lemma 2.

Theorem 7 (K. Nagami and K. Morita, see [15]). *Let X and Y be metric spaces,* $\dim X \leq n$ *and* $f: X \xrightarrow{\text{onto}} Y$ *be a perfect mapping such that* $|f^{-1}(y)|=k$ *for every* $y \in Y$. *Then* $\dim Y \leq n$.

Theorem 8. *Let X be a metric space and let* $\dim X \leq n$. *Then* $X=\cup_{i=0}^{n} A_i$, *where* $\dim A_i \leq 0$ *for every* $i=0, 1, 2, \dots, n$.

Proof. By the particular case when $F_k=X$ for $k=1, 2, \dots$, of theorem 3 we have a subset M of $B(\tau)$ and a perfect mapping $f: M \xrightarrow{\text{onto}} X$, such that $|f^{-1}(x)| \leq n+1$ for every $x \in X$. Define $A_i=\{x \in X \mid |f^{-1}(x)|=i+1\}$ for $i=0, 1, 2, \dots, n$. By theorem 7 $\dim A_i \leq 0$ for every $i=0, 1, 2, \dots, n$. Since $X=\cup\{A_i \mid i=0, 1, 2, \dots, n\}$ the theorem is proved.

Theorems 1 and 6 permit to obtain the following modification of theorem 3 (the proof is somewhat technical and we omit it).

Theorem 9 (K. Morita [14]). *Let X be a metric space of weight τ and $A_1, A_2, \dots$ be a sequence of (not necessarily closed) subsets of X such that* $\dim A_i \leq 0$ *for every* $i=1, 2, \dots$. *Then there are a subset M of $B(\tau)$ and a perfect mapping* $f: M \xrightarrow{\text{onto}} X$ *such that* $|f^{-1}(x)| \leq n$ *for every* $x \in A_n$ *and every* $n=1, 2, \dots$

Theorems 8 and 9 give

Theorem 10. *Let X be a metric space and X' be a (not necessarily closed) subset of X with* $\dim X' \leq n$. *Then there are a subset M of $B(\tau)$ and a perfect mapping* $f: M \xrightarrow{\text{onto}} X$ *such that* $|f^{-1}(x)| \leq n+1$ *for every* $x \in X'$.

Theorems 3 and 10 give

Theorem 11. *Let X and Y be metric spaces, X' a (not necessarily closed) subset of X with* $\dim X' \leq n$. *Then every l. s. c.* $\Phi: X \to \mathcal{F}(Y)$ *has an u. s. c. selection* $\psi: X \to \mathcal{C}(Y)$ *such that* $|\psi(x)| \leq n+1$ *for* $x \in X'$.

P r o o f. Let M and f be as in theorem 10 and consider $\varphi: M \to \mathcal{F}(Y)$, defined by the formula $\varphi = \Phi \circ f$. By theorem 3 (since $\dim M = 0$), there is a single-valued selection $g: M \to Y$ for φ. Define $\psi: X \to \mathcal{C}(Y)$ by setting $\psi = g \circ f^{-1}$. It is obvious that ψ satisfies all our requirements.

E x a m p l e 1. Let X be the closed unit interval. By theorem 4 there are a (closed) subset M of $B(\aleph_0)$ and a continuous open compact mapping $f: M \xrightarrow{\text{onto}} X$. Define $\Phi: X \to \mathcal{C}(M)$ by the formula $\Phi = f^{-1}$. Obviously, Φ is l. s. c. Thus, by theorem 3, there is an u. s. c. selection $\psi: X \to \mathcal{C}(M)$ for Φ such that $|\psi(x)| = 1$ for every rational point $x \in X$. But Φ has not an l. s. c. selection with the same property (i. e. to be single-valued on the set of rational points). Really, suppose $\varphi: X \to \mathcal{C}(M)$ is an l. s. c. selection for Φ with $|\varphi(x)| = 1$ for every rational $x \in X$ and let $\omega = \{U_1, U_2, \ldots, U_n\}$ be an open covering of X, which has not a disjoint open refinement. Denote $M_1 = \cup\{\varphi(x) \mid x \in X\}$ and $V_i = f^{-1}(U_i) \cap M_1$ for every $i = 1, 2, \ldots, n$. Then $\{V_1, V_2, \ldots, V_n\}$ is an open coveriug of M_1. Therefore, because $\dim M_1 = 0$, there is a disjoint open covering $\{W_1, W_2, \ldots, W_n\}$ of M_1 such that $W_i \subset V_i$ for $i = 1, 2, \ldots, n$. Thus, $\gamma = \{\varphi^{-1}(W_1), \varphi^{-1}(W_2), \ldots, \varphi^{-1}(W_n)\}$ is an open refinement of ω with $\mathrm{Ord}(\gamma, Q) = 2$, where Q is the set of rationals in X. But this implies that γ is disjoint (note that Q is dense in X), which is a contradiction.

7. Hannerization. We begin with the following d e f i n i t i o n.

Let $\mathcal{H}$ be a class of topological spaces. For a given space Y we shall say that Y is an absolute-extensor — AE (or absolute-neighbourhood extensor — ANE) [in co-dim *k] with respect to $\mathcal{H}$ iff every continuous mapping $f: F \to Y$, where F is a closed subset of a member X of $\mathcal{H}$ [such that* $\dim(X \setminus F) \leq k$*] has a continuous extension over X (over a neighbourhood OF of F in X).*

T h e o r e m 1 (S. N e d e v, M. Č o b a n, [20]). *Let the metrizable space Y of weight τ be an AE (ANE) [in co-*dim *k] with respect to the class of all {no more than n-dimensional} metric spaces of weight $\leq \tau$. Then Y is a such extensor with respect to the class of all {no more than n-dimensional} τ-collectionwise normal, perfectly normal spaces. If, in addition, Y is Čech complete (i. e. if the topology of Y is induced by some complete metric), then Y is such extensor with respect to the class of all {no more than n-dimensional} τ-collectionwise normal spaces.*

P r o o f. Consider a continuous mapping $g: F \to Y$, where F is a closed subset of a space X, the latter belonging to the corresponding class. Define the l. s. c. $\Phi: X \to \mathcal{C}'(\widetilde{Y})$ by setting $\Phi(x) = \{f(x)\}$ for every $x \in F$ and $\Phi(x) = \widetilde{Y}$ for every $x \in X \setminus F$ (here $\widetilde{Y} = Y$ if Y is complete and $\widetilde{Y}$ is a completion of Y otherwise). By theorem 5.3, Φ has an u. s. c. weak-factorization (Z, g, ψ). Put $H = \{z \in Z \mid |\psi(z)| = 1\}$. It is easily seen that H is a G_δ-subset of Z. Namely, take $U_n(z)$ to be a neighbourhood of z such that $\psi(t) \subset O_{2^{-n}}(\psi(z))$ for every $t \in U_n(z)$ ($U_n(z)$ exists by the upper-semi-continuity of ψ) and let $U_n = \{U_n(z) \mid z \in H\}$ for $n = 1, 2, \ldots$. Then $H = \cap_{n=1}^{\infty} U_n$ and U_n is open in Z for every $n = 1, 2, \ldots$. Thus, $g^{-1}(H)$ is a G_δ-subset of X such that $F \subset g^{-1}(H)$. There exists, therefore, a closed G_δ-subset F_0 of X such that $F \subset F_0 \subset g^{-1}(H)$ (we use the normality of X). Thus, $X \setminus F_0 = \cup_{i=1}^{\infty} F_i$, where F_i is a closed subset of X [and $\dim F_i \leq k$] for every $i = 1, 2, \ldots$ According to theorem 5.3 the mapping g has a factorization (Z_1, g_1, h) such that $P = g_1(F_0)$ is a closed subset of Z_1, F_0

$=g_1^{-1}(g_1(F_0))$ [and $\dim(Z_1\setminus P)\leq k$ by the countable-sum theorem]. Note that in the case when X is perfectly normal we take $F_0=F$. Now $\psi_0 h: P\to Y$ is a continuous mapping having, by the hypothesis of the theorem, an extension, say $\tilde{h}$. Then $\tilde{h}\circ g_1$ is the required extension.

Theorem 2. *If the {no more than n-dimensional} metrizable space Y is* ANE *in* co-dim *zero with respect to the class of all {no more than n-dimensional} paracompact spaces, then Y is Čech-complete.*

Proof (see C. H. Dowker [6]). Let $\tilde{Y}$ be a completion of Y. We define a new topological space X by introducing a new topology on $\tilde{Y}$ in the following manner: a subset U of $\tilde{Y}$ is declared to be open in X iff $U=V\cup W$, where V is open in $\tilde{Y}$ and $W\subset\tilde{Y}\setminus Y$. First of all we show that X is collectionwise normal. Let $\{F_\alpha\mid\alpha\in A\}$ be a discrete collection of closed subsets of X. Then $\{Y\cap F_\alpha\mid\alpha\in A\}$ is a discrete (in Y) collection of closed subsets of Y, so that there is a disjoint collection $\{W_\alpha\mid\alpha\in A\}$ of open subsets of Y with $Y\cap F_\alpha\subset W_\alpha$ for every $\alpha\in A$. Let V_α be an open subset of $\tilde{Y}$ such that $W_\alpha=Y\cap V_\alpha$ for every $\alpha\in A$. The collection $\{V_\alpha\mid\alpha\in A\}$ is also disjoint (remember that Y is dence in $\tilde{Y}$). Now put $U_\alpha=(V_\alpha\cup(F_\alpha\setminus Y))\setminus\cup\{F_\beta\mid\beta\in A,\beta\neq\alpha\}$. It is clear that $\{U_\alpha\mid\alpha\in A\}$ is a disjoint collection of open subsets of X such that $F_\alpha\subset U_\alpha$ for every $\alpha\in A$. Thus, X is collectionwise normal. Next let us show that X is paracompact. Let ω be an open covering of X. The collection $\omega'=\{U\cap Y\mid U\in\omega\}$ is an open covering of Y and let $\gamma'=\{W_U\mid U\in\omega\}$ be a locally finite (in Y) open index-refinement of ω'. We have $W_U=Y\cap V_U$ for some open V_U in $\tilde{Y}$ and let $G_U=U\cap V_U$ for every $U\in\omega$. Thus, $\gamma''=\{G_U\mid U\in\omega\}$ is a collection of open subsets of X, which covers Y and is such that $\operatorname{Ord}(\gamma'',Y)\leq\aleph_0$. Therefore, by lemma 1.6, there is a locally finite (in X) collection $\gamma=\{\Gamma_U\mid U\in\omega.\}$ of open subsets of X such that γ covers Y and refines ω. Hence the collection $\gamma\cup\{\{x\}\mid x\in X\setminus\cup\{\Gamma_U\mid U\in\omega\}\}$ is an open locally finite refinement of ω. Thus, X is paracompact. {Moreover, if $\dim Y\leq n$ then $\dim X\leq n$ by theorem 6.2}. Now, by the hypothesis, the identity mapping of Y can be extended over a neighbourhood OY of Y in X, so that we have a continuous mapping $f: OY\to Y$ such that $f(y)=y$ for every $y\in Y$. Let ϱ be the metric on $\tilde{Y}$ and consider the function $\varphi: Oy\to[0,+\infty)$ defined by the formula $\varphi(x)=\varrho(x,f(x))$ for every $x\in OY$. Obviously, φ is continuous and $Y=\varphi^{-1}(0)$ so that Y is a G_δ-subset of OY. By the definition of X we infer then that Y is a G_δ-subset of $\tilde{Y}$, which completes the proof.

Theorem 3. *If a metrizable space Y is an* ANE *with respect to the class of all* 0-*dimensional perfectly normal τ-paracompact spaces, then* $w(Y)\leq\tau$.

Proof (see C. H. Dowker [6]). Suppose that $w(Y)\geq\tau^+$ and let $\{y_\alpha\mid\alpha\in A\}$ be a discrete collection of poinls in Y with $|A|=\tau^+$. Now remember the 0-dimensional, perfectly normal, τ-paracompact space X, which was constructed in example 1.1, and which contains a discrete collection of points $\{x_\alpha\mid\alpha\in A\}$ with the properties: 1) $|\{x_\alpha\mid\alpha\in A\}|=\tau^+$ and 2) there is no disjoint collection $\{U_\alpha\mid\alpha\in A\}$ of open subsets of X such that $x_\alpha\in U_\alpha$ for every $\alpha\in A$.

There is an obvious one-to-one continuous mapping between the sets $X'=\{x_\alpha\mid\alpha\in A\}$ and $Y'=\{y_\alpha\mid\alpha\in A\}$, say $f: X'\to Y'$. Under the assumption of the theorem, f has a continuous extension $\tilde{f}: OX'\to Y$. But Y is a metric space and, therefore, there is a disjoint collection $\{V_\alpha \mid \alpha \in A\}$ of open subsets of Y such that $y_\alpha \in V_\alpha$ for every $\alpha \in A$. Hence $\{\widetilde{f}^{-1}(V_\alpha) \mid \alpha \in A\}$ is a disjoint collection of open subsets of X such that $x_\alpha = \widetilde{f}^{-1}(y_\alpha) \in \widetilde{f}^{-1}(V_\alpha)$ for every $\alpha \in A$, which is a contradiction.

8. Some remarks and open questions.

1. The presented proof of theorem 1.1 was kindly communicated to the author by R. Engelking in a private letter after that the existence of an analogous proof (somewhat more complicated in details (see [20])) was announced by the author in [19].

2. The lemma 3.1 is characteristic for all the carriers $\Phi: X \to \mathfrak{F}(Y)$ such that: a) X is normal and Y is a metric space and b) there are an u. s. c. $\psi: X \to \mathcal{C}(Y)$ and an l. s. c. $\varphi: X \to \mathcal{C}(Y)$ such that $\varphi(x) \subset \psi(x) \subset \Phi(x)$ for every $x \in X$.

In [19] we called such a couple (φ, ψ) a Michael couple of carriers, subordinated to the carrier Φ.

Question 1. Is every carrier Φ having a Michael's couple subordinated to it s. f. p.?

3. One can find an excellent discussion of the continuous selection problem in Michael's paper [12]. In fact all the results and examples in [12] (except the proof of Theorem 3.2′ a) $\to$ b) which is not correct) are fundamental for the selection theory and it is obvious that here we have exploited a great number of Michael's ideas.

Our next question 2 regards the theorem 3.1‴ in [12] which characterizes the perfect-normality by means of selections. Namely, is the carrier Φ in b) and c) of Theorem 3.1‴ in [12] s. f. p.?

An easy exercise is the following assertion.

The following properties of a T_1-space X are equivalent:

a) X is perfectly normal;

b) every l. s. c. $\Phi: X \to L(R)$ has a continuous single-valued selection;

c) every l. s. c. $\Phi: X \to L(R)$ has an u. s. c. set-valued selection.

Here $L(R) = \{\{0\}, \{1\}, (0, +\infty)\}$ (a three-element collection).

4. Question 3. In [12] the problem of extension of a partial single-valuad selection is discussed. What about extension of a partial set-valued selection and especially in the case when dimension type restrictions are considered?

5. Does the inversion of theorem 4.9 hold? At least in the case when X is metric space and $F_k = X$ and $n_k = n$ for $k = 1, 2, \ldots$ (Smirnov's problem)?

6. The factorization method by which theorem 7.1 is proved allows to obtain various theorems of this type (see, for instance, [20]). The method mentioned above, together with [18; proposition 1] yields some refinements of theorems 4.8 and 4.9, for instance one may assert that $|\widetilde{f}_1(x)| < \aleph_0$ in theorem 8 and $|\widetilde{f}(x)| < \aleph_0$ in theorem 4.9 hold for every $x \in X$ and if $\dim(X \setminus F) \leq n$, $n \geq 0$, then $|\widetilde{f}_1(x)| \leq n+1$ holds for every $x \in X$ in theorem 4.8. Moreover, the completeness of Y is not necessary in the case when F is G_δ-set in X (see again Question 4).

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